Omega as a Performance Measure

Hossein Kazemi [a]
Thomas Schneeweis [b]
Raj Gupta [c]

CISDM
www.cisdm.org

University of Massachusetts, Amherst
Isenberg School of Management
Amherst, MA 01003

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a. Corresponding Author. Associate Professor of Finance and Associate Director of Center for International Securities and Derivatives Markets. 413-545-5629 kazemi@som.umass.edu.
b. Professor of Finance and Director of Center for International Securities and Derivatives Markets. 413-253-4601 schneeweis@som.umass.edu.
c. Research Associate, Center for International Securities and Derivatives Markets. 413-253-4601; bhaswar@som.umass.edu.

1. Introduction

In a recent paper, Shadwick and Keating (2002b) present a new measure of performance called Omega. According to the authors, Omega was developed to overcome the inadequacy of many traditional performance measures when applied to investments that do not have normally distributed return distributions. Unlike other measures of performance, Omega was developed with the intention to take the entire return distribution into account. In this paper we show that Omega is essentially the ratio of a call price to a put price. This result has several implications as far as this measure of performance is concerned.

First, it provides a very intuitive explanation for what Omega is and what its limitations are. Second, it communicates to investment professionals that Omega is not a "significantly new" measure of risk since its building blocks (i.e., call and put options) have long been used to measure potential risk and return of investments. In a sense, Omega takes these well known building blocks and arranges them in a manner that provides an intuitive measure of risk/return. Third, one could use the vast body of research in the area of option pricing to obtain accurate estimates of Omega and to gain better understanding of its properties. In particular, since most investments do no satisfy the basic assumptions of Black-Scholes option pricing model,[1] non-parametric as well as approximate option pricing models (e.g., see Jarrow and Rudd (1982), Rubinstein (1998), Stutzer (1996)) can be used to obtain estimates of Omega. Finally, this paper presents a new version of Omega called Sharpe-Omega. This measure provides *exactly* the same information that Omega provides, but the formulation is very similar to the well known Sharpe ratio.[2] In particular, Sharpe-Omega of an investment is given by:

$$\text{Sharpe-Omega} = \frac{\text{Expected Return} - \text{Threshold}}{\text{Put Option Price}}$$

This measure will always give the same ranking as Omega.

The plan of the paper is as follows. The next section discusses the results in greater detail and Section 3 provides some numerical examples. All mathematical proofs appear in the Appendix.

2. Results

Shadwick and Keating (2002b) define Omega as:

$$\Omega(L) = \frac{\int_{L}^{b}[1-F(x)]dx}{\int_{a}^{L}F(x)dx}, \tag{1}$$

where x is the random one-period rate of return on an investment, $F(y) = \Pr\{x \le y\}$ is cumulative distribution of the one-period return, L is a threshold selected by the investor, and (a,b) represent the upper and lower bounds of the return distribution respectively. We show that Omega can be written as:

$$\Omega(L) = \frac{C(L)}{P(L)}, \tag{2}$$

where $C(L)$ is essentially the price of a European call option written on the investment and

$P(L)$ is essentially the price of a European put option written on the investment (See the Appendix for the proof and a discussion of the difference between $C(L)$ and $P(L)$ and traditional option prices). The maturity of both options is one period (e.g., 1 month) and strike price of both options is basically L. In particular, we show that the numerator and the denominator of Omega can be expressed as (see Appendix)

$$\int_{L}^{b}\left[1 - F(x)\right]dx = \int_{L}^{b}(x - L)f(x)dx = E\left[\max(x - L, 0)\right], \tag{3}$$

$$\int_{a}^{L} F(x)dx = \int_{a}^{L}(L - x)f(x)dx = E\left[\max(L - x, 0)\right], \tag{4}$$

where $f(x)$ is the density function of the one-period rate of return on the investment.

Equations (3) and (4) represent essentially undiscounted call and put prices. To calculate the present values of these two values, we multiply both sides by $\exp(-r_f)$, where r_f is the per-period riskless rate. The results will be $C(L)$ and $P(L)$. In other words,

$$C(L) = e^{-r_f} E\left[\max(x - L, 0)\right],$$
$$P(L) = e^{-r_f} E\left[\max(L - x, 0)\right].$$

Sharpe-Omega: A Better Omega

In this section first we present a variation in Omega that preserves all of its features and at the same time provides a measure of risk that is similar to the Sharpe ratio and thus more intuitive. We show in the Appendix that $(\Omega - 1)$ is proportional to the expected excess return on the investment divided by the price of a put option written on the asset. We call this Sharpe-Omega ratio and it is formally defined as

$$\text{Sharpe-Omega} = \frac{\bar{x} - L}{P(L)}, \tag{5}$$

where $\bar{x}$ the expected rate of return on the investment. Sharpe-Omega is proportional to $\Omega - 1$ and thus provides the same information as Omega measure and it always gives the same ranking of investments that Omega offers. On the other hand, we can see that Sharpe-Omega represents a measure of return/risk that is more intuitive than Omega. Since the price of the put option is the cost of protecting an investment's return below the target ratio, it is a reasonable measure of the investment's riskiness, and similar to Omega, this measure takes the shape of the distribution below the threshold into account. In the next section of the paper we present numerical results concerning the behavior of Omega and Sharpe-Omega.

We can see from equation (5) that when the threshold is set equal to the expected return on investment, the Sharpe-Omega is zero. We now examine two different cases: (1) The expected return is less than the threshold and (2) the most likely case that the expected return is greater than threshold.

1. When $\bar{x} < L$, Sharpe-Omega measure will be negative. In this case the higher the put price the better the investment. For instance, higher volatility will increase the put price and thus increasing the value of Sharpe-Omega (note that its absolute value

declines). This is similar to the case of Omega being less than one. As shown in Shadwick and Keating (2002a), when Omega is less than one, higher volatility increases the value of Omega measure because it increases the likelihood of earning a return above the mean. In practice this is not likely to be a scenario of interest because the threshold is usually set below the mean.

2. When $\bar{x} > L$, Sharpe-Omega ratio will be positive. Unlike the previous case, higher volatility will increase the put price and thus reducing the Sharpe-Omega ratio. This is similar to the case of Omega being greater than 1.

Of course, the prime objective of Shadwick and Keating in the development of Omega was to provide a "universal performance measure." Given the above results, the crucial question is how universal is this measure and what are its properties? First, as a measure of risk or performance, Omega and Sharpe-Omega are not that "new" in the sense that investors have for years worked with call and put options as instruments for capturing return and reducing risk. In this framework, the numerator of the Omega (see equation (2)) represents the "cost" of acquiring the return above a threshold and denominator represents the "cost" of protecting the return below the threshold. Therefore, is seems quite sensible to use the ratio as a measure of return/risk of an investment (a similar argument applies to Sharpe-Omega). Second, given what appears in equation (5), we can see that investments with the same expected excess return will have different Omegas (or Sharpe-Omegas) to the degree that prices of puts written on these investments are different. Even though this definition of risk (i.e., price of a put) is far more general than standard deviation, some investors may still consider it to be a rather narrow definition of risk. Third, an attractive feature of Omega and Sharp-Omega is that under some circumstances the denominator of equation (5) can be approximated by the market price of a put option written on the investment (This is shown through numerical examples in the next section). Fourth, in the original presentation of Omega it was unclear how this measure is affected by changes in higher moments unless the distribution function was parametrically specified. The results presented here can be used to provide approximate answers to such questions. In particular, there is a vast literature on approximate option pricing for arbitrary distributions. For instance Corrado and Su (1996 and 1997) Posner and Milevsky (1998) have provided analytical solutions for option prices in terms of the first 5 and 4 moments of the distribution of an investment's return. These solutions can be used to obtain accurate estimates of Omega or Sharpe-Omega for arbitrary distribution functions.

3. Numerical Examples

In this section we provide some numerical examples. If an investment's return is lognormally distributed, then one could use the Black-Scholes option pricing formula to calculate Omega quickly. The only adjustment that needs to be done is to use the mean return on the investment as the riskless rate. Of course, most investments, and in particular, alternative investments, do not have lognormally distributed returns. For this reason we use a generalization of Black-Scholes formula to obtain numerical values for Omega and Sharpe-Omega (see Jurczenko et. al. (2002)). Also, we use the Black-Scholes formula to examine the differences between market prices of puts and put prices that appear in Omega and Sharp-Omega measures.

In Black-Scholes formula option prices are effectively determined in a risk neutral world. Thus, the expected rate of return on the asset is set equal to the riskless rate. The put price that appears in Omega and Sharpe-Omega has to be calculated in the risk-averse world and thus the true mean return of the investment must be used. This will create a difference between the market price of the put and the price that should appear in performance measures. Of course, given enough information, the appropriate put price that appears in performance measures can be calculated, but it will be beneficial to study the market prices of these put

prices as well.

Exhibit 1 displays the ratio of put prices that appear in the performance measures to the market prices of puts under various conditions. We can see that in general the put price that appears in Omega and Sharpe-Omega will be less than the market price of the put. The difference is generally lower when volatility and/or threshold are high.

Exhibit 1

As shown in the Appendix, there exists a family of distribution functions known as Johnson family (see Johnson (1949)). This family of distributions can match the first 4 moments (i.e., mean, variance, skewness, and kurtosis) of any well defined probability distribution (see Exhibit 2). We use this family of distributions to calculate Omega and then examine its sensitivity to changes in these moments. Further, for illustrative purposes we use this family of distributions to display estimates of Sharpe-Omega for S&P500 index as well as CSFB convertible arbitrage and CSFB equity market neutral index. We use monthly data covering January 1994-May 2003.

Exhibits 2 and 3

We can see that the CSFB-Tremont equity market neutral and the convertible arbitrage hedge fund indices are equally good low levels of the thresholds while the S&P500 index has a higher Sharpe-Omega measure for high values of the threshold. Interestingly, the equity market neutral index is better than the convertible arbitrage index when the threshold is high though they are equally good for low levels of thresholds. For illustrative purposes we present our estimates of Omega for these investments in Exhibit 4. As expected, Omega gives the same ranking as Sharpe-Omega.

Exhibit 4

Finally, in exhibits 5-8 we present results regarding the sensitivity of the Sharpe-Omega measure to changes in the distribution of an investment's return. We use the same Johnson family of distributions to present estimates Sharpe-Omega under various scenarios.

Exhibits 5-8

Similar to results reported in Shadwick and Keating (2002a) for Omega measure, these graphs show that the Sharpe-Omega is most sensitive to changes in mean and variance. Higher skewness and kurtosis have relatively small effect on Omega and Sharpe-Omega unless the threshold is substantially less than the mean return.

4. References

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5. Appendix

Let $f(x)$ and $F(y) = \int_a^y f(x)dx = \Pr\{x \le y\}$ denote the density function and the cumulative distribution function of x, the one-period rate of return on an asset. We now show that Omega, defined as (see Keating and Shadwick (2002b))

$$\Omega = \frac{\int_L^a \left(1 - F(x)\right)dx}{\int_b^L F(x)dx}, \tag{6}$$

where L is the threshold, is essentially equal to ratio of a call price to a put price. That is,

$$\Omega = \frac{C(L)}{P(L)},$$

where $C(L)$ and $P(L)$ are prices of one period call and put prices with the mean return of the underlying asset substituted for the riskless rate in the option pricing formula (i.e., in pricing the options the expectation is taken under true measure rather than the risk neutral measure).

Let (a,b) denote the domain of x, which means that

$$\lim_{x \to a} F(x)x = 0,$$
$$\lim_{x \to b} \left(1 - F(x)\right)x = 0.$$

First, we look at the numerator of Omega. To understand this term, we expand the term $d\left[x\left(1 - F(x)\right)\right]$ and then integrate it over (L,b).

$$\int_L^b d\left[x\left(1 - F(x)\right)\right] = \int_L^b \left(1 - F(x)\right)dx + \int_L^b xd\left(1 - F(x)\right),$$
$$= \int_L^b \left(1 - F(x)\right)dx - \int_L^b xdF(x). \tag{7}$$

Note that $\int_L^b d\left[x\left(1 - F(x)\right)\right]$ can also be written as

$$x\left(1 - F(x)\right)\Big|_L^b = \lim_{x \to b} x\left(1 - F(x)\right) - \lim_{x \to L} x\left(1 - F(x)\right),$$
$$= 0 - L \times \left(1 - F(L)\right), \tag{8}$$
$$= -L \times \Pr\{x \ge L\}$$

Thus, combining equations (7) and (8), we have

$$-L\left(1-F\left(L\right)\right)=\int_{L}^{b}\left(1-F\left(x\right)\right)dx-\int_{L}^{b}xdF\left(x\right)$$

$$\int_{L}^{b}\left(1-F\left(x\right)\right)dx=-L\left(1-F\left(L\right)\right)+\int_{L}^{b}xdF\left(x\right)$$

$$=\int_{L}^{b}\left(x-L\right)f(x)dx$$

$$=\int_{a}^{b}\max\left(x-L,0\right)f(x)dx$$

The last line is essentially the undiscounted price of a call option.

To understand the denominator, we expand the term $d\left[xF(x)\right]$ and then integrate it over (a,L).

$$\int_{a}^{L}d\left[xF\left(x\right)\right]=\int_{a}^{L}F\left(x\right)dx+\int_{a}^{L}xdF\left(x\right) \tag{9}$$

Note that $\int_{a}^{L}d\left[xF\left(x\right)\right]$ can also be written as

$$xF(x)\Big|_{a}^{L}=\lim_{x\to L}xF(x)-\lim_{x\to a}xF(x),$$

$$=L\times F\left(L\right)-0, \tag{10}$$

$$=L\times\Pr\left\{x\le L\right\}.$$

Thus, combining equation (9) and (10), we have

$$L\times F(L)=\int_{a}^{L}F\left(x\right)dx+\int_{a}^{L}xdF\left(x\right),$$

$$\int_{a}^{L}F\left(x\right)dx=L\times F\left(L\right)-\int_{a}^{L}xdF\left(x\right),$$

$$=\int_{a}^{L}\left(L-x\right)f(x)dx, \tag{11}$$

$$=\int_{a}^{b}\max\left(L-x,0\right)f(x)dx.$$

The last line is essentially the undiscounted price of a put option. Given these results, we have

$$\Omega=\frac{\int_{L}^{b}\left(1-F\left(x\right)\right)dx}{\int_{a}^{L}F\left(x\right)dx}$$

$$=\frac{E\left[\max\left(x-L,0\right)\right]}{E\left[\max\left(L-x,0\right)\right]} \tag{12}$$

$$=\frac{e^{-r_{f}}E\left[\max\left(x-L,0\right)\right]}{e^{-r_{f}}E\left[\max\left(L-x,0\right)\right]}=\frac{C(L)}{P(L)},$$

where $C(L)$ and $P(L)$ denote call and put prices where the expectation is taken under the true measure rather than the risk neutral measure.

Since x is the rate of return on the underlying asset, we can think of $C(L)$ and $P(L)$ as call and put prices where the initial price of the asset is 1 and the exercise price is given by $\exp(L)$ (please note that we are assuming continuous compounding). For example, when the value of the underlying asset is lognormally distributed, $P(L)$ will result in the Black and Scholes option pricing formula with mean return on the investment substituted for the riskless rate in the formula. That is,

$$P(L) = \exp(L - r_f) N(-d_2) - \exp(\bar{x} - r_f) N(-d_1),$$
$$d_1 = \frac{\bar{x} - L + 0.5\sigma^2}{\sigma},$$
$$d_2 = d_1 - \sigma,$$
$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}y^2} dy,$$

where $\bar{x}$, is the expected continuously compounded per period rate of return on the investment.

We now discuss Sharpe-Omega ratio and its relationship to Omega. As shown in equation (12), Omega is given by

$$\Omega(L) = \frac{e^{-r_f} E\left[\max(x - L, 0)\right]}{e^{-r_f} E\left[\max(L - x, 0)\right]}. \tag{13}$$

If we subtract 1 from both sides and then rearrange the terms, we will have the Sharpe-Omega ratio.

$$\Omega(L) - 1 = \frac{e^{-r_f} E\left[\max(x - L, 0)\right]}{e^{-r_f} E\left[\max(L - x, 0)\right]} - 1$$
$$e^{r_f}\left(\Omega(L) - 1\right) = \frac{E\left[\max(x - L, 0)\right] - E\left[\max(L - x, 0)\right]}{e^{-r_f} E\left[\max(L - x, 0)\right]}$$
$$e^{r_f}\left(\Omega(L) - 1\right) = \frac{\bar{x} - L}{e^{-r_f} E\left[\max(L - x, 0)\right]}$$

We can see that Sharpe-Omega is given by $e^{r_f}\left(\Omega(L) - 1\right)$ and therefore it always gives the same ranking as Omega does.

The Johnson family of distributions is given by

$$q(z) = \frac{\exp\left(-\dfrac{z^2}{2}\right)}{\sqrt{2\pi}}\left(1 + \frac{s}{6}\left(z^3 - 3z\right) + \frac{k-3}{24}\left(z^4 - 6z + 3\right)\right)$$

This distribution has 0 mean, unit variance, skewness s and kurtosis k. By defining $x = \sigma \times z + \mu$, we can change the mean and the standard deviation as well.

8

6. Exhibits

Exhibit 1
Ratio of $P(L)$ to Market Put Prices for Various L and σ
Monthly Figures
Mean = 1%, Riskless Rate = 0.4%
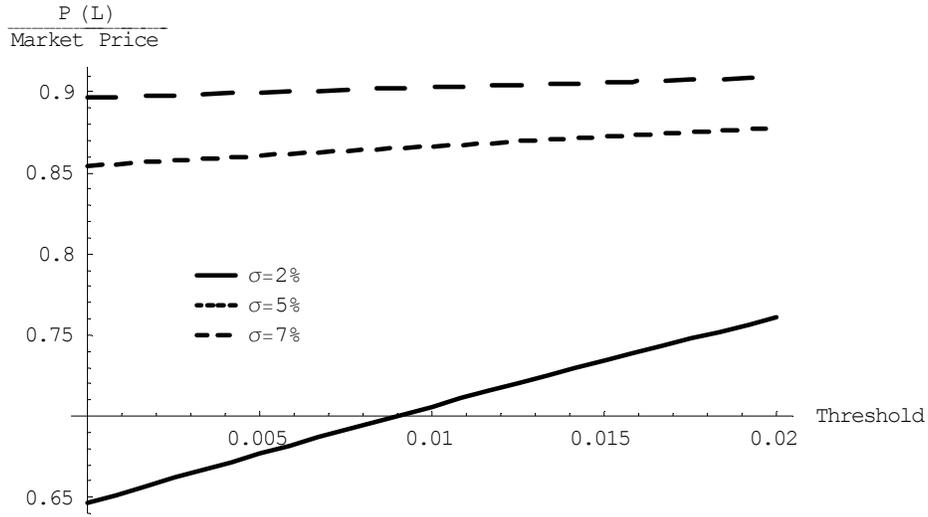
Exhibit 2
Density Function of the Four Parameter Johnson Distribution for
Various Levels of Skewness and Kurtosis
Mean = 0, Variance = 1
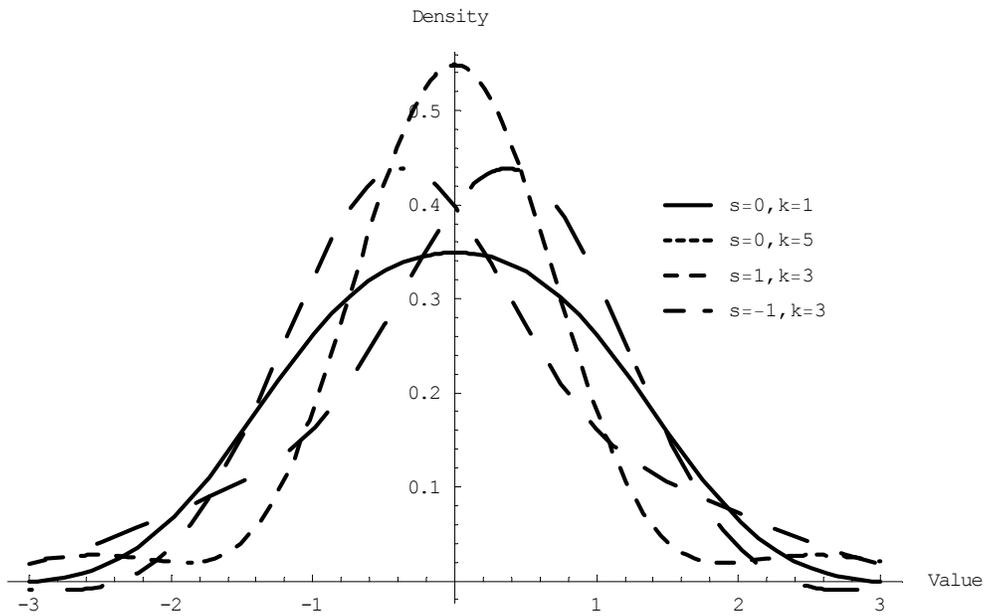
Exhibit 3
Sharpe-Omega Measure for S&P500, CSFB-Tremont Convertible Arbitrage and Equity
Market Neutral Indices
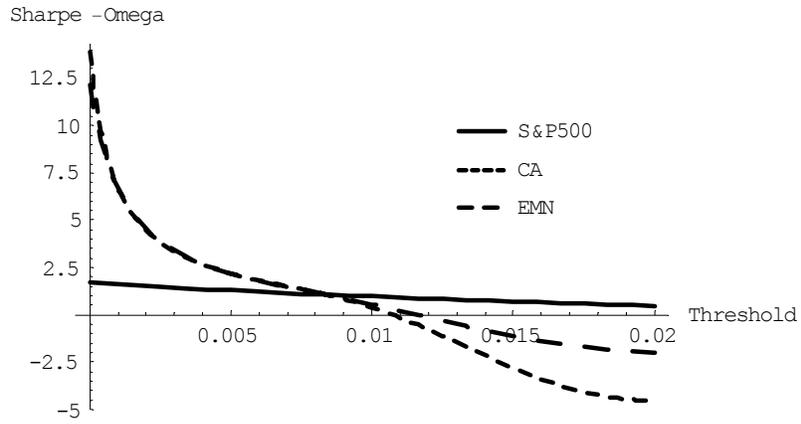
Exhibit 4
Omega Measure for S&P500, CSFB-Tremont Convertible Arbitrage and Equity Market
Neutral Indices
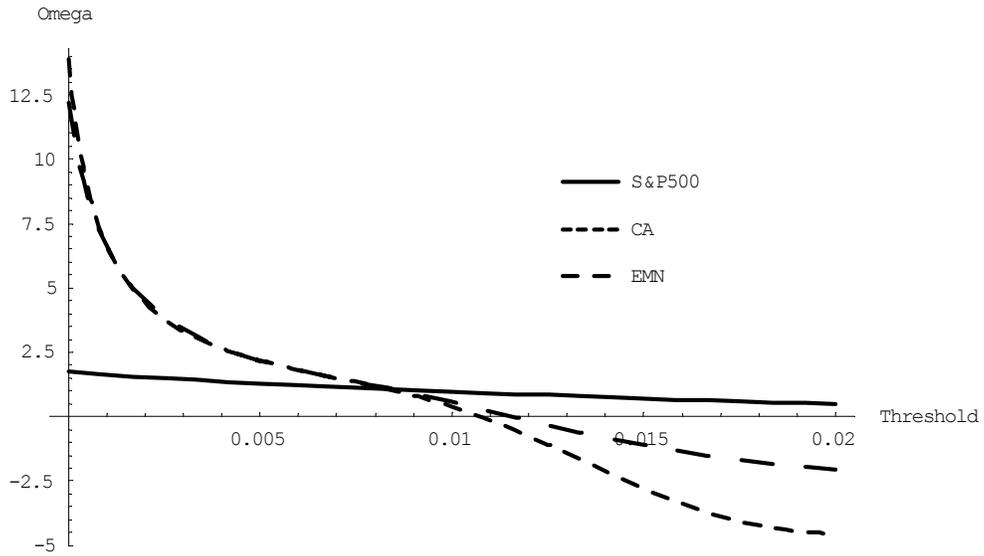
Exhibit 5
Sharpe-Omega Measure for Various Levels of Monthly Mean
(Standard Deviation = 5%; Skewness = 0; Kurtosis = 3)
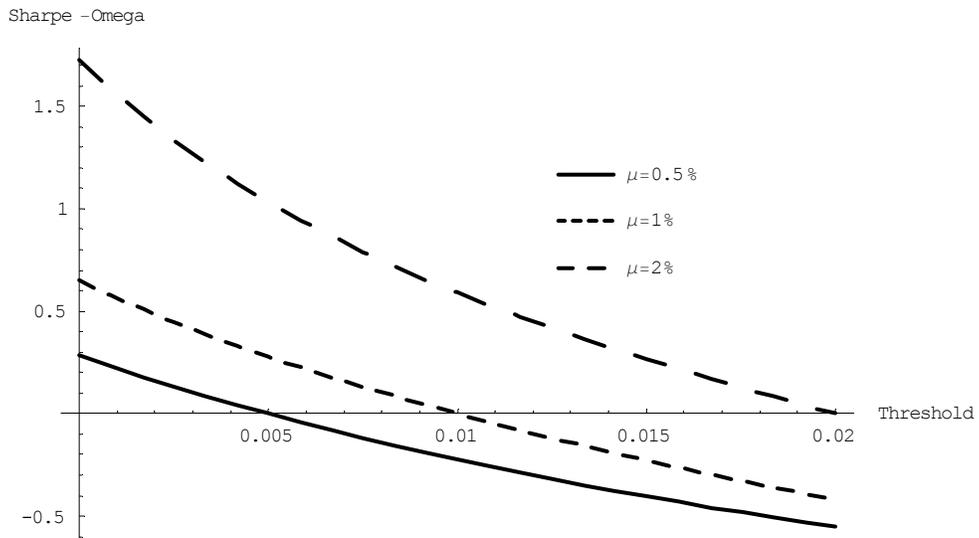
Exhibit 6
Sharpe-Omega Measure Various Levels of Monthly Standard Deviation
(Mean = 1%; Skewness = 0; Kurtosis = 3)
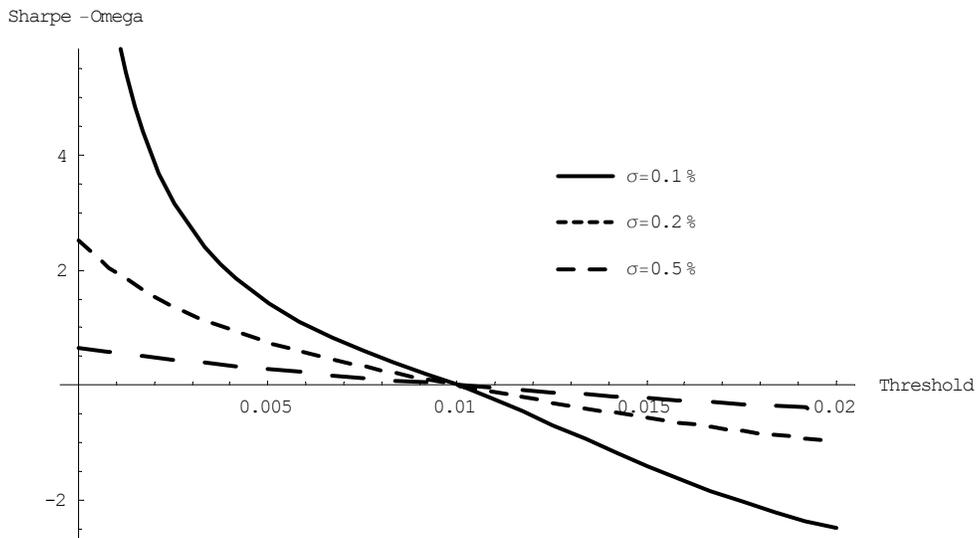
Exhibit 7
Sharpe-Omega Measure Various Levels of Monthly Skewness
(Mean = 1%; Standard Deviation = 5%; Kurtosis = 3)
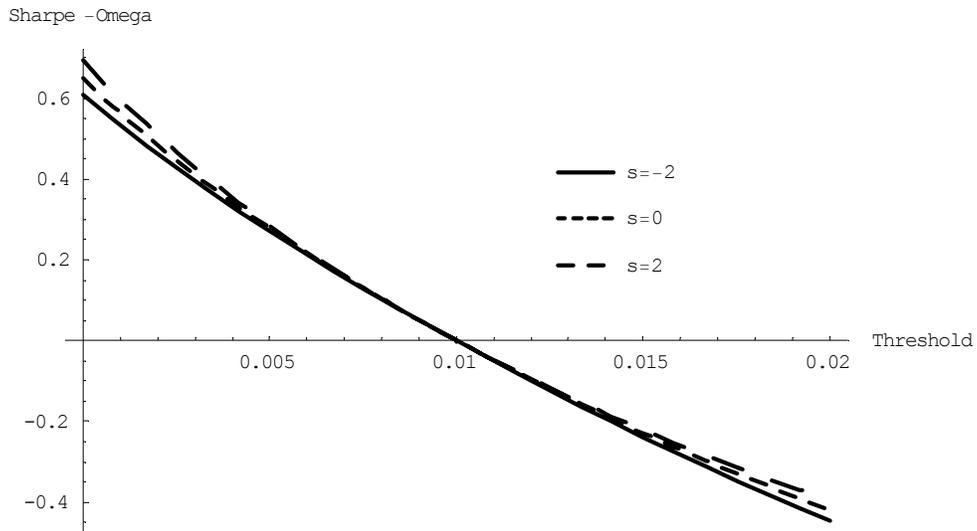
Exhibit 8
Sharpe-Omega Measure Various Levels of Monthly Kurtosis
(Mean = 1%; Standard Deviation = 5%; Skewness = 0)
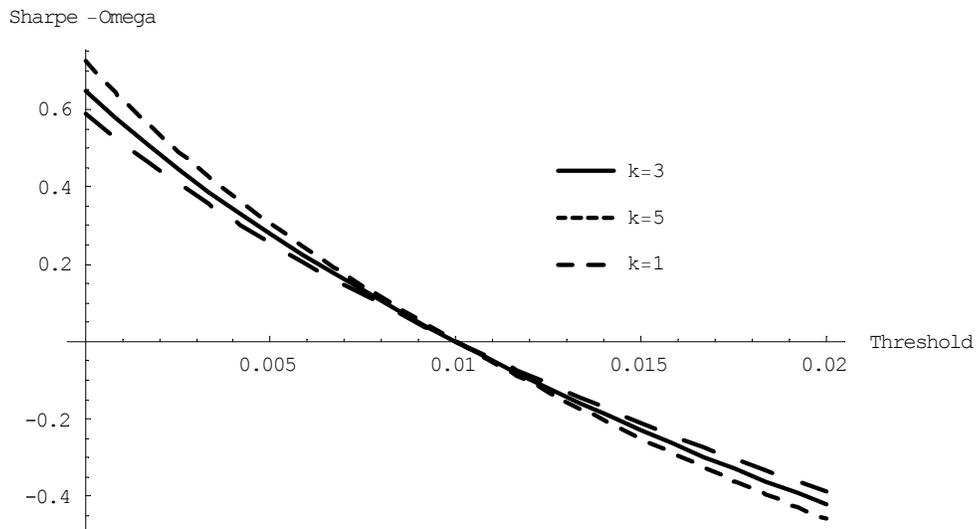
7. Notes

[1] Among others, the Black-Scholes option pricing model assumes that asset values are lognormally distributed.

[2] The Sharpe ratio is defined as:

$$\frac{\text{Expected Return} - \text{Riskless Rate}}{\text{Standard Deviation of Return}}$$