

# School Choice: A Mechanism Design Approach

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*A central issue in school choice is the design of a student assignment mechanism. Education literature provides guidance for the design of such mechanisms but does not offer specific mechanisms. The flaws in the existing school choice plans result in appeals by unsatisfied parents. We formulate the school choice problem as a mechanism design problem and analyze some of the existing school choice plans including those in Boston, Columbus, Minneapolis, and Seattle. We show that these existing plans have serious shortcomings, and offer two alternative mechanisms each of which may provide a practical solution to some critical school choice issues.*  
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*School choice* is one of the widely discussed topics in education.<sup>1</sup> It means giving parents the opportunity to choose the school their child will attend. Traditionally, children are assigned to public schools according to where they live. Wealthy parents already have school choice, because they can afford to move to an area with

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<sup>1</sup> Milton Friedman (1955, 1962) initiates the school choice literature.

good schools, or they can enroll their child in a private school. Parents without such means, until recently, had no choice of school, and had to send their children to schools assigned to them by the district, regardless of the school quality or appropriateness for the children. As a result of these concerns, *intra-district* and *inter-district* choice programs have become increasingly popular in the past ten years.<sup>2</sup> Intra-district choice allows parents to select schools throughout the district where they live, and inter-district choice allows them to send their children to public schools in areas outside their resident districts. In 1987, Minnesota became the first state to oblige all its districts to establish an inter-district choice plan (Allyson M. Tucker and William F. Lauber, 1995). Today, several states offer inter-district and intra-district choice programs.

Since it is not possible to assign each student to her top choice school, a central issue in school choice is the design of a *student assignment mechanism*.<sup>3</sup> While the education literature stresses the need for rigorous student assignment mechanisms and provides guidance

<sup>2</sup> For empirical investigation of various issues in school choice see Mark Schneider et al. (2000). Another line of discussion has been on private school vouchers. See Caroline M. Hoxby (1994), Dennis Epple and Richard E. Romano (1998), Cecilia E. Rouse (1998), Thomas J. Nechyba (2000), and Raquel Fernández and Richard Rogerson (2003).

<sup>3</sup> Indeed, one of the key obstacles identified by the critics of school choice concerns student selection to overdemanded schools (see Donald Hirch, 1994, p. 14).

for design (see, for example, Michael J. Alves and Charles V. Willie, 1990; Office of Educational Research and Improvement, 1992; and Timothy W. Young and Evans Clinchy, 1992, Ch. 6), it does not offer specific mechanisms.<sup>4</sup> Many of the real-life school choice plans have protocols and guidelines for the student assignment without explicit procedures. Saul Yanofsky (see Office of Educational Research and Improvement, 1992, p. 19), the former superintendent of the White Plains Public Schools, states: "You need to have a set of procedures that are very explicit, with rules."

The lack of rigorous procedures invites selective interpretation and it often results in evasive action by the students and their parents. Consider the following statement by the Supreme Court of Mississippi which affirms the judgement of a circuit court against a school district (Supreme Court of Mississippi, 2001)<sup>5</sup>:

We agree that the denial of Gentry's transfer cannot be based on the alleged "middle-school" transfer policy since there is no written record outlining its substance. Such a denial based on this vague policy would clearly be arbitrary and capricious.

Along similar lines consider the following summary of a court case concerning a school choice plan in Wisconsin (Court of Appeals of Wisconsin, 2000)<sup>6</sup>:

*McMorrow v. State Superintendent of Public Instruction*

Respondent applied under open enroll-

<sup>4</sup> This is a major problem for school choice programs in other countries as well. For example, Gulam-Husien Mayet (1997), the former Chief Welfare Adviser for the Inner London Education Authority, indicates that the lack of synchronization and transparency in admissions to London public schools is a major problem for parents and local authorities.

<sup>5</sup> *Pascagoula Municipal Separate School District v. W. Harvey Barton and Renee Barton, as Parents and Next Friends of William Gentry Barton, A Minor*. Supreme Court of Mississippi, No. 2000-CC-00035-SCT, decided on February 1, 2001.

<sup>6</sup> *Michael E. McMorrow, Petitioner-Respondent, v. State Superintendent of Public Instructions, John T. Benson, Respondent-Appellant*. Court of Appeals of Wisconsin, No. 99-1288, decided on July 25, 2000.

ment, Wis. Stat. 118.51 (1997-98), to attend high school in a district where he did not live. His application was denied and appellant affirmed, concluding that the denial was supported by substantial evidence based on lack of class space; thus, it was not arbitrary or unreasonable. The circuit court reversed. When three other continuing students were accepted even though space was not available, reliance on class size guidelines to deny respondent enrollment was arbitrary. The court affirmed. There was no substantial evidence to support appellant's findings of fact, and appellant erroneously interpreted statutory provisions which gave preference to continuing students only when spaces were available in the first place. Under the statute, when there were more applicants than spaces available, admission selection was to be on a random basis. Thus, accepting three students in spite of class size guidelines and denying a fourth that same exception without any explanation was arbitrary and unreasonable.

Outcome: Judgement affirmed. State Superintendent of Public Instructions (SSPI) erred when it found that the open enrollment statute supported preferential treatment of three continuing students when no class space was available; and when based on that finding, SSPI erroneously concluded that denying respondent's application for class space reasons was not arbitrary or unreasonable.

Other school choice programs, such as those in Boston, Minneapolis, and Seattle, are accompanied by explicit procedures. However, each of these procedures has serious shortcomings. Under these procedures students with high priorities at specific schools lose their priorities unless they list these schools as their top choices. Consequently, students and their parents are forced to play very complicated admissions games, and often, misrepresenting their true preferences is in their best interest. This is not only confusing to students and their parents, but also results in inefficient allocation of school seats.

In this paper we propose two competing student assignment mechanisms, each of which may be helpful in dealing with these critical school choice issues. A natural starting point is

studying how similar allocation problems are handled in real life as well as in the mechanism design literature. A closely related problem is the allocation of dormitory rooms (or on-campus housing facilities) to students (Aanund Hylland and Richard Zeckhauser, 1979).<sup>7</sup> The following mechanism, known as the *random serial dictatorship*, is almost exclusively used in real-life applications of these problems (Abdulkadiroğlu and Sönmez, 1998, 1999): Order the students with a lottery and assign the first student her top choice, the next student her top choice among the remaining slots, and so on. This mechanism is not only *Pareto efficient*, but also *strategy-proof* (i.e., it cannot be manipulated by misrepresenting preferences), and it can accommodate any hierarchy of seniorities. So why not use the same mechanism to allocate school seats to students? The key difficulty with this approach is the following: Based on state and local laws, the priority ordering of a student can be different at different schools. For example:

- students who live in the attendance area of a school must be given priority for that school over students who do not live in the school's attendance area;
- siblings of students already attending a school must be given priority; and
- students requiring a bilingual program must be given priority in schools that offer such programs.

Therefore a single lottery cannot be used to allocate school seats to students. It is this school-specific priority feature of the problem that complicates the student assignment process. A student assignment mechanism should be flexible enough to give students different priorities at different schools. This point directs our attention to another closely related problem, namely the *college admissions problem* (David Gale and Lloyd S. Shapley, 1962).

College admissions problems have been ex-

tensively studied (see Alvin E. Roth and Marilda A. O. Sotomayor, 1990, for a survey) and successfully applied in British and American entry-level labor markets (see Roth, 1984, 1991). The central difference between the college admissions and school choice is that in college admissions, schools themselves are agents which have preferences over students, whereas in school choice, schools are merely "objects" to be consumed by the students. This distinction is important because the education of students is not and probably should not be organized in a market-like institution. A student should not be rejected by a school because of her personality or ability level. Despite this important difference between the two models, school preferences and school priorities are similar mathematical objects and the college admissions literature can still be very helpful in designing an appealing student admissions mechanism.

The central notion in the college admissions literature is *stability*: There should be no unmatched student-school pair  $(i, s)$  where student  $i$  prefers school  $s$  to her assignment and school  $s$  prefers student  $i$  to one or more of its admitted students. This mathematical property is equivalent to the following appealing property in the context of school choice, where schools do not have preferences but instead they have priorities: There should be no unmatched student-school pair  $(i, s)$  where student  $i$  prefers school  $s$  to her assignment and she has higher priority than some other student who is assigned a seat at school  $s$ . Therefore a stable matching in the context of college admissions eliminates *justified envy* in the context of school choice. Moreover it is well-known that there exists a stable matching which is preferred to any stable matching by every student in the context of college admissions (Gale and Shapley, 1962). Since only the welfare of students matters in the context of school choice, this matching Pareto-dominates any other matching that eliminates justified envy. This series of observations motivate the following student admissions mechanism to the school choice problem: Interpret school priorities as preferences and select the student optimal stable matching of the induced college admissions problem. We refer to this mechanism as the *Gale-Shapley student optimal stable mechanism*. Since 1998,

<sup>7</sup> See also Lin Zhou (1990), Lars-Gunnar Svensson (1999), Haluk İ. Ergin (2000), Szilvia Pápai (2000), James Schummer (2000), Anna Bogomolnaia and Hervé Moulin (2001), Eiichi Miyagawa (2001, 2002), Lars Ehlers (2002), Ehlers et al. (2002), Andrew McLennan (2002), and Abdulkadiroğlu and Sönmez (forthcoming).

a version of this mechanism has been used in the United States hospital-intern market (Roth and Elliot Peranson, 1997, 1999).

Gale-Shapley student optimal stable mechanism has one additional very appealing feature: It is *strategy-proof*. That is, truthful preference revelation is a dominant strategy for the students. In particular, students and their parents do not need to worry about losing their priorities as a consequence of reporting their truthful preferences. Gale-Shapley student optimal stable mechanism relieves students and their parents of devising complicated admissions strategies.

However, Gale-Shapley student optimal stable mechanism is not “problem free” in the context of school choice. While it Pareto-dominates any other mechanism that eliminates justified envy, its outcome may still be Pareto-dominated. That is because there is a potential conflict between complete elimination of justified envy and Pareto efficiency (see Example 1 in Section II, subsection A). This observation motivates the following question: Could there be a milder interpretation of the priorities which in turn does not cause a conflict with Pareto efficiency? The answer to this question is affirmative. Suppose that if student  $i_1$  has higher priority than student  $i_2$  for school  $s$ , that does not necessarily mean that she is entitled a seat at school  $s$  before student  $i_2$ . It rather represents the opportunity to get in school  $s$ . If  $i_1$  has higher priority than  $i_2$ , then she has a better opportunity to get in school  $s$ , other things being equal. This milder requirement is compatible with Pareto efficiency: Find all students, each of whom has the highest priority at a school. Among these individuals there is a group of students, all of whom can be assigned their top choices by trading their priorities. Assign all such students their top choices and once they are removed, proceed in a similar way starting with the students who have the highest priorities among the remaining students. We refer to this Pareto-efficient mechanism as the *top trading cycles mechanism*. When all schools have the same priority ordering (say an ordering obtained from a common lottery draw), this mechanism reduces to the random serial dictatorship which is commonly used in the allocation of on-campus housing facilities. The top trading cycles mechanism is a natural extension of this mechanism, an extension which allows

for different priorities at different schools: The simple insight of assigning objects to agents one at a time based on their priority can be simply extended by assigning objects to *top trading cycles* one cycle at a time based on priorities. As in the case of the Gale-Shapley student optimal stable mechanism, the top trading cycles mechanism is also strategy-proof. Therefore, the choice between these two competing mechanisms depends on the structure and interpretation of the priorities. In some applications policy makers may rank complete elimination of justified envy before full efficiency, then Gale-Shapley student optimal stable mechanism can be used in those cases. Efficiency may be ranked higher by others, and the top trading cycles mechanism can be used in such applications.

One of the major concerns about the implementation of school choice plans is that they may result in racial and ethnic segregation at schools. Because of these concerns, choice plans in some districts are limited by court-ordered desegregation guidelines. This version of school choice is known as *controlled choice*. In many school districts (such as in Boston prior to 1999, as well as in Columbus and Minneapolis) controlled choice constraints are implemented by imposing racial quotas at public schools. An important advantage of both mechanisms is that they can be easily modified to accommodate controlled choice constraints by imposing racial quotas. Moreover, the modified mechanisms are still strategy-proof and the modified top trading cycles mechanism is constrained efficient.

The mechanism design approach has recently been very fruitful in many real-life resource allocation problems. Important examples include the design of FCC spectrum auctions [see John McMillan (1994); Peter Cramton (1995); R. Preston McAfee and McMillan (1996); Paul Milgrom (2000)] and the redesign of the American hospital-intern market [see Roth and Peranson (1999); Roth (2002)]. This paper, to the best of our knowledge, is the first paper to approach the school choice problem from a mechanism-design perspective. We believe this approach may be helpful in some critical school choice issues.

The organization of the rest of the paper is as follows: In Section I, we introduce the school

choice model, give examples of real-life school admissions mechanisms, and illustrate their shortcomings. In Section II, we introduce the two proposed mechanisms and analyze their properties. In Section III, we introduce controlled choice, modify the proposed mechanisms, and analyze them. In Section IV, we conclude. Finally, we present an example and include the omitted proofs in the Appendix.

### I. School Choice

In a *school choice problem* there are a number of students, each of whom should be assigned a seat at one of a number of schools. Each school has a maximum capacity but there is no shortage of the total seats. Each student has strict preferences over all schools, and each school has a strict priority ordering of all students. Here, priorities do not represent school preferences but they are imposed by state or local laws. For example, in several states a student who has a sibling already attending a school is given priority for that school by the education codes. Similarly, students who live within walking proximity of a school are given priority. For each school, the priority between two students who are identical in each relevant aspect is usually determined by a lottery.

The school choice problem is closely related to the well-known *college admissions problem* introduced by Gale and Shapley (1962). The college admissions problem has been extensively studied (see Roth and Sotomayor, 1990, for a survey) and successfully applied in the American and British entry-level labor markets (see Roth, 1984, 1991). The key difference between the two problems is that in school choice, schools are objects to be “consumed” by the students, whereas in college admissions, schools themselves are agents who have preferences over students.

The outcome of a school choice problem is an assignment of schools to students such that each student is assigned one school and no school is assigned to more students than its capacity. We refer each such outcome as a *matching*. A matching is *Pareto efficient* if there is no other matching which assigns each student a weakly better school and at least one student a strictly better school. A *student assignment mechanism* is a systematic procedure that selects a matching

for each school choice problem. A student assignment mechanism is a *direct mechanism* if it requires students to reveal their preferences over schools and selects a matching based on these submitted preferences and student priorities. A student assignment mechanism is Pareto efficient if it always selects a Pareto-efficient matching. A direct mechanism is *strategy-proof* if no student can ever benefit by unilaterally misrepresenting her preferences.

Since it is not possible to assign each student her top choice, a central issue in school choice is the design of a student assignment mechanism (Hirsch, 1994). In this paper we propose two direct student assignment mechanisms with different strengths. Depending on the priorities of policy makers, either mechanism can be practically implemented in real-life applications of school choice problems. Before we introduce and analyze these mechanisms, we describe and analyze some real-life student assignment mechanisms.

#### A. Boston Student Assignment Mechanism

One of the common mechanisms is the direct mechanism that is used by the city of Boston. The mechanism that we next describe has been in use in Boston since July, 1999. Prior to that another version of the same mechanism that imposed racial quotas was in use (United States District Court for the District of Massachusetts, 2002).<sup>8</sup> Variants of the same mechanism are currently used in Lee County, Florida,<sup>9</sup> Minneapolis (Steven Glazerman and Robert H. Meyer, 1994), and Seattle,<sup>10</sup> among other school districts.

The Boston student assignment mechanism works as follows.<sup>11</sup>

1. Each student submits a preference ranking of the schools.

<sup>8</sup> *Boston's Children First et al. v. Boston School Committee et al.* United States District Court for the District of Massachusetts, Civil Action Number: 99-11330-RGS, decided on January 25, 2002.

<sup>9</sup> See <http://www.lee.k12.fl.us/dept/plan/Choice/faqs.htm#13>.

<sup>10</sup> See page 12 of <http://www.seattleschools.org/area/eso/elementaryenrollmentguide20022003.pdf>.

<sup>11</sup> See <http://boston.k12.ma.us/teach/assign.asp>. See also the case, *Boston's Children First et al. v. Boston School Committee* cited in footnote 8.

2. For each school a priority ordering is determined according to the following hierarchy:

- First priority: sibling and walk zone.
- Second priority: sibling.
- Third priority: walk zone.
- Fourth priority: other students.

Students in the same priority group are ordered based on a previously announced lottery.

3. The final phase is the student assignment based on preferences and priorities:

*Round 1:* In Round 1 only the first choices of the students are considered. For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her first choice.

*Round 2:* Consider the remaining students. In Round 2 only the second choices of these students are considered. For each school with still available seats, consider the students who have listed it as their second choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her second choice.

In general at

*Round k:* Consider the remaining students. In Round  $k$  only the  $k^{\text{th}}$  choices of these students are considered. For each school with still available seats, consider the students who have listed it as their  $k^{\text{th}}$  choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her  $k^{\text{th}}$  choice.

The major difficulty with the Boston student assignment mechanism is that it is not strategy-proof. Even if a student has very high priority at school  $s$ , unless she lists it as her top choice she loses her priority to students who have listed  $s$

as their top choices. Hence the Boston student assignment mechanism gives very strong incentives to students and their parents to misrepresent their preferences by improving ranks of those schools for which they have high priority.<sup>12</sup> This point is also observed by Glazerman and Meyer (1994) for Minneapolis:

It may be optimal for some families to be strategic in listing their school choices. For example, if a parent thinks that their favorite school is oversubscribed and they have a close second favorite, they may try to avoid "wasting" their first choice on a very popular school and instead list their number two school first.

Another difficulty with the Boston student assignment mechanism concerns efficiency. If students submit their true preferences, then the outcome of the Boston student assignment mechanism is Pareto efficient. But since many families are likely to misrepresent their preferences, its outcome is unlikely to be Pareto efficient.

### B. Columbus Student Assignment Mechanism

The mechanism used by Columbus City School District is not a direct mechanism and it works as follows:<sup>13</sup>

1. Each student may apply to up to three different schools.
2. For some schools, seats are guaranteed for students who live in the school's regular assignment area and the priority among remaining applicants is determined by a random lottery. For the remaining schools, the priority among all applicants is determined by a random lottery.

<sup>12</sup> If a mechanism is not strategy-proof, that does not necessarily mean that it can be easily manipulated. For example, Roth and Uriel G. Rothblum (1999) show that although the hospital optimal stable mechanism can be manipulated by the interns, it is unlikely that such an attempt will be successful and hence truthful preference revelation is still in the best interest of the interns. In case of the Boston student assignment mechanism, the situation is quite different and the parents are warned to be careful how they use their top choices.

<sup>13</sup> See [http://www.columbus.k12.oh.us/applications/FAQ.nsf/\(deadline\)?openview#19](http://www.columbus.k12.oh.us/applications/FAQ.nsf/(deadline)?openview#19).

3. For each school, available seats are offered to students with the highest priority by a lottery office and the remaining applications are put on a waiting list. After receiving an offer a student has three days to accept or decline an offer. If she accepts an offer, she is assigned a seat; she then must decline offers from other schools and she is removed from the waiting list of other schools to which she has applied. As soon as seats become available at schools because of declined offers, the lottery office makes offers to students on the waiting lists.

The Columbus student assignment mechanism is similar to the entry-level market for clinical psychologists in the United States (see Roth and Xiaolin Xing, 1997). The market for clinical psychologists is more decentralized and each employer makes its offers via telephone, whereas a centralized lottery office makes all offers on behalf of each school for student assignment at Columbus.

As in the case of the Boston student assignment mechanism, the optimal application strategy of students is unclear under the Columbus student assignment mechanism. When a family gets an offer from its second or third choice, it is unclear whether the optimal strategy is declining this offer or accepting it. Similarly, the optimal list of schools to apply for is unclear. Hence, in Columbus, families are forced to play a very difficult game on a very crucial issue.

Another major difficulty with the Columbus student assignment mechanism concerns efficiency: Consider two students, each of whom hold an offer from the other's first choice. Since they do not know whether they will receive better offers, they may as well accept these offers, and this in turn yields an inefficient matching.

## II. Two Competing Mechanisms

We are now ready to propose two alternative mechanisms to the school choice problem.

### A. Gale-Shapley Student Optimal Stable Mechanism

As we have already emphasized, the school choice problem is closely related to the college

admissions problem: In school choice, schools are not agents and they have priorities over students, whereas in college admissions, schools are agents and they have preferences over students. One promising idea is interpreting school priorities as preferences and applying the following version of the *Gale-Shapley deferred acceptance algorithm* (Gale and Shapley, 1962):

*Step 1:* Each student proposes to her first choice. Each school tentatively assigns its seats to its proposers one at a time following their priority order. Any remaining proposers are rejected.

In general, at

*Step k:* Each student who was rejected in the previous step proposes to her next choice. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students one at a time following their priority order. Any remaining proposers are rejected.

The algorithm terminates when no student proposal is rejected and each student is assigned her final tentative assignment. We refer to the induced direct mechanism as *Gale-Shapley student optimal stable mechanism*. (See the Appendix for a detailed example.)

Gale-Shapley student optimal stable mechanism is used in Hong Kong to assign college seats to high school graduates, and since 1998, a version of it is used in the American hospital-intern market (Roth and Peranson, 1997, 1999).

The central notion in the college admissions literature is *stability*, which is equivalent to the following natural requirement in the context of school choice: There should be no unmatched student-school pair  $(i, s)$  where student  $i$  prefers school  $s$  to her assignment and she has higher priority than some other student who is assigned a seat at school  $s$ . Therefore, a stable matching in the context of college admissions eliminates *justified envy* in the context of school choice.

As it is suggested by its name, Gale-Shapley student optimal stable mechanism is stable in the context of college admissions and therefore

it eliminates justified envy in the context of school choice. It also has a number of additional very plausible properties.

**PROPOSITION 1** (Gale and Shapley, 1962): *Gale-Shapley student optimal stable mechanism Pareto-dominates any other mechanism that eliminates justified envy.*

**PROPOSITION 2** (Lester E. Dubins and David A. Freeman, 1981; Roth, 1982a): *Gale-Shapley student optimal stable mechanism is strategy-proof.*

Nevertheless, Gale-Shapley student optimal stable mechanism is not “problem free.” The following example due to Roth (1982a) shows that there is a potential trade-off between stability and Pareto efficiency.

*Example 1:* There are three students  $i_1, i_2, i_3$ , and three schools  $s_1, s_2, s_3$ , each of which has only one seat. The priorities of schools and the preferences of students are as follows:

$$\begin{array}{ll} s_1 : i_1 - i_3 - i_2 & i_1 : s_2 \ s_1 \ s_3 \\ s_2 : i_2 - i_1 - i_3 & i_2 : s_1 \ s_2 \ s_3 \\ s_3 : i_2 - i_1 - i_3 & i_3 : s_1 \ s_2 \ s_3 \end{array}$$

Let us interpret the school priorities as school preferences and consider the associated college admissions problem. In this case there is only one stable matching:

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}.$$

But this matching is Pareto-dominated by

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{pmatrix}.$$

Here agents  $i_1$  and  $i_2$  have the highest priorities for schools  $s_1$  and  $s_2$ , respectively. So there is no way student  $i_1$  can be assigned a school that is worse than school  $s_1$  and hence she shall be assigned either  $s_2$  or  $s_1$ . Similarly there is no way student  $i_2$  can be assigned a school that is worse than school  $s_2$  and hence she shall be assigned either  $s_1$  or  $s_2$ . Thus students  $i_1$  and  $i_2$  should share schools  $s_1$  and  $s_2$  among themselves. Stability forces them to share these

schools in a Pareto-inefficient way: This is because if students  $i_1$  and  $i_2$  are assigned schools  $s_2$  and  $s_1$  respectively, then we have a situation where student  $i_3$  prefers school  $s_1$  to her assignment  $s_3$  and she has a higher priority for school  $s_2$  than student  $i_2$  does.

As Example 1 shows, complete elimination of justified envy may conflict with Pareto efficiency.<sup>14</sup> If policy makers rank complete elimination of justified envy above Pareto efficiency, then Gale-Shapley student optimal stable mechanism is a very well-behaved mechanism.<sup>15</sup>

### B. Top Trading Cycles Mechanism

The stability notion strictly eliminates all justified envy. Next we consider a milder interpretation of the priorities, which in turn does not cause a conflict with Pareto efficiency. Suppose that if student  $i_1$  has higher priority than student  $i_2$  for school  $s$ , that does not necessarily mean that she is entitled a seat at school  $s$  before student  $i_2$ . It rather represents the opportunity to get into school  $s$ . If  $i_1$  has higher priority than  $i_2$ , then she has a better opportunity to get into school  $s$ , other things being equal.

Next, we introduce a competing mechanism which is Pareto efficient but which does not completely eliminate justified envy. Loosely speaking, the intuition for this mechanism is that it starts with students who have the highest priorities, and allows them to trade the schools for which they have the highest priorities in case a Pareto improvement is possible. Once these students are removed, it proceeds in a similar

<sup>14</sup> In many school districts, strict priorities are obtained with the help of a single tie-breaking lottery in addition to fundamental policy considerations. In others, strict priorities are obtained with the help of several tie-breaking lotteries, typically one for each school. Using a single tie-breaking lottery might be a better idea in school districts that adopt Gale-Shapley student optimal stable mechanism, since this practice eliminates part of the inefficiency: In this case, any inefficiency will be necessarily caused by a fundamental policy consideration and not by an unlucky lottery draw. In other words, the tie-breaking will not result in additional efficiency loss if it is carried out through a single lottery (while that is likely to happen if the tie-breaking is independently carried out across different schools).

<sup>15</sup> Recently, Ergin (2002) characterizes the conditions under which there is no conflict between complete elimination of justified envy and Pareto efficiency.

way starting with the students who have the highest priorities among those who remain. The *top trading cycles mechanism* is a direct mechanism and for any priorities and reported preferences it finds a matching via the following the *top trading cycles algorithm*.<sup>16</sup>

*Step 1:* Assign a *counter* for each school which keeps track of how many seats are still available at the school. Initially set the counters equal to the capacities of the schools. Each student points to her favorite school under her announced preferences. Each school points to the student who has the highest priority for the school. Since the number of students and schools are finite, there is at least one cycle. (A *cycle* is an ordered list of distinct schools and distinct students  $(s_1, i_1, s_2, \dots, s_k, i_k)$  where  $s_1$  points to  $i_1$ ,  $i_1$  points to  $s_2$ ,  $\dots$ ,  $s_k$  points to  $i_k$ ,  $i_k$  points to  $s_1$ .) Moreover, each school can be part of at most one cycle. Similarly, each student can be part of at most one cycle. Every student in a cycle is assigned a seat at the school she points to and is removed. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed. Counters of all other schools stay put.

In general, at

*Step k:* Each remaining student points to her favorite school among the remaining schools and each remaining school points to the student with highest priority among the remaining students. There is at least one cycle. Every student in a cycle is assigned a seat at the school that she points to and is removed. The counter of each school in a cycle is reduced by one and if it reduces to zero the school is also removed. Counters of all other schools stay put.

The algorithm terminates when all students are assigned a seat. Note that there can be no more steps than the cardinality of the set of students. (See the Appendix for a detailed example.)

The top trading cycles algorithm simply

<sup>16</sup> This algorithm is inspired by *Gale's top trading cycles algorithm* which is used to find the unique core allocation (Roth and Andrew Postlewaite, 1977) in the context of *housing markets* (Shapley and Herbert Scarf, 1974).

trades priorities of students among themselves starting with the students with highest priorities. In the very special case where all schools have the same priority ordering (for example, when all schools use the same ordering from a single lottery draw) this mechanism reduces to the *serial dictatorship* induced by this priority ordering. That is, the first student is assigned her top choice, the next student is assigned her top choice among the remaining seats, and so on. Therefore, the *top trading cycles mechanism* is a generalization of this mechanism, extended in a way that allows for different priorities at different schools.

A variant of the top trading cycles mechanism is proposed by Abdulkadiroğlu and Sönmez (1999) in a model of *house allocation with existing tenants*. In that model, there are existing tenants who have squatting rights over their current houses, and there are newcomers and vacant houses. The version proposed by Abdulkadiroğlu and Sönmez (1999) is a special case of the mechanism presented here: In that version, there is a fixed priority ordering for all houses but this ordering is slightly modified for each occupied house by inserting its current occupant at the top.

In a closely related paper, Pápai (2000) independently introduces the *hierarchical exchange rules*, which is a wider class of mechanisms. She characterizes the members of this class to be the only mechanisms that are *Pareto efficient*, *group strategy-proof*, (i.e., immune to preference manipulation by a group of agents) and *reallocation proof* (i.e., immune to manipulation by misrepresenting the preferences and swapping the objects by a pair of agents).

The top trading cycles mechanism has a number of very plausible properties. First of all, unlike the Gale-Shapley student optimal stable mechanism, it is Pareto efficient.

**PROPOSITION 3:** *The top trading cycles mechanism is Pareto efficient.*

Another key desirable feature of the top trading cycles mechanism is that, as in the case of Gale-Shapley student optimal mechanism, it is strategy-proof. Therefore truthful preference revelation is a dominant strategy for all students. In particular, unlike the Boston student assignment mechanism, students do not need to

hesitate on reporting their truthful preferences in fear of losing their priorities. Therefore both our proposed mechanisms release an important burden of finding the optimal application strategy over the shoulders of students and their parents.

**PROPOSITION 4:** *The top trading cycles mechanism is strategy-proof.*

The intuition for the strategy-proofness of the top trading cycles mechanism is very simple. Suppose that a student leaves the algorithm at Step  $k$  when she reports her true preferences. Since she points to the best available seat at each step of the algorithm, all the seats that she prefers leave the algorithm before Step  $k$  and by misrepresenting her preferences she cannot alter the cycles that have formed at any step before Step  $k$ . So these better seats will leave the algorithm before she does whether she reports her true preferences or fake preferences. Thus she can only hurt herself by a manipulation. Strategy-proofness of the core mechanism for housing markets (Roth, 1982b), the top trading cycles mechanism for house allocation with existing tenants (Abdulkadiroğlu and Sönmez, 1999), and the hierarchical exchange functions (Pápai, 2000) are all based on the same critical observation.

**C. Which Mechanism Shall Be Chosen?**

As we have already indicated, both mechanisms are strategy-proof, so the choice between them depends on the structure and interpretation of the priorities. In some applications, policy makers may rank complete elimination of justified envy before full efficiency, and Gale-Shapley student optimal stable mechanism can be used in those cases. University admissions in Turkey is one such application (Michel Balinski and Sönmez, 1999). In Turkey, priorities for university departments are obtained via a centralized exam and complete elimination of justified envy is imposed by law. Depending on the application, Gale-Shapley student optimal stable mechanism may have additional advantages. For example, consider a city which implements separate intra-district choice programs with an eventual target of eliminating the borders and

switching to an inter-district choice program. Furthermore, suppose that cross-district priorities will be lower than within-district priorities in the eventual program. In such applications, transition to an inter-district program is likely to move smoother under the Gale-Shapley student optimal stable mechanism: The outcome produced by the Gale-Shapley student optimal stable mechanism under the inter-district program Pareto-dominates the outcome produced by separate intra-district choice programs (each of which uses the Gale-Shapley student optimal stable mechanism). That is because the outcome produced by separate intra-district choice programs is still stable under the inter-district choice program, provided that cross-district priorities are lower than within-district priorities. Hence, no student can possibly suffer from a transition to the inter-district choice program under the Gale-Shapley student optimal stable mechanism. It is easy to construct an example where the transition to an inter-district choice program hurts some students under the top trading cycles mechanism.<sup>17</sup>

In other applications, the top trading cycles mechanism may be more appealing. School choice in Columbus is one such application. Recall the school priorities in Columbus: For some schools, students in the school's regular assignment area have high priority and they are all guaranteed a seat and the priority among the remaining low-priority students is determined by a random lottery. For the remaining schools, all students are in the same priority group and the priority between them is determined by a random lottery. Under the top trading cycles mechanism, students who have high priorities for their local schools are all guaranteed seats that are at least as good, provided that they truthfully report their preferences. Any instability produced by the top trading cycles mechanism is necessarily due to the randomly obtained priorities and in that case a milder interpretation of the priorities may be more appealing. In other cases the choice between the two mechanisms may be less clear and it depends on the policy priorities of the policy makers.

<sup>17</sup> We are grateful to an anonymous referee who brought this observation to our attention.

### III. Controlled Choice

Controlled choice in the United States attempts to provide choice to parents while maintaining the racial and ethnic balance at schools. In some states, choice is limited by court-ordered desegregation guidelines. In Missouri, for example, St. Louis and Kansas City must observe strict racial guidelines for the placement of students in city schools. There are similar constraints in other countries as well. For example, in England, City Technology Colleges are required to admit a group of students from across the ability range and their student body should be representative of the community in the catchment area (Hirsch, 1994, p. 120).

In many school districts, controlled choice constraints are implemented by imposing racial quotas at public schools. For example, prior to July 1999, a version of the Boston student assignment mechanism which uses racial quotas was in use in the city of Boston. Similarly, in Columbus and Minneapolis, controlled choice constraints are implemented by imposing racial quotas. These quotas may be perfectly rigid or they may be flexible. For example, in Minneapolis, the district is allowed to go above or below the district-wide average enrollment rates by up to 15 percent points in determining the racial quotas. So consider a school district in Minneapolis, where the average enrollment rates of majority students versus minority students are 60 percent and 40 percent, respectively, and consider a school with 100 seats. Racial quotas for this school are 75 for majority students, and 55 for minority students.

Both Gale-Shapley student optimal stable mechanism and the top trading cycles mechanism can be easily modified to accommodate controlled choice constraints by imposing type-specific quotas.

#### A. Gale-Shapley Student Optimal Stable Mechanism with Type-Specific Quotas

Suppose that there are different types of students and each student belongs to one type. If the controlled choice constraints are perfectly rigid then there is no need to modify the Gale-Shapley student optimal stable mechanism. For each type of students, one can separately implement the mechanism in order to allocate the

seats that are reserved exclusively for that type. When the controlled choice constraints are flexible, consider the following modification of the Gale-Shapley student optimal stable mechanism that is studied by Abdulkadiroğlu (2002) in the context of college admissions with affirmative action:<sup>18</sup>

*Step 1:* Each student proposes to her first choice. Each school tentatively assigns its seats to its proposers one at a time following their priority order. If the quota of a type fills, the remaining proposers of that type are rejected and the tentative assignment proceeds with the students of the other types. Any remaining proposers are rejected.

In general, at

*Step k:* Each student who was rejected in the previous step proposes to her next choice. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students one at a time following their priority order. If the quota of a type fills, the remaining proposers of that type are rejected and the tentative assignment proceeds with the students of the other types. Any remaining proposers are rejected.

This modified mechanism satisfies the following version of the fairness requirement: If there is an unmatched student-school pair  $(i, s)$  where student  $i$  prefers school  $s$  to her assignment and she has higher priority than some other student  $j$  who is assigned a seat at school  $s$ , then

1. students  $i$  and  $j$  are of different types, and
2. the quota for the type of student  $i$  is full at school  $s$ .

<sup>18</sup> Abdulkadiroğlu (2002) shows that flexible controlled choice constraints induce substitutable preferences (Alexander S. Kelso, Jr. and Vincent P. Crawford, 1982) in the context of college admissions. That is because the role played by the flexible controlled choice constraints in the present context is analogous to the role of *discriminatory quotas* (see Roth and Sotomayor, 1990, Proposition 5.22) in the context of college admissions problems. Gale-Shapley student optimal stable mechanism for the general case of substitutable preferences is due to Roth (1991).

As an implication, the modified mechanism eliminates all justified envy between students of the same type. The above fairness requirement is equivalent to stability in the context of college admissions with affirmative action. Moreover, truthful preference revelation is a dominant strategy for the students under the modified Gale-Shapley student optimal stable mechanism as well (Abdulkadiroğlu, 2002).

**PROPOSITION 5:** *Gale-Shapley student optimal mechanism with type-specific quotas is strategy-proof.*

In many real-life applications of controlled choice, there are only two types of students (for example, majority students and minority students). In such applications, the modified mechanism is essentially a direct application of the original mechanism with the following twist: Consider a school  $s$  with  $q$  seats and which has quotas of  $q_1, q_2$  for type 1, type 2 students, respectively. Clearly  $q \geq q_1, q \geq q_2$ , and  $q_1 + q_2 \geq q$ . In school  $s$ ,

- $q - q_2$  seats are reserved exclusively for type 1 students,
- $q - q_1$  seats are reserved exclusively for type 2 students,
- and the remaining  $q_1 + q_2 - q$  seats are reserved for either type of students.

So it is as if there are three different schools  $s^1, s^2, s^3$  where

- school  $s^1$  has  $q - q_2$  seats and student priorities are obtained from the original priorities by removing type 2 students and making them unacceptable at school  $s^1$ ,
- school  $s^2$  has  $q - q_1$  seats and student priorities are obtained from the original priorities by removing type 1 students and making them unacceptable at school  $s^2$ , and
- school  $s^3$  has  $q_1 + q_2 - q$  seats and student priorities are the same as the original priorities.

Whenever there are two types of students, our modified mechanism

- divides each school into three schools as explained above,

- extends each student's preferences as follows:
  - for any school  $s$ ,  $s^1$  is preferred to  $s^2$ , which is preferred to  $s^3$ ,
  - for any pair of schools  $s, t$ , if  $s$  is preferred to  $t$  then each of  $s^1, s^2, s^3$  is preferred to each of  $t^1, t^2, t^3$ ,

and

- selects the student optimal stable matching of the induced college admissions problem.

### B. Top Trading Cycles Mechanism with Type-Specific Quotas

As in the case of Gale-Shapley student optimal stable mechanism, there is no need to modify the top trading cycles mechanism when controlled choice constraints are perfectly rigid. One can implement the top trading cycles mechanism separately for each type of students. When the controlled choice constraints are flexible, the top trading cycles mechanism can be modified as follows: For each school, in addition to the original counter which keeps track of how many seats are available, include a type-specific counter for each type of students.

*Step 1:* For each school, set the counter equal to the capacity of the school and set each type-specific counter equal to the quota of the associated type of students. Each student points to her favorite school among those which have room for her type (i.e., with a positive counter reading for her type). Each school points to the student with highest priority for that school. There is at least one cycle. Every student in a cycle is assigned a seat at the school she points to and is removed. The counter of each school in a cycle is reduced by one. Depending on the student it is assigned to, the associated type-specific counter is reduced by one as well. All other counters stay put. In case the counter of a school (not the type-specific ones) reduces to zero, the school is removed as well. If there is at least one remaining student, then we proceed with the next step.

In general, at

*Step k:* Each remaining student points to her favorite remaining school among those which

have room for her type, and each remaining school points to the student with the highest priority among remaining students. There is at least one cycle. Every student in a cycle is assigned a seat at the school that she points to and is removed. The counter of each school in a cycle is reduced by one and depending on the student it is assigned to, the associated type-specific counter is reduced by one as well. All other counters stay put. In case the counter of a school reduces to zero, the school is removed. If there is at least one remaining student, then we proceed with the next step.

There can be efficiency losses in both mechanisms due to the controlled choice constraints. A matching is *constrained efficient* if there is no other matching that satisfies the controlled choice constraints, and which assigns all students a weakly better school and at least one student a strictly better school. The outcome of the modified top trading cycles mechanism is constrained efficient.

**PROPOSITION 6:** *The top trading cycles mechanism with type-specific quotas is constrained efficient.*

Moreover, truthful preference revelation is still a dominant strategy under the modified mechanism.

**PROPOSITION 7:** *The top trading cycles mechanism with type-specific quotas is strategy-proof.*

#### IV. Conclusion

The Office of Educational Research and Improvement (1992, pp. 19–20) emphasizes the following seven factors on which student assignment decisions should be based:

1. *Racial Balance:* Student assignment policies should respect the racial and ethnic proportions of the district.
2. *Instructional Capacity:* Student assignment policies must take into consideration the danger of creating an imbalance in the instructional capacity of a school.
3. *Replication Efforts:* Popular programs should be replicated and undersubscribed

schools be closed and then reopened as distinctive schools created by collective efforts.

4. *Space Availability:* Schools must outline classroom use needs long before the school year begins.
5. *Neighborhood School Priority:* A percentage of slots in a school should be reserved for neighborhood families, as long as racial balance is maintained, to allow continuity for the students and a connection for the school to the neighborhood.
6. *Preference for Siblings:* As a convenience for parents and to promote the sharing of school experience between brothers and sisters, preference for sibling requests should be given some priority.
7. *Gender Balance Considerations for Certain Schools.*

Among these, items 1, 2, 5, 6, and 7 concern the design of a student assignment mechanism. Both mechanisms that we propose respect each of these factors: Racial balance and gender balance can be achieved through type-specific quotas, instructional capacity overload is achieved through regular capacities, neighborhood school priority and preference for siblings can be achieved through school-specific priorities.

In addition to these factors, Alves and Willie (1990), engineers of the Boston Controlled Choice Plan, emphasize the following objectives as essential elements of an effective controlled choice plan:

1. Eliminating, to the extent practical, all individual school attendance boundaries and/or geocodes.
2. Allowing parents and students to make multiple school selections but with no guarantee that they will obtain their first-choice schools or programs of choice.
3. Ensuring complete honesty and integrity in the disposition of all final assignment decisions.

Both mechanisms conform with these objectives as well: Students rank all schools (of course without any guarantee of getting their top choices) and once the policies concerning school priorities are announced and these priorities determined, the final outcome is deterministic and does not leave any room for manipulation.

School choice is becoming increasingly common in the United States. Cities having adopted school choice plans include Boston, Cambridge, Champaign, Columbus, Hartford, Little Rock, Minneapolis, Rockville, Seattle, White Plains, and parts of New York. Recent laws in several states require each school district to establish an intra-district school choice plan. Similarly, recent laws in Florida require each school district to design a school choice plan, even if they do not implement it. An important difficulty in designing such plans is the choice of an appealing student assignment mechanism. Many of the school choice plans that we find have protocols and guidelines for the assignment of students without explicit procedures. This gap offers opportunities to manipulate these controlled choice programs and results in appeals by unsatisfied parents. Jeffrey R. Henig (1994, p. 212) states

The first step that districts must take to ensure fair implementation of choice within and among schools is to make the criteria for accepting and rejecting transfer requests clear and public. Vaguely worded references to "maintaining racial balance," "avoiding overcrowding," and meeting children's "individualized needs" invite selective interpretation unless they are accompanied by practical definitions.

Other school choice programs, such as those in Boston, Columbus, Minneapolis, and Seattle are accompanied by deterministic student assignment mechanisms but these mechanisms are all vulnerable to preference manipulation. As a result, students and their families face a difficult task of finding optimal admissions strategies. Adopting either the Gale-Shapley student optimal stable mechanism or the top trading cycles mechanism may provide a practical solution to some of these critical school choice issues.

## APPENDIX

*Example 2:* This example illustrates the dynamics of the Gale-Shapley student optimal stable mechanism and the top trading cycles mechanism. There are eight students  $i_1, \dots, i_8$  and four schools  $s_1, \dots, s_4$ . Schools  $s_1, s_2$  have two seats each and schools  $s_3, s_4$  have

three seats each. The priorities of the schools and the preferences of the students are as follows:

$$\begin{aligned}s_1 : i_1 - i_2 - i_3 - i_4 - i_5 - i_6 - i_7 - i_8 \\s_2 : i_3 - i_5 - i_4 - i_8 - i_7 - i_2 - i_1 - i_6 \\s_3 : i_5 - i_3 - i_1 - i_7 - i_2 - i_8 - i_6 - i_4 \\s_4 : i_6 - i_8 - i_7 - i_4 - i_2 - i_3 - i_5 - i_1\end{aligned}$$

$i_1$	$i_2$	$i_3$	$i_4$	$i_5$	$i_6$	$i_7$	$i_8$
$s_2$	$s_1$	$s_3$	$s_3$	$s_1$	$s_4$	$s_1$	$s_1$
$s_1$	$s_2$	$s_2$	$s_4$	$s_3$	$s_1$	$s_2$	$s_2$
$s_3$	$s_3$	$s_1$	$s_1$	$s_4$	$s_2$	$s_3$	$s_4$
$s_4$	$s_4$	$s_4$	$s_2$	$s_2$	$s_3$	$s_4$	$s_3$

## GALE-SHAPLEY STUDENT OPTIMAL STABLE MECHANISM:

*Step 1:* Students  $i_2, i_5, i_7, i_8$  propose to school  $s_1$ , student  $i_1$  proposes to school  $s_2$ , students  $i_3, i_4$  propose to school  $s_3$  and student  $i_6$  proposes to school  $s_4$ .

School  $s_1$  tentatively assigns its seats to students  $i_2, i_5$ , and rejects students  $i_7, i_8$ . Since school  $s_1$  is the only school with excess proposals, all other students are tentatively assigned seats at schools that they propose.

*Step 2:* Having been rejected at Step 1, each of students  $i_7, i_8$  proposes to school  $s_2$  which is their next choice. School  $s_2$  considers student  $i_1$  whom it has been holding together with its new proposers  $i_7, i_8$ . School  $s_2$  tentatively assigns its seats to students  $i_8, i_7$ , and rejects student  $i_1$ .

*Step 3:* Having been rejected at Step 2, student  $i_1$  proposes to school  $s_1$  which is her next choice. School  $s_1$  considers students  $i_2, i_5$  whom it has been holding together with its new proposer  $i_1$ . School  $s_1$  tentatively assigns its seats to students  $i_1, i_2$ , and rejects student  $i_5$ .

*Step 4:* Having been rejected at Step 3, student  $i_5$  proposes to school  $s_3$  which is her next choice. School  $s_3$  considers students  $i_3, i_4$  whom it has been holding together with its new proposer  $i_5$ . Since school  $s_3$  has three seats, it tentatively assigns its seats to these students.

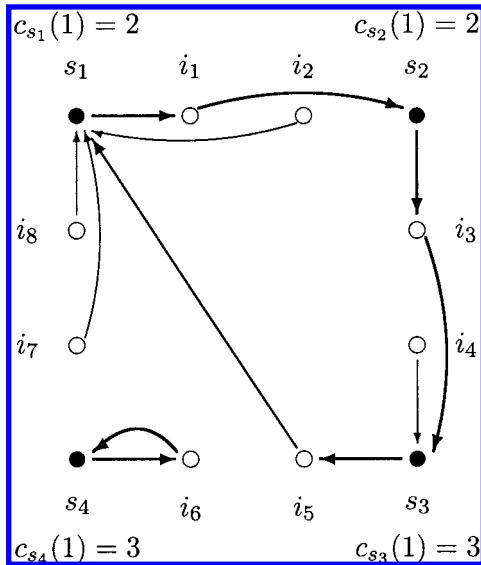


FIGURE A1. TOP TRADING CYCLES ALGORITHM: STEP 1

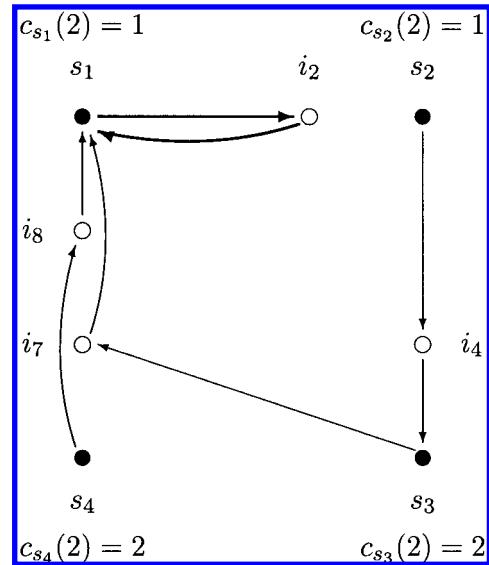


FIGURE A2. TOP TRADING CYCLES ALGORITHM: STEP 2

Since no student proposal is rejected at Step 4, the algorithm terminates. Each student is assigned her final tentative assignment:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ s_1 & s_1 & s_3 & s_3 & s_3 & s_3 & s_2 & s_2 \end{pmatrix}.$$

#### TOP TRADING CYCLES MECHANISM:

Let  $c_{s_1}$ ,  $c_{s_2}$ ,  $c_{s_3}$ , and  $c_{s_4}$  indicate the counters of the schools.

*Step 1:* [see Figure A1].

There are two cycles in Step 1:  $(s_1, i_1, s_2, i_3, s_3, i_5)$  and  $(s_4, i_6)$ . Therefore students  $i_1, i_3, i_5, i_6$  are assigned one slot at schools  $s_2, s_3, s_1, s_4$ , respectively, and removed. Since every school participates in a cycle, all counters are reduced by one for the next step.

*Step 2:* [see Figure A2].

There is only one cycle in Step 2:  $(s_1, i_2)$ . Therefore student  $i_2$  is assigned one slot at school  $s_1$  and removed. The counter of school  $s_1$  is reduced by one to zero and it is removed. All other counters stay put.

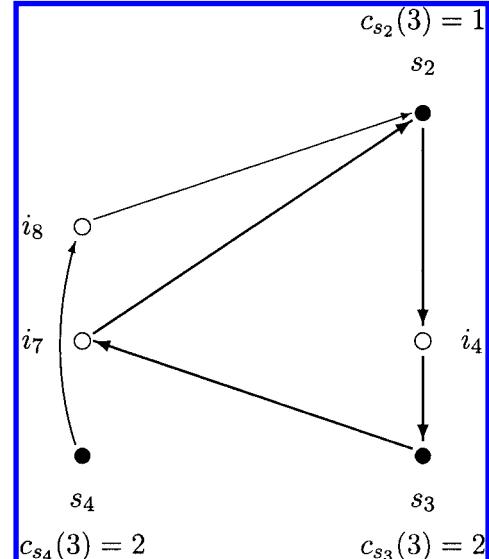


FIGURE A3. TOP TRADING CYCLES ALGORITHM: STEP 3

*Step 3:* [see Figure A3].

There is only one cycle in Step 3:  $(s_3, i_7, s_2, i_4)$ . Therefore students  $i_7, i_4$  are assigned one slot at schools  $s_2, s_3$ , respectively, and removed.

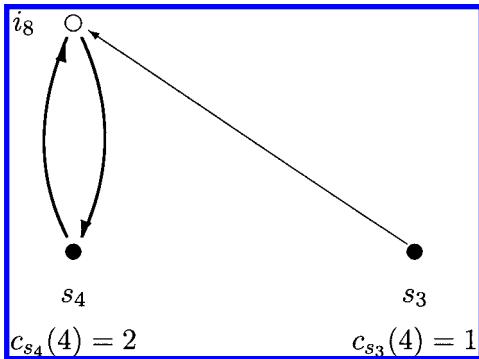


FIGURE A4. TOP TRADING CYCLES ALGORITHM: STEP 4

The counters of schools  $s_2$  and  $s_3$  are reduced by one. Since there are no slots left at school  $s_2$  it is removed. Counters of schools  $s_3$  and  $s_4$  stay put.

*Step 4:* [see Figure A4].

There is only one cycle in Step 4:  $(s_4, i_8)$ . Therefore student  $i_8$  is assigned one slot at school  $s_4$  and removed. There are no remaining students so the algorithm terminates. Altogether the matching it induces is

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ s_2 & s_1 & s_3 & s_3 & s_1 & s_4 & s_2 & s_4 \end{pmatrix}.$$

**PROOF OF PROPOSITION 3:**

Consider the top trading cycles algorithm. Any student who leaves at Step 1 is assigned her top choice and cannot be made better off. Any student who leaves at Step 2 is assigned her top choice among those seats remaining at Step 2 and since preferences are strict she cannot be made better off without hurting someone who left at Step 1. Proceeding in a similar way, no student can be made better off without hurting someone who left at an earlier step. Therefore the *top trading cycles mechanism* is Pareto efficient.

The proof of Proposition 4 is similar to the proof of a parallel result in Abdulkadiroğlu and Sönmez (1999).<sup>19</sup> The following lemma is the key to the proof.

<sup>19</sup> Pápai (2000) independently proves a similar result for a wider class of mechanisms.

**LEMMA:** Fix the announced preferences of all students except  $i$  at  $Q_{-i} = (Q_j)_{j \in I \setminus \{i\}}$ . Suppose that in the algorithm student  $i$  is removed at Step  $T$  under  $Q_i$  and at Step  $T^*$  under  $Q_i^*$ . Suppose  $T \leq T^*$ . Then the remaining students and schools at the beginning of Step  $T$  are the same whether student  $i$  announces  $Q_i$  or  $Q_i^*$ .

**PROOF OF LEMMA:**

Since student  $i$  fails to participate in a cycle prior to Step  $T$  in either case, the same cycles form and therefore the same students and schools are removed before Step  $T$ .

**PROOF OF PROPOSITION 4:**

Consider a student  $i$  with true preferences  $P_i$ . Fix an announced preference profile  $Q_{-i} = (Q_j)_{j \in I \setminus \{i\}}$  for every student except  $i$ . We want to show that revealing her true preferences  $P_i$  is at least as good as announcing any other preferences  $Q_i$ . Let  $T$  be the step at which student  $i$  leaves under  $Q_i$ ,  $(s, i_1, s_1, \dots, s_k, i)$  be the cycle she joins, and thus school  $s$  be her assignment. Let  $T^*$  be the step at which she leaves under her true preferences  $P_i$ . We want to show that her assignment under  $P_i$  is at least as good as school  $s$ . We have two cases to consider:

*Case 1:*  $T^* \geq T$ .

Suppose student  $i$  announces her true preferences  $P_i$ . Consider Step  $T$ . By the Lemma, the same students and schools remain in the market at the beginning of this step whether student  $i$  announces  $Q_i$  or  $P_i$ . Therefore at Step  $T$ , school  $s$  points to student  $i_1$ , student  $i_1$  points to school  $s_1, \dots, s_k$  points to student  $i$ . Moreover, they keep doing so as long as student  $i$  remains. Since student  $i$  truthfully points to her best remaining choice at each step, she either receives an assignment that is at least as good as school  $s$  or eventually joins the cycle  $(s, i_1, s_1, \dots, s_k, i)$  and is assigned a slot at school  $s$ .

*Step 2:*  $T^* < T$ .

By the Lemma the same schools remain in the algorithm at the beginning of Step  $T^*$  whether student  $i$  announces  $Q_i$  or  $P_i$ . Moreover, student  $i$  is assigned a seat at her best choice school remaining at Step  $T^*$  under  $P_i$ . Therefore, in this case too her assignment under

the true preferences  $P_i$  is at least as good as school  $s$ .

#### PROOF OF PROPOSITION 5:

Immediately follows from Abdulkadiroğlu (2002) and a self-contained proof is available upon request. The original strategy-proofness result by Dubins and Freeman (1981) and Roth (1982a) directly carries over to the case with two types of students since the modified mechanism is a direct application of the original mechanism as explained in Section III, subsection A.

#### PROOF OF PROPOSITION 6:

Consider the modified top trading cycles algorithm. Any student who leaves at Step 1 is assigned her top choice and cannot be made better off. Any student who leaves at Step 2 is assigned her top choice among those schools which has room for her type at Step 2 and since preferences are strict she cannot be made better off without hurting someone who left at Step 1. Proceeding in a similar way, no student can be made better off without hurting someone who left at an earlier step. Therefore the top trading cycles mechanism with type-specific quotas is constrained efficient.

#### PROOF OF PROPOSITION 7:

The Lemma preceding the proof of Proposition 4 as well as its proof are valid for the modified mechanism. Moreover the basic elements of the proof of Proposition 4 carry over as well.

Consider a student  $i$  with true preferences  $P_i$ . Fix an announced preference profile  $Q_{-i} = (Q_j)_{j \in \Lambda \setminus \{i\}}$  for every student except  $i$ . We want to show that revealing her true preferences  $P_i$  is at least as good as announcing any other preferences  $Q_i$ . Let  $T$  be the step at which student  $i$  leaves under  $Q_i$ ,  $(s, i_1, s_1, \dots, s_k, i)$  be the cycle she joins, and thus school  $s$  be her assignment. Let  $T^*$  be the step at which she leaves under her true preferences  $P_i$ . We want to show that her assignment under  $P_i$  is at least as good as school  $s$ . We have two cases to consider:

*Case 1:  $T^* \geq T$ .*

Suppose student  $i$  announces her true prefer-

ences  $P_i$ . Consider Step  $T$ . By the Lemma, the same students and schools remain in the market at the beginning of this step whether student  $i$  announces  $Q_i$  or  $P_i$ . Therefore at Step  $T$ , school  $s$  points to student  $i_1$ , student  $i_1$  points to school  $s_1$ ,  $\dots$ , school  $s_k$  points to student  $i$ . Moreover, they keep doing so as long as student  $i$  remains. But at each step student  $i$  truthfully points to her best choice among schools with an available seat for her type. Therefore she either receives an assignment that is at least as good as school  $s$  or eventually joins the cycle  $(s, i_1, s_1, \dots, s_k, i)$  and is assigned a slot at school  $s$ .

*Case 2:  $T^* < T$ .*

By the Lemma the same schools remain in the algorithm at the beginning of Step  $T^*$  whether student  $i$  announces  $Q_i$  or  $P_i$ . Moreover, student  $i$  is assigned a seat at her best choice among schools with an available seat for her type remaining at Step  $T^*$  under  $P_i$ . Therefore, in this case too her assignment under the true preferences  $P_i$  is at least as good as school  $s$ .

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