Judd (1992, *JET*) describes a family of methods in widespread use in the sciences, and described in more detail in his textbook.

Basic idea is to approximate the solution function (or policy function) by a member of a class of parameterized functions.

This makes projection methods equivalent to parameterized expectations for the asset pricing model where the integral and the solution function are the same object.

Variants of the method use different criteria for choosing the approximation.

- Galerkin method
- Collocation method
Judd emphasizes the use of Chebyshev polynomials

The Chebyshev polynomials are orthogonal polynomials that correspond to the weighting function \( f(z) = (1 - z^2)^{-1/2} \) and are defined on the interval \([-1, 1]\).

They obey the recursion

\[
\begin{align*}
T_0(z) &= 1 \\
T_1(z) &= z \\
T_n(z) &= 2zT_{n-1}(z) - T_{n-2}(z) \quad \text{for} \quad n \geq 2.
\end{align*}
\]

If you work with a variable \( x \) defined in an interval \([\underline{x}, \overline{x}]\) define a change of variables

\[
z = 2(x - \underline{x}) / (\overline{x} - \underline{x}) - 1.
\]

Presumably select \( \overline{x} = \mu + c\sigma_x \) and \( \underline{x} = \mu - c\sigma_x \) for some \( c > 0 \), where \( \mu = E(x) \) and \( \sigma_x = \text{var}(x) \):

\[
z = (x - \mu) / (c\sigma_x)
\]
Why not use the modified-Hermite polynomials as we did in discussing discrete state space methods?

They obey the recursion

\[
\begin{align*}
\phi_0(z) & = 1 \\
\phi_1(z) & = z \\
\phi_n(z) & = (1/n)^{1/2} z\phi_{n-1}(z) - [(n-1)/n]^{1/2} \phi_{n-2}(z) \text{ for } n \geq 2.
\end{align*}
\]

If you work with an variable \( x \sim N(\mu, \sigma_x^2) \), define a change of variables

\[
z = (x - \mu)/\sigma_x.
\]
The Euler equation for the price-dividend ratio is given by

\[ v(x) = \beta \int \exp(\alpha y) [v(y) + 1] f(y|x) \, dy. \]

Posit an approximate solution \( \tilde{v}(x) \) defined for \( x \in [\underline{x}, \overline{x}] \):

\[ \tilde{v}(x) = \sum_{i=1}^{N} a_i T_{i-1} \left( \frac{(x - \mu)}{(c\sigma_x)} \right), \]
Posit an approximate solution $\tilde{v}(x)$ defined for $x \in (-\infty, +\infty)$:

$$\tilde{v}(x) = \sum_{i=1}^{N} a_i \phi_{i-1} \left( \frac{x - \mu}{\sigma_x} \right),$$

The rest of both solution methods deals with how to pick the $a_i$ coefficients in so that the approximate solution is as accurate as possible.
The Euler Equation Error

- Judd (1992) suggests forming the Euler equation error corresponding to $\tilde{v}$:

$$
\tilde{v}(x) - \beta \int \exp(\alpha y)[\tilde{v}(y) + 1] f(y|x) \, dy.
$$

which can be written out as

$$
\tilde{v}(x) - \beta \int \exp(\alpha y)[\tilde{v}(y) + 1] \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2\sigma^2} [y - \mu(1 - \rho) - \rho x]^2 \right\}
$$

- Since $y = \mu(1 - \rho) + \rho x + \epsilon$, Judd uses the change of variables transformation $z = [y - \mu(1 - \rho) - \rho x] / \sigma$ with $dz = dy / \sigma$.

- Rewrite the Euler equation error as

$$
\tilde{v}(x) - \beta \int \left\{ \exp \{\alpha[\mu(1 - \rho) + \rho x + \sigma z]\} \times \\
\{ \tilde{v}[\mu(1 - \rho) + \rho x + \sigma z] + 1 \}\right\} (2\pi)^{-1/2} \exp(-z^2 / 2) \, dz
$$
Approximate the residual by performing Gaussian quadrature:

\[ R(x) \equiv \tilde{v}(x) - \beta \sum_{j=1}^{M} \left[ \exp \{ \alpha [\mu (1 - \rho) + \rho x + \sigma z_j]\} \times \right. \]

\[ \left. \{ \tilde{v}[\mu (1 - \rho) + \rho x + \sigma z_j] + 1 \} w_j \right]. \]

Chebyshev method:

\[ R(x) \equiv \tilde{v}(x) - \beta \sum_{j=1}^{M} \left[ \exp \{ \alpha [\mu (1 - \rho) + \rho x + \sigma z_j]\} \times \right. \]

\[ \left. \left\{ \sum_{i=1}^{N} a_i T_{i-1} \left[ \frac{\rho (x - \mu) + \sigma z_j}{c \sigma_x} \right] + 1 \right\} w_j \right]. \]
• Approximate the residual by performing Gaussian quadrature:

\[ R(x) \equiv \tilde{v}(x) - \beta \sum_{j=1}^{M} \left[ \exp \left\{ \alpha [\mu (1 - \rho) + \rho x + \sigma z_j] \right\} \times \left\{ \tilde{v}[\mu (1 - \rho) + \rho x + \sigma z_j] + 1 \right\} w_j \right] . \]

• Modified-Hermite method:

\[ R(x) \equiv \tilde{v}(x) - \beta \sum_{j=1}^{M} \left[ \exp \left\{ \alpha [\mu (1 - \rho) + \rho x + \sigma z_j] \right\} \times \left\{ \sum_{i=1}^{N} a_i \phi_{i-1} \left\{ \frac{[\rho (x - \mu) + \sigma z_j]}{\sigma x} \right\} + 1 \right\} w_j \right] . \]
The true function

\[ r(x) = v(x) - \beta \int \exp(\alpha y)[v(y) + 1] f(y|x) dy \]

obviously has the property \( r(x) = 0 \) for all \( x \).

- Collocation picks \( a \) by setting \( R(x) = 0 \) at \( N \) points

- Also \( r(x)f(x) = 0 \) for any other function \( f(x) \), which implies
  \[ \int r(x)f(x) dx = 0 \]

- Galerkin method: choose \( a \) so that \( R \) is orthogonal to \( N \) mutually orthogonal functions.
Judd suggests choosing the vector of coefficients \( a \) so that

\[
\int R(x, a) T_{i-1} \left[ \frac{(x - \mu)}{(c\sigma_x)} \right] dx = 0 \text{ for } i = 1 \text{ to } N.
\]

I have made the dependence of the Euler error on \( a \) explicit.

Since this integral is hard to evaluate explicitly, instead, form

\[
\sum_{\ell=1}^{m} R(x_{\ell}, a) T_{i-1} \left[ \frac{(x_{\ell} - \mu)}{(c\sigma_x)} \right] = 0 \text{ for } i = 1 \text{ to } N
\]

The \( m \) points \( x_{\ell} \) are chosen so that they correspond to the zeros of the \( m \)th ordered-Chebyshev polynomial; i.e.

\[
x_{\ell} = \mu + c\sigma_x z_{\ell}
\]

Note that the weights for doing quadrature with the Chebyshevs are always 1 (check).
Choose the vector of coefficients $a$ so that

$$\int R(x, a) \phi_{i-1} \left[\frac{(x - \mu)}{\sigma_x}\right] dx = 0 \text{ for } i = 1 \text{ to } N.$$  

To approximate this form

$$\sum_{\ell=1}^{m} R(x_{\ell}, a) \phi_{i-1} \left[\frac{(x_{\ell} - \mu)}{\sigma_x}\right] w_i = 0 \text{ for } i = 1 \text{ to } N$$

The $m$ points $x_{\ell}$ are chosen so that they correspond to the zeros of the $m$th ordered-Modified-Hermite polynomial; i.e.

$$x_{\ell} = \mu + \sigma_x z_{\ell}.$$  

So we have

$$\sum_{\ell=1}^{m} R(\mu + \sigma_x z_{\ell}, a) \phi_{i-1}(z_{\ell}) w_i = 0 \text{ for } i = 1 \text{ to } N.$$
Set $R(x)$ to zero at $N$ points, thus determining the vector $a$.

Usually the points chosen would be the $N$ zeroes of the relevant orthogonal polynomials.

Chebyshev: set $R(x) = 0$ at $x_\ell = \mu + c \sigma_x z_\ell$, $\ell = 1, \ldots, N$ where $z_\ell$ represents a root of the $N$th ordered polynomial.

Modified-Hermite: set $R(x) = 0$ at $x_\ell = \mu + \sigma_x z_\ell$, $\ell = 1, \ldots, N$ where $z_\ell$ represents a root of the $N$th ordered polynomial.
At first glance this stuff seems hard, but it’s actually not too hard in the asset pricing case.

Notice that the equations are linear in the $a$s.

In principle, this is just a linear algebra problem.