School Choice

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ABSTRACT: School choice has presented economists with opportunities to study and design student assignment systems, which in turn have helped push forward the frontiers of mechanism design theory. In this survey, we discuss the student assignment problem in school choice, related issues, assignment mechanisms and new developments in the theory and practice of mechanism design within the context of school choice.

1 Introduction

Good public schools are scarce, and admissions to those will always matter.

Public schools are free of charge and admissions in many districts have been defined by location of schools and the home addresses of pupils. As traditional neighborhood-based assignment has led to the segregation of neighborhoods along socioeconomic lines, recent decades have witnessed a surge

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in programs that offer parental choice over public schools, expanding families' access to schools beyond their residential area. In fact the origins of school choice in the United States can be traced back to Brown v. Board of Education, 1954. Boston's renowned controlled choice program evolved out of a 1974 ruling that enforced desegregation of Boston public schools. Today there are other reasons for public school choice; school districts have been increasingly leaving the one-size-fits-all model of schooling and developing alternative curricula to better meet educational needs of a highly heterogenous student population. As districts offer more options for parents and students, choice and therefore *student assignment* become an integral part of enrollment planning.

Since the introduction of this problem by Abdulkadiroğlu and Sönmez (2003), economists have found great opportunity to study and design student assignment systems around the U.S. Most notable of these are the redesign of the student assignment systems in Boston and New York City. The former was initiated by a Boston Globe article on Abdulkadiroğlu and Sönmez (2003)[1], which described flaws with the student assignment in Boston at the time. The latter was initiated independently when, being aware of his pioneering work on market design in the entry level labor markets (Roth, 1984; Roth and E. Peranson, 1999), the New York City Department of Education (NYC DOE) contacted Alvin E. Roth to inquire about the possibility of adopting a system like the National Residency Matching Program[2] for their high school admissions. The school choice problem and its market design applications have fostered a new line of research in mechanism design theory. The goal of this survey is to summarize recent developments in the field and

[1] See "School assignment flaws detailed" by Cook (2003).

[2] The National Resident Matching Program is a United States-based non-profit non-governmental organization created in 1952 to help match medical school students with residency programs in the U.S.

in mechanism design theory.

We intentionally divide the survey into two parts. The next section discusses the school choice problem and the issues in the canonical model of Abdulkadiroğlu and Sönmez (2003). It also gives a brief discussion to various student assignment mechanisms. The section is intended for the general audience and practitioners in the field. The remainder of the survey expands on the new developments.[3]

2 The School Choice Problem

A *school choice problem* (Abdulkadiroğlu and Sönmez 2003) consists of a finite set of students and a finite set of schools with finitely many seats available for enrollment. In this section, we will refer to students by a, b, c, d and schools by s_1, s_2, s_3 and s. Students have preferences over schools. We represent a student's preferences as a linear order of schools to which she prefers to be assigned rather than accept her outside option. For example, $a : s_1 - s_2 - s_3$ means that student a prefers school s_1 to school s_2 and school s_2 to school s_3; she prefers her outside option to being assigned any other school. Her outside option is not specified in the model, it may be a private school or home schooling or some other option. We assume that students form their preferences based on exogenous school characteristics, such as curricula, extra-curricular activities, distance to home, average test scores and graduation rates in the past years. This rules out, for example, conditioning one's preferences on the composition of the incoming class.

Admissions to schools are usually regulated via assignment priorities. For

[3]The focus of this survey is limited by school choice. Therefore it may miss many important references in matching theory. Naturally, it may also be biased towards my own work on the topic and my experience in the field. For another recent survey on school choice, see Pathak (2011).

instance, for most schools in Boston, for half of the seats at the school, the students are priority-ordered as follows:

1. Students who are guaranteed a space at the school by virtue of already attending that school or a feeder school (guaranteed priority),
2. students who have a sibling at the school and live in the walk zone of the school (sibling-walk priority),
3. students who have a sibling at the school (but who do not live in the walk zone of the school) (sibling priority),
4. students who live in the walk zone of the school (but who do not have a sibling at the school) (walk zone priority), and
5. other students in the zone.

A random lottery number for each student breaks ties in each category (random tie-breaker). For the other half of the seats, walk zone priorities do not apply, and students are priority-ordered based on guaranteed and sibling priority and the random tie-breaker (Abdulkadiroğlu and Sönmez 2003; Abdulkadiroğlu, Pathak, Roth and Sönmez 2006).

Such priority structure may reflect a district's policy choice. Neighborhood priority may be granted to promote involvement of neighborhood parents in school activities; sibling priority may be adopted to reduce transportation and organizational costs for parents and to promote spillover benefits of attending the same school among siblings. Priorities may be determined differently at different schools. In fact, the priority list of a school may even reflect preferences of the school staff over students. For instance, some high schools in New York City can access students' academic records and rank students in a preference order. When priorities are determined by some exogenous rules, such as in Boston, we say that the market is *one-sided.* When priorities at some schools reflect preferences of the school staff, such as in New York City, we say that the market is *two-sided.*

Regardless of its resource, we represent the priority list at a school as a linear order of all the students that are eligible for enrollment at that school. For example, $s_1 : b - a - c$ means that student b has the highest priority at school s_1, a has the next highest priority and c has the lowest priority; student d is not eligible for enrollment at that school. The number of available seats at schools completes the model.

A matching of students and schools determines the assignment of each student. Hereafter, we use *matching, assignment* and *enrollment* interchangeably. Each student is matched with at most one school or remains unmatched. A school can be matched with students up to its capacity. We will utilize examples of the following type in our discussions:

Example 1 *There are three students* $\{a, b, c\}$ *and three schools* $\{s_1, s_2, s_3\}$ *each with one seat. Student preferences and school priorities are given as follows:*

$$\begin{array}{ll} a: & s_2 - s_1 - s_3 \\ b: & s_1 - s_2 - s_3 \\ c: & s_1 - s_2 - s_3 \end{array} \quad \textit{and} \quad \begin{array}{ll} s_1: & a - c - b \\ s_2: & b - a - c \\ s_3: & b - a - c \end{array}$$

We will denote a matching that assigns a *to* s_1, b *to* s_2 *and leaves* c *unmatched as*

$$m_1 = \begin{pmatrix} a & b & c \\ s_1 & s_2 & - \end{pmatrix}$$

2.1 Issues and Policy Goals

What are the goals of a successful choice plan? Are these policy goals compatible with each other? What are the trade-offs and how should one compromise? These questions are closely related to the design of student assignment mechanisms. The education literature provides guidance for the design of assignment mechanisms but does not offer a specific one. Also, flaws in the existing school choice plans result in difficult and stressful decision making for parents, gaming and behind-closed-doors strategies by savvy parents, as well as appeals in the US courts by unsatisfied parents (Abdulkadiroğlu and Sönmez 2003).

Economists' approach to such allocation problems is to translate the relevant policy goals into normative theoretical criteria and look for solutions that meet these criteria, and if no such solution exists, then find one with optimal compromise. The following notions emerge naturally in the context of school choice.

Feasibility

Overcrowding at schools is controlled by school capacities. A matching is deemed feasible in our model if enrollment at each school does not exceed the school capacity and only eligible students are enrolled at every school.

Individual rationality

If a student is assigned a school that is not in her choice list, one may expect her family to opt out for its outside option, which may be a private school, home schooling or some other option. A matching is individually rational if it matches every student only with schools in her choice list, and leaves her unassigned otherwise. Hereafter we consider only feasible and individually rational matchings.

Efficiency

Perhaps the most obvious desideratum that guides a design is that the match process should promote student welfare to the greatest extent possible; that is, it should be efficient for students. We say that a matching *wastes* a seat at school s if there remains an empty seat at s and an eligible student prefers s to her match. In the example above, m_1 wastes a seat at s_3 because student c is unassigned, a seat at s_3 remains available and c prefers s_3 to being unassigned. The matching

$$m_2 = \begin{pmatrix} a & b & c \\ s_1 & s_2 & s_3 \end{pmatrix}$$

improves c's welfare without harming other students. Identifying and remedying such wastefulness is relatively easy. A more subtle wastefulness occurs in the assignment of a and b. Notice that both a and b are assigned their second choices. They become better off if they swap their assignments. In other words, in comparison to m_2, the matching

$$m_3 = \begin{pmatrix} a & b & c \\ s_2 & s_1 & s_3 \end{pmatrix}$$

improves a and b's welfare without harming c. We say that a matching *Pareto dominates* another matching if the former improves some student's welfare without harming others in comparison to the latter. In our example, m_3 Pareto dominates m_2, which Pareto dominates m_1. We say that a matching is *Pareto efficient* or simply *efficient* if it is not Pareto dominated by another matching. In particular, m_3 is efficient in our example; both a and b are assigned their first choices and c cannot be assigned a better choice without harming a or b's assignments.

Note that the following matchings are also efficient:

$$m_4 = \begin{pmatrix} a & b & c \\ s_2 & s_3 & s_1 \end{pmatrix}, \; m_5 = \begin{pmatrix} a & b & c \\ s_3 & s_2 & s_1 \end{pmatrix}$$

Respecting or violating priorities in assignment

An integral input to our model is school priorities. Districts utilize priorities to ration seats when schools are oversubscribed. How priorities restrict assignment is a matter of policy choice.

In their weakest form, priorities simply determine eligibility. If a student is eligible for an empty seat at a school and she prefers it to her match, one might expect her parents to file an appeal to the district. Therefore, a wasteful matching is not desirable from a policy standpoint. However, if determining eligibility were the only role priorities are supposed to play, an unordered list of eligible students would be sufficient. To attribute priorities a broader role in rationing seats in assignment, we say that a matching *violates a student's priority* at school s if the student ranks s higher than her assigned school and has higher priority at s than some other student who is assigned s. We say that a matching is *stable* if it does not violate priorities and not waste any seat.

In the example above, m_3 violates c's priority at s_1, because c prefers s_1 to her assigned school s_3 and she has higher priority at s_1 than b, who is assigned s_1. Therefore it is not stable. In fact m_2 is the only stable matching in this example. Note that students a and b get their second choices at m_2, and would have been better off had they swapped their matchings. In that case c's priority at s_1 would have been violated. This is the first trade-off we encounter: stability comes at the cost of student welfare. A stable matching need not be efficient and an efficient matching need not be stable.

As the following example demonstrates, there may be multiple stable matchings:

Example 2 *There are three students $\{a, b, c\}$ and three schools $\{s_1, s_2, s_3\}$ each with one seat. Student preferences and school priorities are given as follows:*

$$\begin{array}{ll} a: & s_2 - s_1 - s_3 \\ b: & s_1 - s_2 - s_3 \\ c: & s_1 - s_2 - s_3 \end{array} \quad \textit{and} \quad \begin{array}{ll} s_1: & a - b - c \\ s_2: & b - a - c \\ s_3: & b - a - c \end{array}$$

We have only changed the priorities at s_1 from the previous example. Now *there are two stable matchings:*

$$n_1 = \begin{pmatrix} a & b & c \\ s_1 & s_2 & s_3 \end{pmatrix},\ n_2 = \begin{pmatrix} a & b & c \\ s_2 & s_1 & s_3 \end{pmatrix}$$

n_2 would not be stable for Example 1, because c's priority at s_1 would be violated. In this example, c has the lowest priority at every school, so her priority is not violated by n_2.

We say that a stable matching is *student optimal stable* if it is not Pareto dominated by any other stable matchings. In Example 2, n_1 is Pareto dominated by n_2 since it assigns a and b to their higher choices without changing c's assignment. n_2 is not Pareto dominated by any stable matching, so it is student optimal stable.

So far we have only talked about student welfare. The preferences of schools in a two-sided market may also matter. For example, if the priorities reflect school preferences in Example 2, then n_2 no longer Pareto dominates n_1, because while n_2 assigns a and b better, it matches both s_1 and s_2 with their less preferred students. We cannot improve any student's assignment in n_1 without harming the assignment of another student or *school.* In other words, n_1 is efficient when priorities reflect school preferences. In general, stability implies efficiency in such two-sided markets.

School preferences may stem from different comparative advantages. For example, different EdOpt schools in New York city seem to have different preferences even for students with low reading scores, with some schools preferring higher scores and others preferring students who had good attendance. Even when student welfare is the primary concern in such two-sided markets, allowing scope for school preferences via stability may be desirable to utilize such comparative advantages (Abdulkadiroğlu, Pathak and Roth 2009).

Whether or not it is acceptable for priorities to be violated is determined by the circumstances of the specific problem. For instance, during the redesign of student assignment in Boston, violating priorities was initially favored in order to promote student welfare. Boston Public Schools (BPS) decided to respect priorities in the final design. We will discuss these issues in more details later.

Incentives to game the system

If student preferences were known a priori, it would be easy for a district to meet the goal of efficiency or student optimal stability. However, preference data is unknown to the admissions office. Eliciting that information truthfully during application is not a trivial task. Indeed student assignment systems in most school choice programs force parents to submit a choice list that is different than their true preference list. We will see a prevalent example later.

A student assignment system, or simply a mechanism, determines the matching of students with schools for every profile of preferences, priorities and school capacities. Since an assignment mechanism responds to student preferences, a student can presumably affect her assignment by changing the list of schools she submits in her application form. We say that an assignment mechanism is *strategy-proof (for students)* if listing schools in true preference order in the application form is optimal for every student, regardless of the priority structure and other students' applications. In other words, a strategy-proof assignment system ensures that a student gets her best assignment – not necessarily her first choice – under every circumstance by filling in her true preference list. We can define strategy-proofness for schools in a similar manner in two-sided markets, in which schools also rank students in preference order.

Strategy-proofness has at least three policy advantages. First, it simplifies the decision making process for parents by making truthful listing of preferences a best strategy. Under a strategy-proof mechanism, parents may focus solely on determining schools that would best fit their children's educational needs; they do not need to navigate the system via preference manipulation in their application form; indeed doing so may even harm them. This also allows school districts to give straightforward advice on filling in application forms. Second, some parents may lack the information or the ability required to navigate a system that is prone to gaming. By removing the need for gaming, a strategy-proof mechanism levels the playing field among parents. Finally, a strategy-proof mechanism provides reliable demand data for districts, which can play a crucial role in enrollment planning.

When priorities reflect school preferences, if there is a student-school pair that prefer each other to their match, the school has an incentive to circumvent the match to enroll the students it prefers. Stability eliminates such circumstances. Therefore, stability also offers scope for eliminating gaming of the system by schools.

Armed with these notions, next we will discuss and compare three prominent student assignment mechanisms.

2.2 Three student assignment mechanisms

One way to think about these design concerns is that Pareto efficiency for the students is the primary welfare goal, and strategy-proofness in the elicitation of student preferences is an incentive constraint that has to be met. Moreover, stability of the matching may enter as a policy choice when priorities reflect district policies, or as an incentive constraint in two-sided markets in which priorities reflect school preferences. Mechanisms can be evaluated and formulated from this "mechanism design" perspective.

The Boston Mechanism

Probably the most prevalent student assignment mechanism is the so called

Boston mechanism that was developed in Cambridge in 1980s. The Boston mechanism tries to assign as many students as possible to their first choices, assigning higher priority students to overdemanded schools; and only after first choice assignments are made, it considers unassigned students at their second choices in the same fashion, and so on. That is, given student preferences and school priorities, the matching is determined by the following algorithm:

> Step 1: For each school, consider the students who have listed it as their first choice in the application form. Assign seats of the school to these students one at a time in the order of priority at that school until either there are no seats left or there is no student left who has listed it as her first choice.
>
> In general, in step k: Consider only the kth choices of the students who are not assigned in an earlier step. For each school with still available seats, assign the remaining seats to the students who have listed it as their kth choice in the order of priority until either there are no seats left or there is no student left who has listed it as her kth choice.

The algorithm terminates when no more students are assigned. Let us apply this in Example 1: In the first step, student a is considered for and assigned s_2; b and c are considered for s_1; since there is only one seat and c has higher priority, c is assigned s_1. b remains unassigned. Since there is no seat available at s_2, b is not considered for s_2 in the second step. She is considered for and assigned s_3 in the third step and the algorithm terminates. The Boston matching is

$$m_{Boston} = \begin{pmatrix} a & b & c \\ s_2 & s_3 & s_1 \end{pmatrix}$$

Notice that b is assigned her third choice even though she has the highest priority at her second choice s_2. Therefore the Boston mechanism is not stable. Moreover, by ranking s_2 as second choice, b loses her priority to a, who ranks s_2 as first choice. If she instead ranked s_2 as her first choice, she would have been assigned s_2, which she prefers to s_3. That is, the Boston mechanism is not strategy-proof and a student can improve her odds of getting into a school by ranking it higher in her application. Indeed, The BPS school guide [2004, p3] explicitly advised parents to follow that strategy when submitting their preferences (quotes in original):

> For a better chance of your "first choice" school . . . consider choosing less popular schools. Ask Family Resource Center staff for information on "underchosen" schools.

The feature that one may gain from manipulating her choice list in the Boston mechanism is also recognized by parents in Boston and elsewhere. Indeed the West Zone Parent Group (WZPG), a parent group in Boston, recommends strategies to take advantage of the mechanism:[4]

> One school choice strategy is to find a school you like that is undersubscribed and put it as a top choice, OR, find a school that you like that is popular and put it as a first choice and find a school that is less popular for a "safe" second choice.

Efficient Transfer Mechanism

The Efficient Transfer Mechanism (ETM), proposed by Abdulkadiroğlu and Sönmez (2003)[5], lines up students at schools with respect to their priorities.

[4] For more references to anectodal evidence see Abdulkadiroğlu and Sönmez 2003, Ergin and Sönmez 2006, and Abdulkadiroğlu, Pathak, Roth and Sönmez 2006.

[5] ETM is known as Top Trading Cycles mechanism (TTC) in the literature. "Efficient

It tentatively assigns one empty seat at a time to highest priority students. If a students is happy with her assignment, she keeps it. Otherwise, ETM looks for welfare enhancing transfers among those students. Once such transfers are exhausted, it continues in the same fashion by assigning seats to the next highest priority student. In slightly different but more formal language, given student preferences and school priorities, the matching is determined by the following algorithm:

> Step 1: Every school points to its highest priority student, every student points to her most preferred school. A transfer cycle is an ordered list of schools and students (school 1 - student 1 - school 2 - ... - school k - student k) with school 1 pointing to student 1, student 1 to school 2, ..., school k to student k, and student k pointing to school 1. All the cycles are found. Every student in a cycle is assigned a seat at the school she points to and is removed; the number of seats at that school is decreased by one.
>
> In general, in step k: Every school with still available seats points to its highest priority student, every student points to her most preferred school with still available seats. All the cycles are found. Every student in a cycle is assigned a seat at the school she points to and is removed; the number of seats at that school is decreased by one.

The algorithm terminates when no more students are assigned. Applying this in Example 1, s_1 points to a, both s_2 and s_3 point to b; a points to s_2

transfers" reflect the nature of the algorithm equally well, if not better than "top trading cycles". In our experience in the field, parents tend to have a dislike for the word "trade", complicating an objective discussion of the mechanisms for policy makers. Therefore we will refer to the mechanism as Efficient Transfer Mechanism.

and b and c both point to s_1. (s_1, a, s_2, b) form a cycle, a is assigned s_2, b is assigned s_1, they are removed, there are no more available seats at s_1 and s_2. In the second step, only s_3 has an available seat; s_3 points to c, the highest priority student among remaining students, and c points back to s_3, her most preferred school among all with still available seats; (s_3, c) forms a cycle, c is assigned s_3. Note that the ETM matching

$$m_{ETM} = \begin{pmatrix} a & b & c \\ s_2 & s_1 & s_3 \end{pmatrix}$$

is efficient. In fact, ETM is a strategy-proof and efficient mechanism (Abdulkadiroğlu and Sönmez 2003). However, m_3 violates c's priority at s_1, so ETM does not guarantee stability.

The Student Optimal Stable Matching Mechanism

Gale-Shapley's Student Optimal Stable Matching mechanism (SOSM) operates like the Boston mechanism (Gale and Shapley 1962). However a student does not lose her priority at a school to those who rank it higher in their choice lists. To achieve this, SOSM makes *tentative* assignments and reconsiders them at every step. Formally, given student preferences and school priorities, the matching is determined by the following algorithm:

> Step 1: Each students applies to her first choice. Each school tentatively assigns its seats to its applicants one at a time in their priority order until capacity is reached. Any remaining applicants are rejected.
>
> In general, in step k: Each student who was rejected in the previous step applies to her next best choice if one remains. Each school considers the set consisting of the students it has been holding from previous steps and its new applicants, and tentatively assigns its seats to these students one at a time in priority

order. Any students in the set remaining after all the seats are filled are rejected.

The algorithm terminates when no more students are assigned, then tentative assignments are finalized. Let us find the SOSM matching in Example 1: In the first step, student a applies to and is tentatively assigned s_2; b and c apply to s_1; since there is only one seat and c has higher priority, c is tentatively assigned s_1. b is rejected. Then b applies to s_2, which considers b along with a. Since b has a higher priority, b is tentatively assigned s_2 and a is rejected. Then a applies to s_1, which considers a along with c. a is tentatively assigned and c is rejected. Then c applies to and is rejected by s_2, and finally she applies to and is tentatively assigned s_3. Since no more students are assigned, the tentative assignments are finalized and the SOSM produces

$$m_{SOSM} = \begin{pmatrix} a & b & c \\ s_1 & s_2 & s_3 \end{pmatrix}$$

In contrast with the Boston algorithm, SOSM assigns seats only tentatively at each step and students with higher priorities may be considered in subsequent steps. That feature guarantees that SOSM is stable in the sense that there is no student who loses a seat to a lower priority student and receives a less-preferred assignment. More importantly, all students prefer their SOSM outcome to any other stable matching (Gale and Shapley 1962) and SOSM is strategy-proof (Dubins and Freedman, 1981; Roth 1982b). When priorities reflect school preferences, stability eliminates the need for schools to circumvent the match to enroll the students they would prefer. However, in general, there is no stable matching mechanism, student-optimal or not, that is strategy-proof for schools in two-sided markets (Roth 1985).

2.3 Comparison of the mechanisms

The Boston mechanism is not stable. Notice that b's priority at s_2 is violated at m_{Boston} above. On the other hand, it is not possible to improve the assignment of a student who gets her first choice at the Boston matching, since she is already getting her first choice. Consider a student who gets his second choice. His first choice is filled with students who rank it as first choice. Therefore, it is not possible to assign him his first choice without assigning another student at that school lower in her choice list. In general, a student cannot be assigned better than her Boston matching without harming another student's assignment. That is, the Boston mechanism is efficient with respect to the *submitted* preferences.

However, the Boston mechanism is not strategy-proof. As a result, parents are forced to play a complicated game of navigating the system through preference manipulation during applications. Therefore a more important question from a policy point of view is whether the outcome resulting from this strategic interaction will be efficient or stable with respect to *true* preferences. When every parent has access to full information, and therefore knows the true preferences of other parents and the priority orderings at schools, and this is common knowledge among parents, the outcome of the Boston mechanism that emerges from parents' strategic interaction[6] is stable with respect to the true preference profile even though some parents manipulate their preferences (Ergin and Sönmez 2006). This implies that SOSM is preferred to the Boston mechanism by students in such full information environments, since SOSM produces the stable matching that students prefer to any other stable matching. Furthermore, it is easy to find examples for the failure of stability and efficiency with the Boston mechanism when that

[6]Formally, we are referring to a Nash equilibrium outcome of the complete information game induced by the Boston mechanism

assumption is violated.[7]

Both ETM and SOSM are strategy-proof. ETM is efficient but not stable; SOSM is not efficient but it is student optimal stable. We summarize these in the following table.

[INSERT TABLE 1 HERE]

Note that the ETM outcome Pareto dominates the SOSM outcome in Example 1. However, despite its superior efficiency property, the ETM outcome is not always better for every student:

Example 3 *There are three students $\{a, b, c\}$ and three schools $\{s_1, s_2, s_3\}$ each with one seat. Student preferences and school priorities are given as follows:*

$$\begin{array}{rl} a: & s_2 - s_1 - s_3 \\ b: & s_1 - s_3 - s_2 \\ c: & s_1 - s_2 - s_3 \end{array} \quad \textit{and} \quad \begin{array}{rl} s_1: & a - c - b \\ s_2: & b - a - c \\ s_3: & b - a - c \end{array}$$

The SOSM outcome is

$$n_{SOSM} = \begin{pmatrix} a & b & c \\ s_2 & s_3 & s_1 \end{pmatrix},$$

and the ETM outcome is

$$n_{ETM} = \begin{pmatrix} a & b & c \\ s_2 & s_1 & s_3 \end{pmatrix}$$

Student c prefers n_{SOSM} and b prefers n_{ETM}.

[7] Ergin and Sönmez (2006) provide an example with informational asymmetry among parents, in which the resulting outcome of the Boston mechanism fails to be stable with respect to the true preferences. Failure of efficiency is apparent even in the full information game, since a full information equilibrium is stable and stability does not imply efficiency.

As noted above, SOSM and ETM simplify the task of advising parents in filing applications. All an official needs to recommend to parents is that they identify the best fit schools for their child and rank them in the order of their preferences.

A second concern for school districts is to explain the match to parents whose children are not assigned one of their higher choices. The outcome of SOSM is easily justified. If a student does not get into, say, her first choice under SOSM, it is because every student that is enrolled in her first choice has higher priority than she does. The outcome of the ETM can also be justified in a similar fashion. Whereas SOSM tentatively assigns seats to applicants in the order of their preferences, ETM tentatively assigns seats to students in the order of school priorities. Therefore, each seat is associated with the priority of the student that it is initially assigned. If a student does not get into her first choice under ETM, it is because every seat at her first choice was initially assigned to a student with higher priority than hers. Furthermore, she could not be transferred to her first choice because she did not have high enough priority at other schools to qualify for such a transfer.

We discuss these mechanisms in further detail later. For now, a brief discussion of mechanism choices in Boston and NYC will illuminate the interplay between theory and the design.

3 Market Design at Work

The differences in the initiation of the redesign efforts and the decision making processes in Boston and NYC illuminate the contrasting features and challenges in both markets.

School choice in Boston has been partly shaped by desegregation. In 1974, Judge W. Arthur Garrity ordered busing for racial balance. In 1987, the U.S. Court of Appeals freed BPS to adopt a new, choice-based assignment plan with racial preferences. In 1999, BPS eliminated racial preferences in assignment. Despite its poor incentive properties, the Boston mechanism continued to clear the market for public school choice until 2003. Although the gaming aspect of the mechanism had apparently been known in certain Boston parent circles, it was brought to light by Abdulkadiroğlu and Sönmez (2003). A public debate initiated by a Boston Globe piece on the article led to the redesign of the system. In December 2003, the Boston School Committee initiated an evaluation of all aspects of student assignment, which yielded a task-force report with a recommendation of adopting ETM. After intensive discussions, public meetings organized by BPS and analysis of the existing school choice system and the behavior it elicited, in July 2005, the Boston School Committee voted to replace the existing school choice mechanism with SOSM. It is the first time that "strategy-proofness," a central concept in the game theory literature on mechanism design, has been adopted as a public policy concern related to transparency, fairness, and equal access to public facilities. (Abdulkadiroğlu, Pathak, Roth and Sönmez, 2005).

In contrast, the New York City was failing to assign more than 30,000 of the approximately 100,000 incoming high school students to a school of their choice, yielding public outcry during the assignment period every March.[8] The NYC DOE was aware of the matching process for American physicians, the National Resident Matching Program (Roth, 1984). They contacted Alvin E. Roth in the fall of 2003 to inquire if it could be appropriately adapted to the city's schools. After an intense sequence of meetings with economists, the NYC DOE adopted a new system by January 2004 (Abdulkadiroğlu, Pathak and Roth 2005). In this respect, "Boston was like a patient with high blood pressure, a potentially deadly disease that has no easily visible symptoms"; the New York City high school admission process was like "a patient with a heart attack, where the best treatment might not be obvious,

[8] See Goodnough (2003), *New York Times.*

but there was little dispute that treatment was needed" (Abdulkadiroğlu, Pathak, Roth and Sönmez, 2006).

Two features of the NYC High School choice favored SOSM over ETM. The first was that schools withheld capacity to match with students they preferred. The fact that school administrators gamed the system indicated they were strategic players. Stable assignments eliminate part of the incentives for gaming the system. Furthermore, empirical observations suggest that centralized matching mechanisms in two-sided markets are most often successful if they produce stable matchings (Roth 1991). Second, principles of EdOpt schools can express preferences over students. Discussions indicated that principals of different EdOpt schools had different preferences even for students with low reading scores, with some schools preferring higher scores and others preferring students who had good attendance. If schools have different comparative advantages, allowing scope for their preferences seemed sensible.

The performance of the mechanisms also differs across markets. SOSM generates greater efficiency loss in NYC, where as it is almost efficient in Boston. We will compare the two mechanisms in more detail later.

A special form of ETM is utilized in the supplementary round of the NYC high school match.[9] The Louisiana Recovery School District adopted ETM

[9]The supplementary round is designed to match students that have not been matched in the main round of the process. Those students fill out a new application form in which they rank from the list of schools that still have available seats at the end of the main round. Due to the time constraint, priority information is no longer collected from schools in that round. Instead, students are ordered randomly, students are matched one by one in that order with their most preferred school that still has available seats. This mechanism is a special form of ETM, therefore it is strategy-proof and efficient.

in 2012. Also, after consulting with economists,[10] the San Francisco Board of Education unanimously approved a new system based on ETM in March 2010.[11]

As in any market design exercise, choice programs offered by school districts may involve distinctive features that are not captured by the basic model. Next we discuss some of those features brought to light by applications and the developments in the literature led by them.

4 Extensions

For the sake of completeness, we provide the formal definitions in mathematical notation in this section. A (school choice) *problem* consists of

- a finite set of students I,
- a finite set of schools S,
- school capacities $q = (q_s)_{s \in S}$, where q_s is the number of available seats at school $s \in S$,
- a profile of student preferences $P = (P_i)_{i \in I}$,
- and a profile of school priorities $\succsim = (\succsim_s)_{s \in S}$.

Each student $i \in I$ has a strict preference relation P_i over schools and her outside option o.[12] sP_is' means i prefers s to s'. Let R_i denote the weak

[10] Clayton Featherstone, Muriel Niederle, Parag Pathak, Alvin Roth and I teamed up to assist the San Francisco Unified School District (SFUSD) in the redesign. Featherstone and Niederle lead the discussions with SFUSD.

[11] The SFUSD decided to develop the matching software on their own without consulting us any further. Their decision was due to concerns about sharing confidential data for monitoring the effects of the new system.

[12] Formally, P_i is a complete, irreflexive and transitive binary relation over $S \cup \{o\}$.

preference relation induced by P_i, that is, sR_is' if and only if sP_is' or $s = s'$. A school s is *acceptable* for i if i prefers s to her outside option.

Each school $s \in S$ has a weak priority relation $\succsim_s$ over $I \cup \{\varnothing\}$, where $\varnothing$ represents leaving a seat empty.[13] A student i is *eligible* for school s if $i \succ_s \varnothing$. A student i is either eligible for school s or not, that is either $i \succ_s \varnothing$ or $\varnothing \succ_s i$ for all i, s.

A *matching* of students to schools is a set valued function $\mu : I \cup S \rightrightarrows 2^{I \cup S}$ such that

- $\mu(i) \subset S \cup \{o\}$, $|\mu(i)| = 1$ for all $i \in I$,
- $\mu(s) \subset I$, $|\mu(s)| \leq q_s$ for all $s \in S$, and
- $s \in \mu(i)$ if and only if $i \in \mu(s)$ for all $i \in I$ and $s \in S$.

That is, a student is matched with a school or her outside option, the number of students matched with a school cannot exceed its capacity, and a student is matched with a school if and only if the school is also matched with the student. We will equivalently use $\mu(i) = s$ for $s \in \mu(i)$.

Given $(\succsim_S, P_I)$, μ *violates i's priority at s* if i prefers s to her match and another student with lower priority is matched with s, that is, $sP_i\mu(i)$ and there is a student $j \in \mu(s)$ such that $i \succ_s j$.[14]

In the *one-sided* matching models of school choice, priorities can be violated to promote student welfare. In contrast, the *two-sided* matching models

[13] When $\succsim_s$ represents the preferences of s over students, we extend $\succsim_s$ over subsets of I as follows: each $\succsim_s$ is **responsive** (to its restriction on $I \cup \{\varnothing\}$). That is, for every $I' \subset I$ and $i, j \in I \backslash I'$, (i) $I' \cup \{i\} \succsim_s I'$ if and only if $\{i\} \succsim_s \varnothing$, and (ii) $I' \cup \{i\} \succsim_s I' \cup \{j\}$ if and only if $\{i\} \succsim_s \{j\}$ (Roth 1985).

[14] In the standard two-sided matching literature, such (i, s) pair is said to block μ and it is referred as a blocking pair. Alternatively, Balinski and Sönmez (1999) refer it as envy by i at s. The naming of violating priorities is due to Ergin (2002).

do not allow priority violations at *any* school. To provide a unified treatment (Abdulkadiroğlu 2011), in addition to the standard model, we say that a school has *a strict priority policy* if priorities may not be violated at the school, and has *a flexible priority policy* otherwise. If the priority list of a school reflects its preferences, one may assume the school has a strict priority policy. We assume that S is partitioned into S_{strict}, the set of schools with a strict priority policy, and $S_{flexible}$, the set of schools with a flexible priority policy. Formally, $S = S_{strict} \cup S_{flexible}$ and $S_{strict} \cap S_{flexible} = \varnothing$.

Next we define the policy-relevant mathematical properties, or axioms, that a matching may possess.

A matching μ is *feasible* if every student that is matched with a school is eligible for that school. We restrict our attention to feasible matchings only. A matching μ is *individually rational* if every student weakly prefers her match to her outside option. To simplify the exposition, we assume that a student can rank a school only if she is eligible for that school, that is, if $s \succ_i o$ then $i \succ_s \varnothing$. Then individual rationality implies feasibility.

In our unified model, a priority violation at school s is a cause of concern only if s has a strict priority policy. Accordingly, a matching μ is *pseudo-stable* if it is individually rational and it does not violate priorities at any school with a strict priority policy, i.e. there is no $i \in I$, $s \in S_{strict}$ and $j \in \mu(s)$ such that $sP_i\mu(i)$ and $i \succ_s j$. The null matching that matches every student to her outside option is trivially pseudo-stable.

A matching μ *wastes a seat at* s if $|\mu(s)| < q_s$ and there is a student who is eligible for s and prefers it to her match (Balinski and Sönmez 1999), i.e. there exists $i \in I$ such that $i \succ_s \varnothing$ and $sP_i\mu(i)$.[15] A matching μ is *stable* if it is pseudo-stable and it does not waste any seat. Although the null matching is trivially pseudo-stable, it wastes all the seats so it is not stable.

[15]Such a pair is also refered as a blocking pair in the two-sided matching literature. The renaming of it as wastefulness is due to Balinski and Sönmez (1999).

A matching μ *Pareto dominates* another matching ν if every student weakly prefers her μ-match to her v-match and some strictly, i.e. $\mu(i)R_i\nu(i)$ for all $i \in I$ and $\mu(i)P_i\nu(i)$ for some $i \in I$. A matching is *Pareto efficient* if it is not Pareto dominated by another matching. A matching is *student optimal stable* if it is stable and not Pareto dominated by another pseudo-stable matching.

Our unified model reduces to the standard two-sided matching model when $S = S_{strict}$, that is, every school has a strict priority policy. It reduces to the the standard one-sided matching model of school choice when $S = S_{flexible}$, that is, every school has a flexible priority policy. In that case, every matching is pseudo-stable and every student optimal stable matching is Pareto efficient. In other words, Pareto efficiency becomes a special case of our notion of student optimal stable matching when priority violations are allowed at all schools.

A student admissions procedure is defined as a mechanism. A (deterministic) *mechanism* selects a matching for every problem. The definitions for matching trivially extend for a mechanism. For example, a mechanism is stable if it selects a stable matching for every problem. Suppressing school priorities, let $\varphi(P)$ denote the matching selected by a mechanism φ. A mechanism φ is *strategy-proof for students* if reporting true preferences is a dominant strategy for every student in the preference revelation game induced by φ, that is

$$\varphi(P)(i)R_i\varphi(P_i', P_{-i})(i)$$

for all P, $i \in I$ and P_i', where $P_{-i} = (P_j)_{j\in I\setminus\{i\}}$. Strategy-proofness for schools is defined similarly. A mechanism φ *Pareto dominates* another mechanism φ' if for every problem $< I, S, q, P, \succsim >$, every student prefers her φ-match to her φ'-match and some strictly, that is $\varphi(P)(i)R_i\varphi'(P)(i)$ for all i and $\varphi(P)(i)P_i\varphi'(P)(i)$ for some i.

4.1 Further discussion of the mechanisms

When all schools have a strict priority policy, the problem turns into a two-sided matching problem. In that case, SOSM is the unique stable mechanism that is strategy-proof for students (Alcalde and Barberà 1994). When priorities do not reflect school preferences, the notion of respecting priorities can be interpreted as the elimination of justified envy (Balinski and Sönmez 1999). When a student's standing in the priority list of school improves, the student is assigned a weakly better school by SOSM. In fact, SOSM is the only stable mechanism with that property (Balinski and Sönmez 1999).

SOSM is not efficient from students' perspective. Ergin (2002) shows that the outcome of SOSM is efficient if and only if school priorities satisfy a certain acyclicity condition. Ehlers and Erdil (2010) generalize that result when school priorities are coarse. Although, the ETM outcome may Pareto dominate the SOSM outcome for some problems, no Pareto efficient and strategy-proof mechanism Pareto dominates SOSM when school priorities do not involve ties (Kesten 2010). Kesten (2010) proposes a new algorithm that eliminates the efficiency loss associated with SOSM by allowing students to give up certain priorities whenever it does not hurt them to do so.

When all schools have a flexible priority policy, the problem turns into a one-sided matching problem. Starting with Shapley and Scarf (1974), ETM has mostly been studied in exchange markets for indivisible objects. That model corresponds to a special case of our model in which each school has a single seat and a student is ranked highest by at most one school. In that environment, ETM is strategy-proof (Roth 1982a) and it is the only mechanism that is Pareto efficient, strategy-proof and that guarantees every student that is top ranked at a school an assignment that she weakly prefers to that school (Ma 1994). When students are allowed to be ranked highest by more than one school, ETM is a special subclass of Pápai's (2000) hierarchical exchange rules. In that case, Pápai characterizes hierarchical exchange rules by Pareto efficiency, group strategy-proofness (which rules out beneficial preference manipulation by groups of individuals), and reallocation-proofness (which rules out manipulation by two individuals via misrepresenting preferences and swapping objects ex post). ETM is a hierarchical exchange rule defined by the priority lists of schools. In a similar vein, Pycia and Ünver (2010) introduce and characterize trading cycles with brokers and owners by Pareto efficiency and group strategy-proofness. Bogomolnaia, Deb and Ehlers (2005) provide a characterization for a general class of Pareto efficient and strategy-proof mechanisms for the case in which schools have multiple seats and no priorities.

Despite the lack of a Pareto ranking between SOSM and ETM, there exists a clear-cut comparison between SOSM and Boston when market participants have full information about others' preferences and priorities and that is common knowledge. In particular, given strict school priorities, every Nash equilibrium outcome of the Boston mechanism is stable under true preferences. Therefore the dominant strategy equilibrium of SOSM weakly Pareto dominates every Nash equilibrium outcome of the Boston mechanism (Ergin and Sönmez 2006).[16]

Further characterizations of SOSM and the Boston mechanism are provided via monotonicity conditions on preferences by Kojima and Manea (2010) and Kojima and Unver (2010), respectively. Roth (2008) provides a survey of the history, theory, and practice of SOSM.

4.2 Ties in school priorities

Much of the earlier theory of two-sided matching focuses on the case where all parties have strict preferences, mainly because indifferences in preferences

[16] Kojima (2008) generalizes this finding to more complicated priority structures that, for instance, can favor specific student populations via quotas.

were viewed as a "knife-edge" phenomenon in applications like labor markets (Roth and Sotomayor 1990). In contrast, a primary feature of school choice is that there are indifferences -"ties"- in how students are ordered by at least some schools. How to break these ties raises some significant design decisions, which bring in new trade-offs among efficiency, stability, and strategy-proofness (Erdil and Ergin 2008, Abdulkadiroğlu, Pathak and Roth 2009).

The mechanism of choice must specify how to order equal-priority students from the point of view of schools with limited space. For instance, one can assign each student a distinct number, breaking ties in school priorities according to those assigned numbers – single tie breaker –, or one can assign each student a distinct number at each school – multiple tie breakers –, breaking ties according to school specific numbers. Since any non-random assignment of such numbers can be incorporated into the priority structure at the outset, we will consider randomly generated tie breakers.

4.2.1 Ex post efficiency

ETM remains Pareto efficient and strategy-proof with single and multiple tie breakers. Furthermore, when there are no priorities at schools, i.e. all students tie in priority at every school, ETM produces the same probability distribution over matchings when a single or a multiple tie breaker is drawn uniformly randomly (Pathak and Sethuraman 2011).

If one applies SOSM to the strict priorities that result from tie breaking, the stability and strategy-proofness of SOSM is preserved. However, tie breaking introduces artificial stability constraints (since, after tie breaking, schools appear to have strict rankings between equal priority students), and these constraints can harm student welfare. In other words, when SOSM is applied to the strict priorities that result from tie breaking, the outcome it produces may not in fact be a student-optimal stable matching in terms of the original priorities.

When school priorities are weak, there may be multiple student optimal stable matchings that are not Pareto ranked with each other. Every student-optimal stable matching can be obtained by SOSM with some tie breakers (Ehlers 2006). However, some forms of tie breaking may be preferable to others. For instance, during the course of designing the New York City high school match, policy makers from the Department of Education were concerned with the fairness of tie breaking; they believed that each student should receive a different random number at each program they applied to and this number should be used to construct strict preferences of schools for students. Their rationale was that if a student draws a bad number in a single tie breaker, her bad luck would apply to every school of her choice, whereas multiple tie breakers would give a new life line at her lower ranked schools when the student is rejected by a school. However, we show via simulations with NYC High School Match data that significantly more students get their first choices when ties are broken by a single lottery (Abdulkadiroğlu, Pathak and Roth 2009). Table 2 summarizes our simulation results for 250 random draws of tie breakers for grade 8 applicants in 2006-07. In particular, on average SOSM with single breakers matches about 2,255 more students to their first choices. Note also that SOSM with single breakers leaves about 186 more students unassigned, which implies that there is no comparison between SOSM with single breakers and SOSM with multiple tie breakers in terms of first order stochastic dominance.

[INSERT TABLE 2 HERE]

Some theoretical insight for that observation come from that, when school priorities are weak, all student optimal stable matchings can be found by SOSM with single breakers (Abdulkadiroğlu, Pathak and Roth 2009; Erdil 2006). In other words, if there is a matching produced by SOSM with multiple breakers that cannot be produced by any SOSM with single breakers, then it is not a student-optimal stable matching.

However, a single lottery is not sufficient for student-optimality (Erdil and Ergin 2008; Abdulkadiroğlu, Pathak and Roth 2009). Given a matching, a stable improvement cycle of students $\{a_1, ..., a_{n+1} \equiv a_1\}$ is such that every student in the cycle is matched with a school, every a_k, $k = 1, ..., n$, prefers a_{k+1}'s match to her match and she has the highest priority among all students who prefer a_{k+1}'s match to their match (Erdil and Ergin 2008). If the cycle is implemented by transferring a_k to a_{k+1}'s matched school, the resulting matching is stable and Pareto dominates the original matching. Based on this novel observation, Erdil and Ergin (2008) show that a stable matching μ is student optimal if and only if it does not admit a stable improvement cycle. They also introduce stable improvement cycles (SIC) algorithm which starts with an arbitrary stable matching and finds and implements a cycle until no cycle is found. SIC is student optimal stable. Employing SIC on top of SOSM with single breakers, Table 2 shows that about 596 more students can be matched with their first choices.

4.2.2 Incentives and ex post efficiency

More interestingly, ties in school priorities introduce a trade-off between efficiency and strategy-proofness. In particular, there is no strategy-proof mechanism that always selects a student optimal stable matching (Erdil and Ergin 2008). Therefore SOSM with any breakers may yield inefficient outcomes and removal of such inefficiency harms students' incentives. Furthermore, given a set of tie breakers, the associated SOSM is not Pareto-dominated by any strategy-proof mechanism (Abdulkadiroğlu, Pathak and Roth 2009). This observation generalizes two earlier results: SIC is not strategy-proof (Erdil and Ergin 2008) and no Pareto efficient and strategy-proof mechanism Pareto-dominates SOSM when school priorities are strict (Kesten 2010).

In other words, SOSM with a tie breaker lies on the Pareto frontier of strategy-proof mechanisms. This theoretical observation gives us an empirical strategy to assess the cost of strategy-proofness. In particular, the additional 596 students who get their first choices under SIC in Table 2 can be interpreted as the efficiency cost of strategy-proofness for students in SOSM with single breakers.

In Table 2, when students start with their SIC matches and welfare improving transfers are exhausted among students via ETM[17], on average an additional 2,006 students can be matched with their first choice. Similarly, this number can be interpreted as the welfare cost of limiting the scope of manipulation for schools in NYC.

4.2.3 Ex ante efficiency

The earlier literature, in particular all the results stated so far, relies on a notion of efficiency from an ex post point of view, that is, after the resolution of all potential uncertainties. When too many students demand a seat at a school, admissions to the school is regulated by priorities. When priorities are strict, both ETM and SOSM uniquely determine the outcome. In contrast, with weak priorities, there remains a great deal of freedom in placing students according to their preferences. Furthermore a new scope of efficiency from an ex ante point of view emerges. These points are illustrated in the following example by Abdulkadiroğlu, Che and Yasuda (2011).

Example 4 *There are three students,* $\{1, 2, 3\}$ *and three schools,* $\{s_1, s_2, s_3\}$*, each with one seat. Schools have no intrinsic priorities over students, and*

[17]That is, start with the SIC matching. Run the following version of ETM: Every student points to her most preferred school among remaining. Every school points to remaining students that it currently enrolls. Cycles are found, every student in a cycle is transferred to the school she points to and she is removed. Continue in the same fashion until no more students are transferred.

student i has a von-Neumann Morgenstern (henceforth, vNM) utility value of v^i_j when she is assigned to school j:

	v^1_s	v^2_s	v^3_s
$s = s_1$	0.8	0.8	0.6
$s = s_2$	0.2	0.2	0.4
$s = s_3$	0	0	0

Every feasible matching is stable due to schools' indifferences. More importantly, any such assignment is ex post *Pareto efficient, and hence student optimal stable, since students have the same ordinal preferences.*

Since SOSM with any tie breaker is strategy-proof, all three students submit true (ordinal) preferences of $s_1 - s_2 - s_3$. SOSM with a single tiebreaker that is drawn uniformly randomly matches every student to each school with equal probability of $\frac{1}{3}$, which yields an expected payoff of $\frac{1}{3}$ for each student. This random matching is *ex ante* Pareto-dominated by the following random matching: Assign student 3 to s_2, and students 1 and 2 randomly between s_1 and s_3, which yields expected payoff of $0.4 > \frac{1}{3}$ for every student. This Pareto-dominating random matching arises as the unique equilibrium outcome of the Boston mechanism. In fact, this observation holds more generally. Suppose that all students tie in priorities at every school, students have the same ordinal ranking of schools and their cardinal utilities are private information that are drawn from a commonly known distribution. Consider the Boston mechanism and SOSM with a single tie breaker that is drawn uniformly randomly. Then each student's expected utility in every symmetric Bayesian equilibrium of the Boston mechanism is weakly greater than her expected utility in the dominant-strategy equilibrium of SOSM (Abdulkadiroğlu, Che and Yasuda 2011). This finding contrasts but does not contradict with Ergin and Sönmez (2006), who analyze a complete information setup with strict school priorities and heterogenous ordinal preferences for students.

SOSM is strategy-proof and therefore in the dominant strategy equilibrium of SOSM, every student submits her true preference list to the mechanism regardless of her cardinal utilities. In contrast, a student takes her cardinal utilities into account while submitting her equilibrium strategy under the Boston mechanism. That allows the Boston mechanism break ties based on cardinal information, as opposed to the fully random tie breaking under SOSM.

Independently, Featherstone and Niederle (2008) show that truth-telling becomes a Bayesian Nash equilibrium of the Boston mechanism when informational asymmetry on student preferences are introduced in a symmetric environment, in which all schools have the same capacity, all students tie in priorities at every school, and preferences of each student are drawn uniformly randomly on the set of all possible rank orderings of the set of schools. Then more students are matched with their first choices in the truth-telling equilibrium of the Boston mechanism than in the dominant-strategy truth-telling equilibrium of SOSM.

Troyan (2011) take a more ex ante approach and examines welfare before students know their cardinal utilities and priorities. He shows that, from this perspective, the Boston mechanism ex-ante Pareto dominates any strategy-proof and anonymous mechanism, including SOSM and ETM, even with arbitrary priority structures.

These complimentary works draw a picture of the Boston mechanism that has been overlooked by the earlier literature that relies on the complete information assumption.

Motivated by their observation for the Boston mechanism, Abdulkadiroğlu, Che and Yasuda (2008) propose an SOSM with "preferential" tie breaking. Every student submits her ordinal preference list and picks one school as a target, at which she will be favored in tie breaking. When two students tie at a school, the one who picks it as a target is favored in tie breaking, otherwise the ties are broken randomly. It is still a dominant strategy to submit true preferences to their mechanism and gaming is limited to the choice of the target school. They show that their modified mechanism results in ex ante efficiency gains in large economies. In a similar vein, Miralles (2008) shows that a variant of the Boston mechanism that utilizes a new lottery in every round of the assignment algorithm obtains similar efficiency gains over SOSM in a continuum economy.

Budish, Che, Kojima and Milgrom (forthcoming), on the other hand, generalize the theory of randomized assignment to accommodate multi-unit allocations and various real-world constraints, including group-specific quotas in school choice. They also provide new mechanisms that are ex-ante efficient and fair.

4.2.4 Ex ante stability

When school priorities are weak, random tie breaking with SOSM yields randomization over stable matchings. In that setup, Kesten and Ünver (2010) introduce two notions of stability from an ex ante point of view: A random matching is ex ante stable if there are no students a, b and a school s such that a has a higher priority at s than b, b is matched with s with positive probability and a is matched with positive probability with a school that she prefers less than s. An ex ante stable random matching is strongly ex ante stable if it avoids the following case among equal priority students, which they refer as ex-ante discrimination: a and b have equal priority at s, b enjoys a higher probability of being assigned to s than a, and a is matched with positive probability with a school that she prefers less than s. Kesten and Ünver (2010) propose an algorithm to select the strongly ex ante stable random matching that is ordinally Pareto dominant among all strongly ex ante stable random matchings.

4.3 Leveling the playing field

Strategy-proofness has emerged as a major public policy concern related to transparency, fairness, and equal access to public facilities in the redesign of the Boston school assignment system (Abdulkadiroğlu, Pathak, Roth and Sönmez 2006). In July 2005, the Boston School Committee voted to adopt SOSM which removes the incentives to "game the system" that handicapped the Boston mechanism. In his memo to the School Committee on May 25, 2005, Superintendent Payzant wrote:

> The most compelling argument for moving to a new algorithm is to enable families to list their true choices of schools without jeopardizing their chances of being assigned to any school by doing so... A strategy-proof algorithm levels the playing field by diminishing the harm done to parents who do not strategize or do not strategize well.

Pathak and Sönmez (2008) investigate this issue by studying a complete information model with strict school priorities and with both sincere students, who always submit their true preference rankings, and sophisticated students, who respond strategically. They find that the Nash equilibrium outcomes of the Boston mechanism is equivalent to the set of stable matchings of a modified economy where sincere students lose their priorities to sophisticated students at all but their first choice schools; furthermore, every sophisticated student weakly prefers her assignment under the Pareto-dominant Nash equilibrium outcome of the Boston mechanism to the dominant-strategy outcome of SOSM.

A second issue raised by Abdulkadiroğlu, Che and Yasuda (2010) is related to neighborhood priorities, a common feature of many school choice programs. For instance, BPS gives priority to students who live within 1 mile from an elementary school, within 1.5 miles from a middle school, and within 2 miles from a high school in attending those schools. At the same time, one of the major goals of public school choice is to provide equal access to good schools for every student, especially for those in poor neighborhoods with failing schools. This goal is compromised by neighborhood priority. The extent to which the neighborhood priority inhibits the access to good schools by students in failing schools districts differs across mechanisms. Under the SOSM, one does not need to give up her neighborhood priority when applying for other (better) schools. This is in sharp contrast to what happens under the Boston mechanism. When a student does not rank her neighborhood school as first choice under the Boston mechanism, she loses her neighborhood priority at that school to those who rank it higher in their choice list. Similarly, if she ranks her neighborhood school as first choice, then she gives up priority at the other schools. In either case, another student would be able to improve her odds at that school or some other school. Abdulkadiroğlu, Che and Yasuda (2011) provide examples in which this feature of the Boston mechanism provides greater access to good schools for students without neighborhood priority at those schools.

4.4 Controlled Choice

Controlled school choice in the United States attempts to provide parental choice over public schools while maintaining racial, ethnic and socioeconomic balance at schools. Boston's renowned controlled choice program emerged out of concerns for economically and racially segregated neighborhoods that were a consequence of traditional neighborhood-based assignment to public schools. Today many school districts adopt desegregation guidelines either voluntarily or because of a court order. Other forms of control exist in choice programs in the US. Miami-Dade County Public Schools control for the socioeconomic status of students in order to diminish concentrations of low-income students at schools. In New York City, Educational Option (EdOpt) schools have to accept students of wide-ranging abilities. In particular, 16 percent of students that attend an EdOpt school must score above grade level on the standardized English Language Arts test, 68 percent must score at grade level, and the remaining 16 percent must score below grade level (Abdulkadiroglu, Pathak, and Roth, 2005).

It is easy to modify the mechanisms when each student can be of one type from a finite set, such as {Asian, Black, Hispanic, White, Other} and the number of students of a type matched with a school cannot exceed a type specific quota at that school. In ETM, a student points to her most preferred school among all schools at which there is an available seat and the quota for her type is not met yet. ETM with quotas is Pareto efficient and strategy-proof (Abdulkadiroğlu and Sönmez 2003).

In SOSM, a school tentatively admits students in the order of priority up to its capacity among those students for whom the type specific capacity has not met yet. Given strict school priorities and quotas, SOSM with quotas produces a stable matching that respects quotas and is weakly preferred by every student to any other stable matching that respects quotas (Roth 1984). Under the same assumptions, it is also strategy-proof (Abdulkadiroğlu 2005). These properties extend to a more general setting with substitutable preferences (Hatfield and Milgrom 2005).[18]

Ehlers (2009) introduces quotas for the *minimum* number of students

[18]Let the choice of school s from a set of students X is defined as $Ch(X;\succ_s) \subset X$ such that $Ch(X;\succ_s) \succ_s Z$ for all $Z \subset X$, $Z \neq Ch(X;\succ_s)$. Then a preference relation $\succ_s$ has the property of substitutability if $i \in Ch(X\backslash\{j\};\succ_s)$ for every $X \subset I$, $i \in Ch(X;\succsim_s)$, $j \in Ch(X;\succsim_s)\backslash\{i\}$ (Kelso and Crawford 1982, Roth 1984). That is, whenever i is chosen from a set, i will be chosen even if some other student is removed from the set.

of each type that have to be assigned to schools. He shows that minimum quotas are incompatible with stability, relaxes the stability requirement and studies student optimal stable matchings.

Kojima (2010) shows that affirmative action quotas can make majority students as well as every minority student worse-off under both SOSM and ETM. Hafalir, Yenmez and Yildirim (2011) offer an alternative policy that gives preferential treatment to minorities for a number of reserved seats at each school. They also provide a group strategy-proof mechanism, which gives priority to minority students for reserved seats at schools. Their mechanism also Pareto dominates SOSM with quotas. Westcamp (2011) offers a strategy-proof SOSM for the allocation of German public universities for medicine and related fields, in which floating quotas are employed to prioritize students according to their grades or wait-time.

The generalized theory of randomized assignment with minimum as well as maximum type-specific quotas by Budish, Che, Kojima and Milgrom (forthcoming) applies to the controlled school choice problem when student assignment involves randomization.

4.5 Short preference lists

Some school districts impose a limit on the number of schools that can be listed in an application. For instance students could list at most five schools in Boston before 2005; and NYC High School Admissions process allows students to rank at most twelve schools in their applications.

Haeringer and Klijn (2009) study the preference revelation game induced by different mechanisms when students can only list up to a fixed number of schools. They focus on the stability and efficiency of the Nash equilibrium outcomes in a model with strict school priorities. They find that, when students can list a limited number of schools, (i) SOSM may have a Nash equilibrium in undominated strategies that produce a matching that is not stable under true preferences; (ii) ETM may have a Nash equilibrium in undominated strategies that produce a matching that is not Pareto efficient under true preferences.

Pathak and Sönmez (2011) show that an SOSM with a cap of maximum k choices in more manipulable than an SOSM with a cap of maximum $l > k$ choices in the sense that the former mechanism can be manipulated at a larger set of preference profiles.

4.6 Large Markets

Size matters. Some of the trade-offs vanish as the number of participants increases. Whereas the number of stable matchings can be arbitrarily large in finite economies, Roth and Peranson (1999) observe that the set of stable matchings has been small in the NRMP, which they explain via simulations by the short preference lists submitted by the applicants in relatively large markets.[19] In contrast, Azevedo and Leshno (2012) give general conditions under which a model with finitely many schools and continuum of students admits a unique stable matching.

There is no stable mechanism that is strategy-proof for students as well as schools (Roth 1982). Also, when schools have more than one seat, there is no stable mechanism that is strategy proof for schools (Roth 1985). These results can be proved via examples with a few students and schools. However, in a model with one seat at every school, Immorlica and Mahdian (2005) show that as the size of the market becomes large, the set of stable matchings shrinks. Kojima and Pathak (2009) generalize this finding to the model with multiple seats at schools and strict school priorities which reflect school

[19]We discuss the large market findings withing the context of school choice, although some of them have been formulated outside the school choice context.

preferences. They show that when schools are also strategic, reporting true preferences becomes an approximate Bayesian equilibrium for schools as the market power of schools vanishes in large markets.

Several applications, including the school choice programs in Korea and the second round of the NYC high school match, involve no priorities on the school side. In that a case, the random priority mechanism (RP)[20], which assigns every student her most preferred school among the remaining schools one at a time in the order of a randomly drawn order of students, is strategy proof and ex post Pareto efficient. Bogomolnaia and Moulin (2001) observe that RP allocation can be improved for some students in the sense of first order stochastic dominance without harming other students' allocations. An allocation which cannot be improved that way is ordinally efficient. Bogomolnaia and Moulin (2001) provide an ordinally efficient probabilistic serial mechanism (PS). However they also show that no ordinally efficient mechanism is strategy-proof for students. Che and Kojima (2010) show that, as the number of students and school capacities grow, the RP becomes equivalent to the PS mechanism, of which the former is strategy-proof and the latter is ordinally efficient. Therefore, the trade-off between strategy-proofness and ordinal efficiency vanishes in such large markets.

Azevedo and Budish (2012) introduce a new notion to study incentives in large markets. Accordingly, a mechanism is strategy-proof in the large if all of its profitable manipulations vanish with market size. They show that the outcomes of a large class of mechanisms can be implemented approximately by mechanisms that are strategy-proof in the large. Budish (2011) studies an assignment problem in which all the known mechanisms are either unfair ex post or manipulable even in large markets. He introduces a slightly different strategy-proofness in the large notion and proposes a combinatorial assign-

[20]This mechanism is also known as random serial dictatorship and can be implemented as SOSM with a uniformly randomly drawn single tie breaker.

ment mechanism that is strategy-proof in the large, approximately efficient and fair.

4.7 Hybrid Matching Problems

A close look at the real-life cases reveals that the school choice problem exhibits features of one-sided matching and two-sided matching simultaneously. For instance, many school districts offer admissions to some selective *exam* schools via an entrance exam. Violating priorities induced by an entrance exam proves to be a political and legal challenge even when such violations are justified by court ordered desegregation guidelines (Abdulkadiroğlu 2011). On the other hand, as in the case of Boston, violating priorities at regular schools may be considered in order to promote student welfare (Abdulkadiroğlu, Pathak and Roth 2005). A similar issue may arise when some schools are strategic and have preferences over students while others are not, as in the case of NYC High School Match (Abdulkadiroğlu, Pathak, Roth, 2005, 2010). In that case, violating the preferences of a strategic school would create an instance at which the school would prefer to circumvent the assignment to match with a more preferred student who also prefers the school to her match.

Ehlers and Westcamp (2010) study a school choice problem with exam schools and regular schools. They assume that exam schools rank students in strict priority order and regular schools are indifferent among all students. Their model is a special case of Erdil and Ergin (2008) and Abdulkadiroğlu, Pathak, Roth (2010), however their scope is quite different. In particular, they identify conditions on priorities of exam schools under which strategy-proofness is preserved.

Abdulkadiroğlu (2011) studies a generalized matching model that encompasses one-sided and two-sided matching as well as their hybrid. In his model, every school is endowed with a priority list that may involve ties. However, a school may have a strict or flexible priority policy, and a stable matching may violate priorities at schools with a flexible priority policy. He characterizes student optimal stable matchings via stable transfer cycles. A stable transfer cycle is an application of SIC (Erdil and Ergin 2008). It operates like ETM but puts restrictions on schools with strict priority policies as in SIC. In particular, in a stable transfer cycle, a student can point to any school that she prefers to her current match as long as the school has a flexible priority policy. Otherwise, in order to be able to point to it, she has to be ranked highest among all students who prefer that school to their current match. Schools, on the other hand, point to the highest priority students among remaining.

4.8 Experiments

Starting with Chen and Sönmez (2006), there is a growing experimental literature with a focus on school choice. Consistent with theory, Chen and Sönmez (2006) observe a high preference manipulation rate under the Boston mechanism. They also find that efficiency under Boston is significantly lower than that of ETM and SOSM. However, contrary to theory, they find that SOSM outperforms ETM in terms of efficiency in their experimental environment.

Pais and Pinter (2007), on the other hand, show that, when the experiment is conducted in an incomplete information setup, ETM outperforms both SOSM and Boston in terms of efficiency. Moreover it is slightly more successful than SOSM regarding the proportion of truthful preference revelation and manipulation is stronger under the Boston mechanism; even though agents are much more likely to revert to truth-telling in lack of information about the others' payoffs, ETM results are less sensitive to the amount of information that participants hold.

Calsamiglia, Haeringer, and Klijn (2009) analyze the impact of imposing limit on the number of schools in choice lists. They show that manipulation is drastically increased, which is consistent with Pathak and Sönmez's (2011) theoretical argument; including a safety school in the constrained list explains most manipulations; both efficiency and stability of the final allocations are also negatively affected.

Featherstone and Niederle (2008) observe that, when school priorities involve ties and are broken randomly, and preferences are private information, the Boston mechanism obtains better efficiency than SOSM.

Klijn, Pais and Vorsatz (2010) study how individual behavior is influenced by risk aversion and preference intensities. They find that SOSM is more robust to changes in cardinal preferences than the Boston mechanism independently of whether individuals are allowed to submit a complete or a restricted ranking over the set of schools and subjects with a higher degree of risk aversion are more likely to play "safer" strategies under the SOSM but not under the Boston mechanism.

5 Conclusion

School choice has provided economists with new opportunities to study and design student assignment systems, which in turn have helped push forward the frontiers of mechanism design theory. This survey aims at demonstrating this point. Many interesting questions remain open. To what extent is the stable improvement cycles mechanism manipulable in the field? How restrictive are the minimum quotas for minorities in controlled school choice programs? To what extent do they preclude stability and foster gaming in the field? Can we design and implement mechanisms with better efficiency properties? Are there simple mechanisms that elicit not only ordinal preferences but also some information on the underlying cardinal preferences? In fact, how do we define a simple mechanism; are they robust (Milgrom 2009)? Theory gives impossibilities for some of these questions, and it is silent on others. Designing better market mechanisms will require not only further new theory, but also new engineering approaches that rely on careful synthesis of the theory, empirical analysis and experiments (Roth 2002).

In addition, in contrast to other market design applications, school choice has a direct public policy appeal. For example, how does information impact choice patterns and academic achievement for disadvantaged students (Hastings and Weinstein 2008); does school choice foster competition among schools, does it help eliminate achievement gap (Hastings, Kane and Staiger 2008)? Second, school choice programs in the US present economists with unprecedented data with randomized assignments. Such data allows researchers to study the impact of different schooling options on student outcomes without suffering from selection bias issues, such as charter schools and their public school alternatives (Abdulkadiroğlu, Angrist, Dynarski, Kane and Pathak 2010; Angrist, Cohodes, Dynarski, Fullerton, Kane, Pathak and Walters 2011; Hoxby and Murarka 2009) and small schools (Bloom, Thompson, Unterman 2010). While designing student assignment systems as market designers, we can also think about and address such broader questions as social scientists. Can we also incorporate sound econometric tools into our designs that would help districts evaluate their schooling alternatives beyond simple descriptive statistics and free of selection bias?

Table 1: Properties of the mechanisms

	Boston	ETM	SOSM
Strategy-proof	No	Yes	Yes
Efficient	No	Yes	No
Stable	No	No	Yes
Student Optimal Stable	No	No	Yes

Table 2: Welfare Consequences of Tie Breaking and Strategy-proofness for Grade 8 Applicants in NYC in 2006-07

Choice	Single tie breakers	Multiple tie breakers	SIC	Efficient
1	29,849.9 (67.7)	32,105.3 (62.2)	32,701.5 (58.4)	34,707.8 (50.5)
2	14,562.3 (59.0)	14,296.0 (53.2)	14,382.6 (50.9)	14,511.4 (51.1)
3	9,859.7 (52.5)	9,279.4 (47.4)	9,208.6 (46.0)	8,894.4 (41.2)
4	6,653.3 (47.5)	6,112.8 (43.5)	5,999.8 (41.4)	5,582.1 (40.3)
5	4,386.8 (39.4)	3,988.2 (34.4)	3,883.4 (33.8)	3,492.7 (31.4)
6	2,910.1 (33.5)	2,628.8 (29.6)	2,519.5 (28.4)	2,222.9 (24.3)
7	1,919.1 (28.0)	1,732.7 (26.0)	1,654.6 (24.1)	1,430.3 (22.4)
8	1,212.2 (26.8)	1,099.1 (23.3)	1,034.8 (22.1)	860.5 (20.0)
9	817.1 (21.7)	761.9 (17.8)	716.7 (17.4)	592.6 (16.0)
10	548.4 (19.4)	526.4 (15.4)	485.6 (15.1)	395.6 (13.7)
11	353.2 (12.8)	348.0 (13.2)	316.3 (12.3)	255.0 (10.8)
12	229.3 (10.5)	236.0 (10.9)	211.2 (10.4)	169.2 (9.3)
Unassigned	5,426.7 (21.4)	5,613.4 (26.5)	5,613.4 (26.5)	5,613.4 (26.5)

[a] Data from the main round of the New York City high school admissions process in 20062007 for students requesting an assignment for grade 9 (high school). The table reports the average choice received distribution of applicants from SOSM with single tie breakers, SOSM with multiple tie breakers, Stable Improvement Cycles algorithm and Efficient matchings which are produced by TTC by using the SIC assignment as endowment. The averages are based on 250 random draws. Simulation standard errors are reported in parentheses. Reproduced from Abdulkadiroğlu, Pathak, Roth (2009).

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