

The Optimal Chips Mechanism in a Model of Favors

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Abstract

We interpret an “equality matching” relationship as a chips mechanism, and we characterize the optimal chips mechanism in a repeated favor-exchange game. We then compare the optimal chips mechanism with a more sophisticated favor-exchange relationship in which the size of a favor owed may decline over time. Abdulkadirođlu and Bagwell (2012) show that the “highest symmetric self-generating line” of payoffs in the favor-exchange game is supported by such a relationship. We find sufficient conditions for a sophisticated favor-exchange relationship of this kind to produce higher levels of cooperation and exchange among players than they achieve in the optimal chips mechanism.

1 Introduction

Psychological and anthropological studies report that an important category of human social interactions emphasizes trust and reciprocity. Fiske (1992) surveys ethnographic field work and experimental studies and argues that virtually all human social interactions can be described in terms of four patterns, each with a distinctive psychological basis. One pattern is called “equality matching” (EM). As Fiske (1992, p. 703) states, “The operating principle is that when people relating in an EM mode receive a favor, they feel

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obligated to reciprocate by returning a favor.” In EM relationships, people keep track of the imbalances between them and engage in score keeping, with the restoration of balance being a primary aspiration. As Fiske (1992, p. 705) puts it, “People think about *how much* they have to give to reciprocate or compensate others or come out even with them. EM always entails some kind of additive tally of who owes what and who is entitled to what.”

We study EM relationships in a *repeated favor-exchange game with private information*. The game has two players. Players a and b play the following stage game every period: either player a is given income, player b is given income, or neither player is given income. Each player is privately informed as to whether or not he has income. Thus, if a player does not receive income, then the player does not observe whether neither player received income or the other player received income. Next, if one player receives income, then that player may choose to send some or all of his income to the other player. If a transfer is made, then the level of the transfer is publicly observed. Motivated by the trust game studied in the experimental literature (Berg, Dickhaut and McCabe, 1995), we assume that the transfer is value enhancing.; that is, the benefit of the transfer to the recipient exceeds the cost of the transfer to the sender.

Our game may be understood as a model of favors. When a player receives income and makes a transfer to the other player, the former player provides a favor to the latter player. An important feature of our game is then that a player privately observes whether he has the capacity to provide a favor in a given period. Our game is a discrete-time version of the continuous-time game considered by Mobilius (2001). Our stage game is essentially equivalent to Mobilius’ with a different normalization of payoffs. We choose this normalization in order to compare our results with those in our companion paper (Abdulkadiroğlu and Bagwell, 2012).

We model an EM relationship as a chips mechanism. The players own a certain number of chips, which are worthless on their own. The players utilize chips only to keep an “additive tally of who owes what and who is entitled to what.” When player i receives income and the other player j owns some chips, i sends all of his income to the other player, who then gives one chip to i in return. When a player has all of the chips in the game, he does not send any income to the other player.

Our findings shed light on the mechanics and limitations of EM relationships. First, a chips mechanism with a positive number of chips can be supported as an equilibrium if players are sufficiently patient. An $(N + 1)$ -

chips mechanism provides a player with $n \leq N$ chips with higher expected payoff than an N -chips mechanism does. The presence of private information, however, imposes an upper bound on the number of chips that can be used in equilibrium. Indeed, if a player had infinitely many chips, he would clearly not have an incentive to give his income to the other player in exchange for an additional chip.

It is well known that repeated interaction can foster cooperation; however, our interpretation of EM relationships requires further that players are privately informed. To see this point, suppose instead that both players commonly observe when one of them receives income. Given that favors are value enhancing, the players recognize a gain from cooperation: if in all periods any player that receives income were to send all income to the other player, then the players would both enjoy higher payoffs in comparison to the payoffs that they would receive were instead they always to keep all income. For sufficiently patient players, this cooperative behavior can be enforced as a subgame perfect equilibrium, if the players threaten that any deviation induces a reversion to the autarkic Nash equilibrium of the stage game. The repeated favor-exchange game with full information thus produces a behaviorally simple cooperative equilibrium in which score keeping an additive tallies play absolutely no role.

By contrast, when players are privately informed, a player must be given incentive to reveal that he has received income and to make a corresponding transfer. An EM relationship (or chips mechanism) can provide this incentive by ensuring that such a player receives a future benefit. In particular, in an N -chips mechanism, a player's expected utility increases with the number of chips he owns. Thus, when a player gives up his current income in exchange for a chip, he is entitled to a higher expected payoff next period than he was at the beginning of the current period. We refer to this as the incentive compatibility constraint (IC) for the player. For an N -chips mechanism to be an equilibrium, IC must hold for every player with $n < N$ chips. We refer to such an equilibrium mechanism as incentive compatible (IC).

It turns out that the only binding incentive compatibility constraint is the one for the player with $N - 1$ chips. That is, if the player with $N - 1$ chips has incentive to transfer all of his income, then so does any player with a smaller number of chips. In fact, as we state above, the players' payoffs increase as the number of chips, N , in the game increases. However, the maximum feasible payoff, and therefore the payoff of a player with all of the chips, is bounded from above. Also, as N increases, the payoff of a player

with n chips, $n \leq N - 1$, increases more than the payoff of a player with $n - 1$ chips. So it is the player with $N - 1$ chips whose IC constraint is likely to be violated as number of chips increases. Furthermore, if an N -chips mechanism is IC, then so is an $(N - 1)$ -chips mechanism. And if an N -chips mechanism fails to be IC, so does the $(N + 1)$ -chips mechanism. This observation yields a simple algorithm to find the optimal equilibrium chips mechanism that is characterized by the maximum number of chips. Start with $N = 1$.

- Given N -chips, write the expected payoffs recursively. Then applying Blackwell's theorem, obtain a contraction mapping. Calculate the associated expected payoffs as the unique fixed point of the mapping.
- If the IC constraint for the player with $N - 1$ chips is violated, then the optimal number of chips is $N - 1$. Otherwise repeat the two steps with $N + 1$.

Since the number of chips in equilibrium is bounded from above, the algorithm converges in finite time.

Understanding the limitations of an EM relationship is equally important as understanding its mechanics. To this end, we provide a sufficient condition on the discount factor for an N -chips mechanism to fail to be IC. If the discount factor takes an intermediate value, then the optimal chips mechanism utilizes only one chip. That is, when a player gives his income to the other player, he does not send any additional income until the other player pays back the favor. We will refer to the chip mechanism with one chip ($N = 1$) as the *simple EM relationship*. In a more general model, Abdulkadiroğlu and Bagwell (2012) study equilibria that can be characterized using symmetric self-generating lines of payoffs, which correspond to lines of payoffs that have slope minus 1 and are symmetric around the 45° line. The simple EM relationship can be represented as such a self-generating line. In the equilibrium that characterizes a simple EM relationship, the players switch between the end points of the associated self-generating line. Thus, when a player provides a favor by sending all of his income to the other player, the former player becomes the favored player in the next period. The expected payoff for the favored player takes the highest value on the line. In a simple EM relationship, the favored player makes no further transfers until after the other player returns the favor by transferring all of his income. At that point, the expected payoffs of the players switch to the other extreme of the line, which favors the other player.

Abdulkadiroğlu and Bagwell (2012) characterize the highest symmetric self-generating line (HSSGL). This is the symmetric self-generating line that gives the highest total expected payoff to the players. On the HSSGL, and as in the simple EM relationship, when the disfavored player sends all of his income, his expected payoff in the next period takes the highest value on the line. In contrast the simple EM relationship, however, if the favored player receives income, then he transfers a positive portion of this income (i.e., provides a partial favor) and thereby ensures that his payoff in the next period takes the highest value on the line. Furthermore, in the implementation of the HSSGL, if no income transfer is observed in a period, then in the next period the players’ expected payoffs move towards the center on the line. To achieve this movement in payoffs, the implementation requires that the transfer from the disfavored (favored) player decreases (increases) following a period with no income, although the required transfer from the disfavored player is always larger than that required from the favored player. Thus, and as Abdulkadiroğlu and Bagwell (2012) discuss, in such a “sophisticated favor-exchange relationship,” the size of the favor that is owed by the disfavored player declines over time, as neutral (no-income) phases of the relationship are experienced. Hauser and Hopenhayn (2008) independently observe the same pattern, which they call “forgiveness,” in their simulations of the Pareto frontier of the continuous-time version of our game that is analyzed by Möbius (2001). This finding implies that a sophisticated favor-exchange relationship may foster more cooperation than an EM relationship.

Möbius (2001) introduces the repeated favor-exchange game in continuous time and discusses the chips mechanism. Hauser and Hopenhayn (2008) show that the Pareto frontier of Möbius’ game is self-generating. As noted, they observe forgiveness in their simulations of the Pareto frontier and conjecture that this property also holds on the frontier. In a discrete-time model, Nayar (2009) reports parameter restrictions under which the implementation of payoffs on the Pareto frontier requires that continuation values are drawn from the outer boundary of the equilibrium set, where the outer boundary includes the Pareto frontier but is potentially larger. She also provides a partial characterization of the strategies that support payoffs on the Pareto frontier. Kalla (2010) studies two important extensions in discrete time.¹

¹Lau (2011) also studies a model with favor exchange. In his model, the costs and benefits of favors are stochastic, and a player may have private information as to the costs of providing a favor.

First, he introduces incomplete information regarding players' discount factors. He characterizes sufficient conditions under which patient players can separate from impatient players and then implement a favor-exchange relationship. He shows that separation under symmetric equilibria has to take place within a finite time period, after which beliefs diverge and separation becomes impossible. Second, in a complete-information setting, Kalla introduces scope for risk sharing via concave utility functions. He shows that some form of a favor-exchange relationship then becomes possible for all discount factors. Finally, the chips mechanism has been studied in the context of collusion by Skryzpacz and Hopenhayn (2004) and Olszewski and Safronov (2012), while HSSGL's are studied in the context of collusion by Athey and Bagwell (2001).

We introduce the model in the next section. We characterize the optimal chips mechanism in Section 3 and compare it in Section 4 with sophisticated favor-exchange relationships. We conclude in Section 5.

2 Model

We follow the notation of Abdulkadiroğlu and Bagwell (2012) for the sake of comparability of our results. Otherwise, the stage game of our game is isomorphic to Möbius (2001) via payoff normalization.

We study a stylized model with two players, a and b . In the stage game, either player a is given an income of \$1, player b is given an income of \$1, or neither player is given an income. The former two events each occur with probability $p \in (0, 1/2)$ and the latter event thus occurs with probability $1 - 2p$. Each player is privately informed as to whether or not he receives income. Thus, if a player does not receive income, then the player does not observe whether neither player received income or the other player received income. If a player receives income, then that player may send any $x \in [0, 1]$ to the other player. The transferred income becomes qkx . We assume $qk > 1$; that is, the transfer is value enhancing. We assume risk neutral players in order to abstract from insurance arrangements, and we let $\beta \in (0, 1)$ denote the players' common discount factor.

One can interpret x as investment, q as the probability of success of the investment and k as the productivity of a successful investment. Abdulkadiroğlu and Bagwell (2012) provide this interpretation and allow for immediate reciprocity after a successful investment, the success of which is

observed privately.

In this paper, we focus on EM relationships. When people interacting in an EM mode receive a favor, they feel obligated to reciprocate by returning a favor. In EM relationships, people keep track of the imbalances between them and engage in score keeping, with the restoration of balance being a primary aspiration. This kind of behavior is referred as a chips mechanism in game theory. The players own a certain number of chips, which are worthless on their own. The players utilize chips only to keep an “additive tally of who owes what and who is entitled to what.” When player i receives income and the other player j owns some chips, i gives his income to the other player, who then gives one chip to i in return. When a player has all of the chips in the game, he does not send any income to the other player.

In the following, we characterize the optimal chips mechanism and relate it to the HSSGL’s of Abdulkadiroğlu and Bagwell (2012).

3 Optimal Chips Mechanism

Consider an N -chips mechanism:

- Each player i holds $c_i \in \{0, 1, \dots, N\}$ chips such that $c_a + c_b = N$.
- If player i gets an income of \$1, i gives \$1 to j if $c_i < N$ and j gives one to i in return. If $c_i = N$, i.e. i already holds all the chips, then i consumes \$1 himself. No chips are transferred in this case.

Payoffs and Constraints

Let V_n^N denote a player’s expected discounted payoff when the player holds n chips. By symmetry, V_n^N is also player b ’s expected payoff when b holds n chips. Furthermore, if a holds n chips then b holds $N - n$ chips. $V^N = (V_0^N, \dots, V_N^N)$ is the unique solution of the following equation system (for existence and uniqueness see Lemma 1 below):

$$V_0^N = p\beta V_1^N + p\beta V_0^N + (1 - 2p)\beta V_0^N \quad (1)$$

$$V_n^N = p\beta V_{n+1}^N + p(qk + \beta V_{n-1}^N) + (1 - 2p)\beta V_n^N \text{ for } n = 1, \dots, N - 1 \quad (2)$$

$$V_N^N = p(1 + \beta V_N^N) + p(qk + \beta V_{N-1}^N) + (1 - 2p)\beta V_N^N \quad (3)$$

The individual rationality (IR) constraint for a player with $n = 0, \dots, N$ chips is

$$IR_n : V_n^N \geq \frac{p}{1 - \beta}$$

The incentive compatibility (IC) constraint for a player with $n = 0, \dots, N - 1$ chips is

$$IC_n : V_{n+1}^N - V_n^N \geq \frac{1}{\beta}$$

Results

Lemma 1 *There exists a unique solution V^N to the equations system (1)-(3). Furthermore, $V_n^N < V_{n+1}^N$ for all N and $n = 0, \dots, N - 1$.*

Proof. *Given $N \geq 1$, define a mapping $\Gamma : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1}$ as follows: For any $V = (V_0, \dots, V_N) \in \mathbb{R}^{N+1}$, let $V' = \Gamma(V) \in \mathbb{R}^{N+1}$ be defined by*

$$\begin{aligned} V'_0 &= p\beta V_1 + p\beta V_0 + (1 - 2p)\beta V_0 \\ V'_n &= p\beta V_{n+1} + p(qk + \beta V_{n-1}) + (1 - 2p)\beta V_n \text{ for } n = 1, \dots, N - 1 \\ V'_N &= p(1 + \beta V_N) + p(qk + \beta V_{N-1}) + (1 - 2p)\beta V_N \end{aligned}$$

Then V^N is a fixed point of Γ , i.e. $V^N = \Gamma(V^N)$. Note that Γ is monotone, i.e. $\hat{V} \geq V \Rightarrow \Gamma(\hat{V}) \geq \Gamma(V)$. Also $\Gamma(V + \bar{a}) = \Gamma(V) + \beta\bar{a}$ for any $\bar{a} = (a, \dots, a) \in \mathbb{R}^{N+1}$. Therefore Γ is a contraction mapping by Blackwell's theorem (Stokey, Lucas and Prescott 1989) so that Γ has a unique fixed point.

Define $\Gamma^0 = \Gamma$ and $\Gamma^n = \Gamma \circ \Gamma^{n-1}$. Then $V^N = \lim_{n \rightarrow \infty} \Gamma^n(V)$ for every $V \in \mathbb{R}^{N+1}$.

Now suppose that $V = (V_0, \dots, V_N) \in \mathbb{R}^{N+1}$ is such that $V_n \leq V_{n+1}$ for every $n = 0, \dots, N - 1$. Let $V' = (V'_0, \dots, V'_N) = \Gamma(V)$. We will show that $V'_n \leq V'_{n+1}$ for every $n = 0, \dots, N - 1$.

First note that

$$\begin{aligned} V'_0 &= p\beta V_1 + p\beta V_0 + (1 - 2p)\beta V_0 \\ &< p\beta V_1 + p(qk + \beta V_0) + (1 - 2p)\beta V_0 \\ &\leq p\beta V_2 + p(qk + \beta V_0) + (1 - 2p)\beta V_1 = V'_1 \end{aligned} \tag{4}$$

where the first inequality follows from $pqk > 0$ and the last inequality follows from $V_0 \leq V_1 \leq V_2$. Next note for $n = 1, \dots, N - 2$ that

$$\begin{aligned} V'_n &= p\beta V_{n+1} + p(qk + \beta V_{n-1}) + (1 - 2p)\beta V_n \\ &\leq p\beta V_{n+2} + p(qk + \beta V_n) + (1 - 2p)\beta V_{n+1} = V'_{n+1} \end{aligned} \tag{5}$$

where the last inequality follows from $V_{n-1} \leq V_n \leq V_{n+1}$. Finally note that

$$\begin{aligned} V'_N &= p(1 + \beta V_N) + p(qk + \beta V_{N-1}) + (1 - 2p)\beta V_N \\ &> p\beta V_N + p(qk + \beta V_{N-2}) + (1 - 2p)\beta V_{N-1} = V'_{N-1} \end{aligned} \quad (6)$$

where the last inequality follows from $V_{N-2} \leq V_{N-1} \leq V_N$ and $p > 0$. So $V'_n \leq V'_{n+1}$ for every $n = 0, \dots, N - 1$.

Now pick some $V \in \mathbb{R}^{N+1}$ such that $V_n \leq V_{n+1}$ for every $n = 0, \dots, N - 1$. Then the above result and $V^N = \lim_{n \rightarrow \infty} \Gamma^n(V)$ together imply that $V_n^N \leq V_{n+1}^N$ for every $n = 0, \dots, N - 1$.

Next, given that $V_n^N \leq V_{n+1}^N$ for every $n = 0, \dots, N - 1$, repeating the argument in (4) with $V_0^N \leq V_1^N$, i.e. using V_0^N and V_1^N on the right hand side of the first line of (4), proves that $V_0^N < V_1^N$. Repeating the argument in (5) with $V_0^N < V_1^N \leq V_2^N$ proves that $V_1^N < V_2^N$. Then repeating it again with $V_{n-1}^N < V_n^N \leq V_{n+1}^N$ proves $V_n^N < V_{n+1}^N$ for all $n = 1, \dots, N - 2$. Finally $V_{N-1}^N < V_N^N$ follows from repeating the argument in (6) with $V_{N-1}^N \leq V_N^N$. ■

Lemma 2 Consider an N -chips mechanism. Let V_n^N denote a player's expected discounted payoff when the player holds n chips. Then

$$\sum_{n=0}^N V_n^N = \frac{Npqk + p}{1 - \beta}.$$

Proof. Summing up (1)-(3) for $n = 0, \dots, N$ gives the desired result. ■

The following lemma implies that we can ignore the individual rationality constraints.

Lemma 3 IC_0 implies IR_n for every $n = 0, \dots, N$

Proof. IC_0 is equivalent to $V_1^N \geq V_0^N + \frac{1}{\beta}$. Combining this inequality with $V_0^N = p\beta V_1^N + p\beta V_0^N + (1 - 2p)\beta V_0^N$, we obtain IR_0 . Since $V_{n+1}^N > V_n^N$, the rest follows. ■

The following lemma gives an upper bound for V_N^N in an IC N -chips mechanism.

Lemma 4 *If IC_{N-1} holds, then $V_N^N \leq \frac{pqk}{1-\beta}$.*

Proof. *IC_{N-1} is equivalent to $V_{N-1}^N \leq V_N^N - \frac{1}{\beta}$. Combining this with $V_N^N = p(1 + \beta V_N^N) + p(qk + \beta V_{N-1}^N) + (1 - 2p)\beta V_N^N$, we obtain $V_N^N \leq \frac{pqk}{1-\beta}$. ■*

This is intuitive because the largest symmetric payoff that the players can achieve is $\frac{pqk}{1-\beta}$, which obtains when the players always give their income to the other. This provides an upper bound for V_N^N . Note that the upper bound cannot be supported as an equilibrium outcome.

Using Lemmas 2 and 4, we may immediately establish the following corollary.

Corollary 5 *If IC_N holds in an $(N+1)$ -chips mechanism, then $\sum_{n=0}^N V_n^{N+1} \geq \sum_{n=0}^N V_n^N$.*

The next lemma states we need to check IC_{N-1} only in order to check IC of an N -chips mechanism.

Lemma 6 *IC_{N-1} implies IC_n for all $n = 0, \dots, N - 2$*

Proof. *Suppose that IC_{N-1} holds. Then by Lemma 4, $V_N^N \leq \frac{pqk}{1-\beta}$. As an inductive step, assume that IC_n holds for $1 < n \leq N - 1$. We will show that IC_{n-1} holds as well.*

Applying IC_n, \dots, IC_{N-1} , we obtain $V_n^N \leq \frac{pqk}{1-\beta} - \frac{N-n}{\beta}$ which gives an upper bound for V_{n-1}^N :

$$V_{n-1}^N < V_n^N \leq \frac{pqk}{1-\beta} - \frac{N-n}{\beta} \quad (7)$$

Combining $V_n^N = p\beta V_{n+1}^N + p(qk + \beta V_{n-1}^N) + (1 - 2p)\beta V_n^N$ and IC_n , we obtain

$$V_n^N \geq \frac{p(qk + 1) + p\beta V_{n-1}^N}{1 - (1-p)\beta}$$

Then IC_{n-1} is satisfied if

$$\frac{p(qk + 1) + p\beta V_{n-1}^N}{1 - (1-p)\beta} \geq V_{n-1}^N + \frac{1}{\beta}$$

equivalently

$$V_{n-1}^N \leq \frac{\beta pqk - (1-\beta)}{\beta(1-\beta)} \quad (8)$$

Note that

$$\frac{pqk}{1-\beta} - \frac{N-n}{\beta} \leq \frac{\beta pqk - (1-\beta)}{\beta(1-\beta)}$$

is equivalent to $n < N$, which holds. So (7) and (8) imply IC_{n-1} . The proof is completed by induction. ■

The following lemma is another way of stating Lemma 6. It states that, in an IC chips mechanism, the IC constraint becomes more relaxed for a player if the player holds less chips.

Lemma 7 *If an N -chips mechanism is IC for some $N \geq 2$, then $V_{n+1}^N - V_n^N < V_n^N - V_{n-1}^N$ for every $n = 1, \dots, N-1$.*

Proof. Define $\Delta_n^N = V_{n+1}^N - V_n^N$ for $n = 0, \dots, N-1$. Then $\Delta^N = (\Delta_0^N, \dots, \Delta_{N-1}^N) \in \mathbb{R}^N$ is solved from

$$\begin{aligned} \Delta_0^N &= \alpha \frac{qk}{\beta} + \alpha \Delta_1^N \\ \Delta_n^N &= \alpha \Delta_{n-1}^N + \alpha \Delta_{n+1}^N \text{ for } n = 1, \dots, N-2 \\ \Delta_{N-1}^N &= \alpha \Delta_{N-2}^N + \alpha \frac{1}{\beta} \end{aligned}$$

where $\alpha = \frac{p\beta}{1-(1-2p)\beta} < \frac{1}{2}$. Since the N -chips mechanism is IC, $\frac{1}{\beta} \leq \Delta_n^N$ for $n = 0, \dots, N-1$. Then

$$\Delta_{N-1}^N = \alpha \Delta_{N-2}^N + \alpha \frac{1}{\beta} \leq 2\alpha \Delta_{N-2}^N < \Delta_{N-2}^N$$

Next suppose that $\Delta_{n+1}^N < \Delta_n^N$ for some $0 < n < N-1$. Then

$$\Delta_n^N = \alpha \Delta_{n-1}^N + \alpha \Delta_{n+1}^N < \alpha \Delta_{n-1}^N + \alpha \Delta_n^N$$

equivalently $\Delta_n^N < \frac{\alpha}{1-\alpha} \Delta_{n-1}^N < \Delta_{n-1}^N$, where the last inequality follows from $\frac{\alpha}{1-\alpha} < 1$ since $\alpha < \frac{1}{2}$. This proves inductively that $V_{n+1}^N - V_n^N = \Delta_n^N < \Delta_{n-1}^N = V_n^N - V_{n-1}^N$ for every $n = 1, \dots, N-2$. ■

Proposition 8 *Suppose that an N -chips mechanism is IC for $N \geq 2$. Then for every $\frac{N}{2} \leq n < N$,*

$$V_{n+1}^N + V_{N-(n+1)}^N < V_n^N + V_{N-n}^N$$

Proof. *Suppose that an N -chips mechanism is IC for some $N \geq 2$. Let $\frac{N}{2} \leq n < N$, then $n + 1 > N - n$ so we have $V_{n+1}^N - V_n^N < V_{N-n}^N - V_{N-(n+1)}^N$, equivalently $V_{n+1}^N + V_{N-(n+1)}^N < V_n^N + V_{N-n}^N$, by Lemma 7. ■*

That is, the sum of expected utilities $V_n^N + V_{N-n}^N$ decreases as n increases, in other words, as the discrepancy between the numbers of players' chips increases. This is because one of the players becomes closer to holding all the chips, which is the main cause of inefficiency with the chips mechanism. A visual reading of this proposition will be instructive later: Plot the (V_n^N, V_{N-n}^N) on a two dimensional graph for all n . Then the payoff pairs will follow a concave curve.

The following is an immediate corollary to Lemma 6 and Lemma 7.

Corollary 9 *If an N -chips mechanism is not IC, then the IC_{N-1} is violated in the N -chips mechanism.*

In addition to Corollary 5, the following lemma states that a player with n chips achieves a higher payoff in an IC $(N + 1)$ -chips mechanism than in the N -chips mechanism.

Lemma 10 *If an $(N + 1)$ -chips mechanism is IC, then $V_n^{N+1} \geq V_n^N$ for every $n = 0, \dots, N$.*

Proof. *If $N = 0$ then the claim follows trivially. Let $N \geq 1$. Suppose that an $(N + 1)$ -chips mechanism is IC. Let*

$$V_n^{\tilde{N}} = A_n^{\tilde{N}} V_0^{\tilde{N}} - B_n^{\tilde{N}} p q k \text{ for } \tilde{N} = N, N + 1.$$

Then $A_0^N = A_0^{N+1} = 1$, $B_0^N = B_0^{N+1} = 0$, and from equation (1), $A_1^N = \frac{1-(1-p)\beta}{p\beta} = A_1^{N+1}$ and $B_1^N = 0 = B_1^{N+1}$. Given $V_j^{\tilde{N}} = A_j^{\tilde{N}} V_0^{\tilde{N}} - B_j^{\tilde{N}} p q k$ for every $j = 1, \dots, n - 1$ where $1 \leq n - 1 < N$, $\tilde{N} = N, N + 1$, solve for $V_n^{\tilde{N}}$ from

$$V_{n-1}^{\tilde{N}} = p\beta V_n^{\tilde{N}} + p(qk + \beta V_{n-2}^{\tilde{N}}) + (1 - 2p)\beta V_{n-1}^{\tilde{N}}$$

and obtain

$$V_n^{\tilde{N}} = A_n^{\tilde{N}} V_0^{\tilde{N}} - B_n^{\tilde{N}} pqk = \frac{(1 - (1 - 2p)\beta)V_{n-1}^{\tilde{N}} - p\beta V_{n-2}^{\tilde{N}} - pqk}{p\beta}$$

so that

$$A_n^N = A_n^{N+1} = \frac{(1 - (1 - 2p)\beta)A_{n-1}^N - p\beta A_{n-2}^N}{p\beta}$$

and

$$B_n^N = B_n^{N+1} = \frac{(1 - (1 - 2p)\beta)B_{n-1}^N - p\beta B_{n-2}^N + 1}{p\beta} \quad (9)$$

Note that $A_n^N = A_n^{N+1}$ and $B_n^N = B_n^{N+1}$ follow from the fact that we solve for $A_n^{\tilde{N}}$ and $B_n^{\tilde{N}}$ from the same set of equations since $n < N + 1$ for all $n = 0, \dots, N$.

Note that $B_0^{\tilde{N}} = B_1^{\tilde{N}} = 0$, so $B_2^{\tilde{N}} = 1/p\beta$ from (9). Applying equation (9), we obtain

$$B_{n+1}^{\tilde{N}} - B_n^{\tilde{N}} = \frac{1 - (1 - 2p)\beta}{p\beta} (B_n^{\tilde{N}} - B_{n-1}^{\tilde{N}}) - (B_{n-1}^{\tilde{N}} - B_{n-2}^{\tilde{N}})$$

Then noting that (i) $B_2^{\tilde{N}} - B_1^{\tilde{N}} > B_1^{\tilde{N}} - B_0^{\tilde{N}} = 0$ and (ii) $\frac{1 - (1 - 2p)\beta}{p\beta} > 2$ so that $B_{n+1}^{\tilde{N}} - B_n^{\tilde{N}} > 2(B_n^{\tilde{N}} - B_{n-1}^{\tilde{N}}) - (B_{n-1}^{\tilde{N}} - B_{n-2}^{\tilde{N}})$, we recursively obtain $B_{n+1}^{\tilde{N}} - B_n^{\tilde{N}} > B_n^{\tilde{N}} - B_{n-1}^{\tilde{N}} > 0$. Then $B_n^{\tilde{N}} > B_{n-1}^{\tilde{N}}$ follows for all $n = 2, \dots, N$ and $B_1^{\tilde{N}} = B_0^{\tilde{N}} = 0$. Then $A_n^{\tilde{N}} > A_{n-1}^{\tilde{N}} > 0$ for all $n = 1, \dots, N$ since $V_n^{\tilde{N}} > V_{n-1}^{\tilde{N}}$ for all $n = 1, \dots, N$ and $V_n^{\tilde{N}} = A_n^{\tilde{N}} V_0^{\tilde{N}} - B_n^{\tilde{N}} pqk$. Then apply Corollary 5,

$$\begin{aligned} \sum_{n=0}^N V_n^{N+1} &= \left(\sum_{n=0}^N A_n^{N+1} \right) V_0^{N+1} - \left(\sum_{n=0}^N B_n^{N+1} \right) pqk \\ &\geq \sum_{n=0}^N V_n^N = \left(\sum_{n=0}^N A_n^N \right) V_0^N - \left(\sum_{n=0}^N B_n^N \right) pqk \end{aligned}$$

and simplify by using $A_n^N = A_n^{N+1} > 0$ and $B_n^N = B_n^{N+1}$ to obtain $V_0^{N+1} \geq V_0^N$. Then applying $V_n^{\tilde{N}} = A_n^{\tilde{N}} V_0^{\tilde{N}} - B_n^{\tilde{N}} pqk$, $A_n^N = A_n^{N+1} > 0$ and $B_n^N = B_n^{N+1}$, we obtain $V_n^{N+1} \geq V_n^N$ for every $n = 0, \dots, N$. ■

The following lemma gives us a method to compute the optimal IC chips mechanism. Namely, we increase the number of chips by one as long as the resulting chips mechanism is IC; we stop when it fails to be IC.

Lemma 11 *If N -chips mechanism is not IC, then $(N + 1)$ -chips mechanism is not IC either.*

Proof. *Rewriting (3) for $N + 1$ and N ,*

$$\begin{aligned} V_{N+1}^{N+1} &= p(1 + \beta V_{N+1}^{N+1}) + p(qk + \beta V_N^{N+1}) + (1 - 2p)\beta V_{N+1}^{N+1} \\ V_N^N &= p(1 + \beta V_N^N) + p(qk + \beta V_{N-1}^N) + (1 - 2p)\beta V_N^N \end{aligned}$$

subtracting side by side and obtain and rearranging, we obtain

$$V_{N+1}^{N+1} - V_N^N = \frac{p\beta}{1 - (1-p)\beta} (V_N^{N+1} - V_{N-1}^N)$$

Suppose to the contrary that the $(N+1)$ -chips mechanism is IC. Then $V_N^{N+1} \geq V_N^N$ by Lemma 10. Also $V_N^N > V_{N-1}^N$ by Lemma 1. We thus obtain that $V_N^{N+1} - V_{N-1}^N > V_N^{N+1} - V_N^N \geq 0$. Then $\frac{p\beta}{1-(1-p)\beta} < 1$ implies

$$V_{N+1}^{N+1} - V_N^N < V_N^{N+1} - V_{N-1}^N \Leftrightarrow V_{N+1}^{N+1} - V_N^{N+1} < V_N^N - V_{N-1}^N$$

By Corollary 9, IC_{N-1} is violated in the N -chips mechanism so that $V_{N+1}^{N+1} - V_N^{N+1} < V_N^N - V_{N-1}^N < \frac{1}{\beta}$, which in turn implies that IC_N is violated in the $(N + 1)$ -chips mechanism, a contradiction with IC of the $(N + 1)$ -chips mechanism. ■

The next is an immediate corollary.

Corollary 12 *If an N -chips mechanism is IC, then for every $N' = 1, \dots, N - 1$, the N' -chips mechanism is IC.*

These findings yield the following algorithm for finding the optimal chips mechanism: Start with $N = 1$.

- Given N -chips, write the expected payoffs recursively. Then applying Blackwell's theorem, obtain a contraction mapping. Calculate the associated expected payoffs as the unique fixed point of the mapping.
- If the IC constraint for the player with $N - 1$ chips is violated, then the optimal number of chips is $N - 1$. Otherwise repeat the two steps with $N + 1$.

Since the number of chips in equilibrium is bounded from above, the algorithm converges in finite time.

The following results provide further intuition.

Lemma 13 *If $V_N^{N+1} = V_N^N$ for some N , then the $(N + 2)$ -chips mechanism is not IC.*

Proof. *Suppose to the contrary that $V_N^{N+1} = V_N^N$ for some N and the $(N+2)$ -chips mechanism is IC. Then the $(N + 1)$ -chips mechanism is also IC by Corollary 12. Then applying the same argument in the proof of Lemma 10, we obtain $V_n^{N+1} = V_n^N$ for all $n = 0, \dots, N$ so that*

$$\sum_{n=0}^N V_n^{N+1} = \sum_{n=0}^N V_n^N = \frac{Npqk + p}{1 - \beta}$$

where the last equality follows from Lemma 2. Applying Lemma 2 one more time, we obtain $V_{N+1}^{N+1} = \frac{pqk}{1-\beta}$. Since the $(N + 2)$ -chips mechanism is assumed to be IC, $V_{N+1}^{N+2} \geq V_{N+1}^{N+1} = \frac{pqk}{1-\beta}$ by Lemma 10 and $V_{N+2}^{N+2} \leq \frac{pqk}{1-\beta}$ by Lemma 4, which together imply the violation of IC_{N+1} in the $(N + 2)$ -chips mechanism, a contradiction. ■

Let N^{opt} be such that N^{opt} -chips mechanism is IC but the $(N^{opt} + 1)$ -chips mechanism is not IC. We refer the N^{opt} -chips mechanism as the optimal IC chips mechanism.

The following is an immediate corollary to Lemma 13.

Corollary 14 *$V_n^{N+1} > V_n^N$ for every $N < N^{opt} - 1$ and $n = 0, \dots, N$.*

The next lemma states that the payoff of a player with more chips increases more as N increases. This lemma and Lemma 4 together show us where the difficulty is coming from in sustaining IC as the number of chips increases. In particular, Lemma 4 states that the payoff of the player with the maximum number of chips is always bounded from above. However, the next lemma states that the payoff of the player with all of the chips but one increases the most as the number of chips increases. So it is this latter player whose IC constraint is violated as number of chips increases.

Lemma 15 *If $N < N^{opt} - 1$, then $V_n^{N+1} - V_n^N \leq \frac{p\beta}{1-(1-p)\beta}(V_{n+1}^{N+1} - V_{n+1}^N) < V_{n+1}^{N+1} - V_{n+1}^N$ for all $N, n = 0, \dots, N-1$. If $N = N^{opt} - 1$, then $V_n^{N+1} - V_n^N \leq \frac{p\beta}{1-(1-p)\beta}(V_{n+1}^{N+1} - V_{n+1}^N) \leq V_{n+1}^{N+1} - V_{n+1}^N$ for all $N, n = 0, \dots, N-1$.*

Proof. *By Corollary 12, the N -chips mechanism is IC for every $N < N^{opt}$. First assume that $N < N^{opt} - 1$. The inequality $\frac{p\beta}{1-(1-p)\beta}(V_{n+1}^{N+1} - V_{n+1}^N) < V_{n+1}^{N+1} - V_{n+1}^N$ follows from $\frac{p\beta}{1-(1-p)\beta} < 1$ and Corollary 14. From (1), $V_0^N = \frac{p\beta}{1-(1-p)\beta}V_1^N$ for all N so that $V_0^{N+1} - V_0^N \leq \frac{p\beta}{1-(1-p)\beta}(V_1^{N+1} - V_1^N)$. For the inductive step, assume that $V_n^{N+1} - V_n^N \leq \frac{p\beta}{1-(1-p)\beta}(V_{n+1}^{N+1} - V_{n+1}^N) < V_{n+1}^{N+1} - V_{n+1}^N$ for some $n = 0, \dots, N-2$. Rewriting (2)*

$$\begin{aligned} V_{n+1}^{N+1} &= p\beta V_{n+2}^{N+1} + p(qk + \beta V_n^{N+1}) + (1-2p)\beta V_{n+1}^{N+1} \\ V_{n+1}^N &= p\beta V_{n+2}^N + p(qk + \beta V_n^N) + (1-2p)\beta V_{n+1}^N \end{aligned}$$

and subtracting side by side, we obtain

$$V_{n+1}^{N+1} - V_{n+1}^N = p\beta(V_{n+2}^{N+1} - V_{n+2}^N) + p\beta(V_n^{N+1} - V_n^N) + (1-2p)\beta(V_{n+1}^{N+1} - V_{n+1}^N)$$

Substituting $V_n^{N+1} - V_n^N < V_{n+1}^{N+1} - V_{n+1}^N$ and rearranging, we obtain

$$V_{n+1}^{N+1} - V_{n+1}^N < \frac{p\beta}{1-(1-p)\beta}(V_{n+2}^{N+1} - V_{n+2}^N)$$

which proves the inductive step.

The proof for the case of $N = N^{opt} - 1$ follows the same arguments, except now we only have $V_{N^{opt}-1}^{N^{opt}} \geq V_{N^{opt}-1}^{N^{opt}-1}$ by Lemma 10 so that $\frac{p\beta}{1-(1-p)\beta}(V_{n+1}^{N^{opt}} - V_{n+1}^{N^{opt}-1}) \leq V_{n+1}^{N^{opt}} - V_{n+1}^{N^{opt}-1}$ follows from $\frac{p\beta}{1-(1-p)\beta} < 1$. Now substituting $V_n^{N^{opt}} - V_n^{N^{opt}-1} \leq V_{n+1}^{N^{opt}} - V_{n+1}^{N^{opt}-1}$ in

$$V_{n+1}^{N^{opt}} - V_{n+1}^{N^{opt}-1} = p\beta(V_{n+2}^{N^{opt}} - V_{n+2}^{N^{opt}-1}) + p\beta(V_n^{N^{opt}} - V_n^{N^{opt}-1}) + (1-2p)\beta(V_{n+1}^{N^{opt}} - V_{n+1}^{N^{opt}-1})$$

we inductively prove that

$$V_{n+1}^{N^{opt}} - V_{n+1}^{N^{opt}-1} \leq \frac{p\beta}{1-(1-p)\beta}(V_{n+2}^{N^{opt}} - V_{n+2}^{N^{opt}-1})$$

■

4 Comparison with HSSGL

We utilize the following cut-off values:

$$\beta^* = \frac{1}{1 + p(qk - 1)} < \beta_N^* = \frac{1}{1 + \frac{2}{N(N+1)}p(qk - 1)}, \quad N \geq 2$$

Note that β^* is the cutoff that Abdulkadiroğlu and Bagwell (2012) use. Also $\beta^* = \beta_1^*$.

The following lemma gives a sufficient condition for an N -chips mechanism to fail incentive compatibility.

Lemma 16 *For any $N \geq 2$, if $\beta < \beta_N^*$ then the N -chips mechanism is not incentive compatible.*

Proof. *Suppose that an N -chips mechanism is IC. Then $V_N^N \leq \frac{pqk}{1-\beta}$ by Lemma 4 so that $\sum_{n=0}^{N-1} V_n^N \geq \frac{(N-1)pqk+p}{1-\beta}$ by Lemma 2. Now we can find a lower bound for V_{N-1}^N by stacking as much value as possible to V_0^N, \dots, V_{N-2}^N . To do that, first note that IC_0, \dots, IC_n imply $V_n^N \geq V_0^N + \frac{n}{\beta}$. Then set $\hat{V}_n^N = \hat{V}_0^N + \frac{n}{\beta}$ for $n = 1, \dots, N-1$, which gives $\sum_{n=0}^{N-1} \hat{V}_n^N = N\hat{V}_0^N + \frac{(N-1)N}{2\beta}$. Then equating this value to $\frac{(N-1)pqk+p}{1-\beta}$, we obtain $\hat{V}_0^N = \frac{1}{N}(\frac{(N-1)pqk+p}{1-\beta} - \frac{(N-1)N}{2\beta})$ and*

$$\hat{V}_{N-1}^N = \hat{V}_0^N + \frac{N-1}{\beta} = \frac{(N-1)pqk+p}{N(1-\beta)} + \frac{N-1}{2\beta}$$

Now we claim that $V_{N-1}^N \geq \hat{V}_{N-1}^N$. If for any $n < N-1$ it were the case that $V_n^N \geq \hat{V}_n^N$, then this inequality would also have to hold at $n = N-1$, since under IC the $\{V_n^N, V_{n+1}^N, \dots, V_{N-1}^N\}$ sequence can rise no slower than the $\{\hat{V}_n^N, \hat{V}_{n+1}^N, \dots, \hat{V}_{N-1}^N\}$ sequence. Likewise, if for all $n < N-1$ it were the case that $V_n^N \leq \hat{V}_n^N$, then $V_{N-1}^N \geq \hat{V}_{N-1}^N$ is implied by

$$\begin{aligned} V_{N-1}^N &\geq \frac{(N-1)pqk+p}{1-\beta} - \sum_{n=0}^{N-2} V_n^N \\ &\geq \frac{(N-1)pqk+p}{1-\beta} - \sum_{n=0}^{N-2} \hat{V}_n^N = \hat{V}_{N-1}^N \end{aligned}$$

This completes the proof of $V_{N-1}^N \geq \hat{V}_{N-1}^N$.

Since also $V_N^N \leq \frac{pqk}{1-\beta}$, we obtain $V_N^N - V_{N-1}^N \leq \frac{pqk}{1-\beta} - \hat{V}_{N-1}^N$ so that IC_{N-1} will be violated if

$$\frac{pqk}{1-\beta} - \hat{V}_{N-1}^N < \frac{1}{\beta}$$

which is equivalent to $\beta < \beta_N^*$. This completes our proof. ■

Note that $\beta_1^* = \beta^*$ so that, under the assumption that $\beta > \beta^*$ the sufficient condition of Lemma 16 for the 1-chip mechanism does not hold. We will refer to the 1-chip mechanism as a simple EM relationship.² Then the next result follows immediately.

Corollary 17 *If $\beta \in (\beta^*, \beta_2^*)$, the simple EM relationship (i.e. the 1-chip mechanism) is the optimal IC chips mechanism, and it is dominated by HSSGL.*

Abdulkadiroğlu and Bagwell (2012) show that the simple EM relationship can be implemented on a set of payoffs that lie on a symmetric -45° line around the 45° line. They also characterize the highest such line as a HSSGL. The next proposition gives us a parameter space for which a HSSGL is more efficient in terms of long-run average investment than the optimal incentive compatible chips mechanism.

Proposition 18 *Suppose that $\beta > \beta^*$ and $\beta(1-\beta) > 2\beta^*(1-\beta^*)$. Then a HSSGL is more efficient than the optimal IC chips mechanism.*

Proof. *First note that there exists such β as long as $\beta^* < \frac{1}{2}(1 - \sqrt{\frac{1}{2}})$.*

By Lemma 16, $\beta > \beta_N^$ must hold for an N -chips mechanism to be IC. This is equivalent to*

$$N(N+1) < \frac{2\beta p(qk-1)}{1-\beta}$$

Substituting $N^2 < N(N+1)$, we obtain an upper bound for N ,

$$N < N^* = \sqrt{\frac{2\beta p(qk-1)}{1-\beta}}$$

²In our companion paper (Abdulkadiroğlu and Bagwell, 2012), we refer to the simple EM relationship as a simple favor-exchange relationship.

In an N -chips mechanism, investment loss occurs when one of the players have all the N chips. Then, since every state is visited with probability $\frac{1}{N+1}$ in the long-run, the average loss of investment is computed as $\frac{2p}{N+1}$.

$x + y$ is constant on HSSGL, therefore the average loss of investment can be computed as the expected loss of investment at any corner, which is equal to $p(1 - y) = \frac{2p\beta^*}{\beta + \beta^*}$.

HSSGL is more efficient than the N -chips mechanism if the investment loss is smaller on HSSGL, i.e. $\frac{2p}{N+1} > \frac{2p\beta^*}{\beta + \beta^*}$, equivalently

$$\beta > N\beta^*$$

Since IC implies an upper bound of N^* for N , the above inequality holds if

$$\beta > N^*\beta^*$$

which is equivalent to $\beta(1 - \beta) > 2\beta^*(1 - \beta^*)$. This completes our proof. ■

In other words, forgiving may produce more cooperation and investment in the long run.

This sufficiency condition implies that β is away from 1, which is consistent with our small β analysis. However, a HSSGL can still be more efficient than the optimal chips mechanism as β goes to 1. Using Lemma 11, we can provide some numerical examples for that. Also, as k gets bigger, β^* gets smaller so that the result holds.

5 Conclusion

We interpret an equality matching relationship as a chips mechanism, and we characterize the optimal chips mechanism in a repeated favor-exchange game. We also compare the optimal chips mechanism with a more sophisticated favor-exchange relationship in which the size of a favor owed may decline as the relationship passes through neutral periods. As we show in a companion paper (Abdulkadiroğlu and Bagwell, 2012), the highest symmetric self-generating line of payoffs in the repeated favor-exchange game is implemented by such a sophisticated favor-exchange relationship. We find sufficient conditions for a sophisticated favor-exchange relationship of this kind to produce higher levels of cooperation and exchange among players than they achieve in the optimal chips mechanism.

6 References

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