

A Nonlinear Stochastic Rational Expectations Model of Exchange Rates*

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Abstract

This paper constructs an example of a nonlinear stochastic rational expectations exchange rate with an explicit solution, which is consistent with nonlinearities in short term movements in exchange rates. The model consists of risk neutral agents, who know the intervention rule of the central bank. The resulting exchange rate switches between two linear stochastic processes, one when intervention is present, and another when intervention is absent. Nonlinearity enters through the probability of intervention, which is time varying and depends on past outcomes. This model is consistent with the empirical observations that the rate of change of the exchange rate has little autocorrelation, but it exhibits strong nonlinear dependence, and its variance changes over time.

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This paper presents a theoretical example of a nonlinear stochastic rational expectations model in discrete time, which has an explicit solution and is consistent with short term movements in exchange rates. The motivation stems from the inability of linear time series models to reproduce important features of short term (daily and weekly) rates of change of nominal exchange rates. The stylized facts are these. One, exchange rate changes have little autocorrelation or linear dependence.^{1/} Two, there is strong evidence of nonlinear dependence.^{2/} In particular, the absolute values and squares of exchange rate changes are autocorrelated,^{3/} and the variances of exchange rate changes are varying over time in a predictable manner.^{4/} These stylized facts cannot be explained by linear models of exchange rates, which can generate only serial correlation in exchange rate changes, but not serial correlation in their squares. This has led to empirical modeling of exchange rate data using nonlinear time series models, such as Engle's (1982) autoregressive conditional heteroscedasticity (ARCH) and related models.^{5/}

A number of theoretical models have been proposed to explain the nonlinear dynamics in observed data. These can be classified into two groups. In the first group, the economic system is intrinsically linear, and nonlinearity results from the way data are observed. For example, in Stock's (1987) time deformation model, economic variables are linear in economic time. Nonlinearity in observed data is caused by a nonlinear relation between economic time and calendar time. In Hamilton's (1988) model, the mean and variance of an observed economic model depend linearly on the state of the economy, which evolves according to markovian transition probabilities unrelated to economic events. Nonlinearity in the univariate representation of

observed data is caused by the fact that the state variable is not observable. Both models obtain nonlinearity from events exogenous to the economic system.

In the second group, nonlinear dynamics comes from an equilibrium model. In a typical rational expectations model with optimizing behavior, the dynamics of asset prices are given by the solution to nonlinear difference equations, which are the first order conditions of agents' maximization problems. Closed form solutions are rarely found, since there is no general method to solve nonlinear stochastic difference equations analytically. This paper takes a different approach. By specifying a central bank intervention rule, we are able to solve analytically for the nonlinear rational expectations exchange rate dynamics.

This approach is similar to that of the target zone literature (Krugman (1991)), where the central bank enforces a known and credible band within which the exchange rate is allowed to move. Intervention occurs to keep the exchange rate from reaching the edges of the band. This results in rational expectations equilibria in which the exchange rate exhibits nonlinear dynamics. In this paper, the central bank enforces a known and credible monetary policy which is neutral when exchange rate depreciation is small but leans against the wind when the wind is blowing hard. In other words, there is a target zone on the rate of change of the exchange rate. This also delivers nonlinear dynamics in the exchange rate. As in the target zone literature, an explicit solution is achieved only by assuming a specific distribution of the exogenous shocks. There are, however, several aspects of this paper which distinguish it from the target zone literature. First, this paper is done in discrete time, which avoids some of the difficulties of applying continuous time models to discrete time data. Second, this paper is designed to study

countries under managed float, such as Israel (Pessach and Razin (1991)) and the Latin American experience (Harberger(1966)), while the target zone literature is most applicable to the countries in the EMS arrangement.

The paper is organized as follows. Section I describes the basic model, which is a standard monetary model of exchange rate determination in a small country. It consists of a money demand equation, deviations from purchasing power parity, and uncovered interest parity. The rational expectations equilibrium is obtained for two cases, one in which the central bank does not intervene at all, and the other in which the central bank follows a deterministic intervention rule. Under these circumstances, the exchange rate is a linear stochastic process. Section II introduces a stochastic intervention rule, in which the central bank intervenes with a given probability. The rational expectations exchange rate turns out to switch randomly between two linear stochastic processes, one in which the central bank is intervening, and one in which the central bank is not intervening. This does not generate any interesting nonlinear dynamics, because the switching probabilities do not depend on past outcomes. This is remedied in Section III, which discusses a conditional intervention rule, in which the central bank intervenes only if the exchange rate moves too much in one direction. Here, the probability of intervention is endogenously determined, and gives rise to interesting nonlinear dependence.

The idea of using switching model to explain nonlinearity in economic data goes back as far as Goldfeld and Quandt (1973). The switching model in Section II does not give rise to interesting nonlinear dynamics, while the one in Section III does. Hence switching models do not always deliver interesting nonlinear structure.

This idea of process switching is also not new in the international economics literature. Flood and Garber (1983) show that the anticipation of a random future event, such as a once-and-for-all shift from a floating to a fixed rate regime, can influence the current behavior of a floating exchange rate, even though the central bank is currently not intervening in the foreign exchange market.^{6/} In contrast, the present paper describes the behavior of the exchange rate under a "managed" float, with repeated stochastic intervention by the central bank.

I. The Basic Model

Throughout this paper, I shall use a simple monetary model of exchange rate determination in a small open economy, which consists of the following three equations:

$$(1) \quad h_t - p_t = \alpha - \beta i_t + \epsilon_t$$

$$(2) \quad p_t = p_t^* + s_t - \eta_t$$

$$(3) \quad i_t = i_t^* + s_{t+1|t} - s_t$$

Equation (1) is the money demand function, where h_t is the logarithm of domestic high powered money at time t , p_t the logarithm of the domestic aggregate price level (measured in units of domestic currency), i_t the one period domestic nominal interest rate, ϵ_t an exogenous shock to money demand, and α and β positive constants. Equation (2) is the definition of the real exchange rate (η_t), which is the deviation from purchasing power parity, where p_t^* is the logarithm of the foreign price level (measured in units of foreign currency), and s_t the logarithm of the exchange rate (measured in units of domestic currency per unit of foreign currency). Equation (3) is "uncovered" or "open" interest parity, where i_t^* denotes the one period foreign nominal

interest rate. This holds whenever agents are risk neutral, which is assumed in my model. The notation $x_{t+1|t}$ for any variable x_t is defined as $E_t[x_{t+1}]$, where the expectation is conditional on all information available as of time t . This assumes that agents have rational expectations in the sense of Muth (1961).

These three equations are standard in monetary models of exchange rate determination. They are the small country, discrete time versions of the model in Flood and Garber (1983). Combining the three equations, I obtain the basic equation which is the starting point of the analysis:

$$(4) \quad (1+\beta) s_t - \beta s_{t+1|t} = h_t + u_t,$$

where $u_t = \beta i^*_t - p^*_t - \epsilon_t + \eta_t$. For the remainder of the paper, I shall assume that u_t is independently and identically distributed (iid), with zero mean and finite variance.

Equation (4) can now be used to solve for the time path of the exchange rate. In the basic model, the central bank does not intervene in the foreign exchange market. By this, I mean that the central bank sets a path for money, $\{h_t\}$, which is econometrically exogenous with respect to the exchange rate. In this case, a non-explosive solution for the exchange rate processes is:

$$(5) \quad s_t = \frac{1}{(1+\beta)} \left[u_t + \sum_{k=0}^{\infty} \delta^k h_{t+k|t} \right],$$

$$\text{where } \delta = \frac{\beta}{(1+\beta)}.$$

The method of undetermined coefficients [see McCallum (1983)] can be used to solve for the s_t for a given money supply process. For instance, suppose the central bank fixes the money supply, given by:

$$(6) \quad h_t = h_{t-1} = h_0,$$

then (5) becomes:

$$(7) \quad s_t = s_0 + \frac{1}{(1+\beta)} u_t ,$$

where $s_0 = h_0$. Thus s_t itself is white noise with mean s_0 .

Now, suppose the central bank follows a deterministic intervention rule:

$$(8) \quad h_t = h_{t-1} - \phi (s_t - s_{t-1} - A)$$

where $\phi > 0$ and A is some arbitrary constant. Here, the central bank "leans against the wind" by reducing the money supply when the exchange rate is depreciating beyond A , and by increasing the money supply otherwise. Note that only non-sterilized intervention will have any effect on the exchange rate in this kind of monetary model.

I assume that there is no information lag or information asymmetry in the model. More precisely, agents know the intervention rule in (8) as well as all current variables, such as the money supply h_t and the exchange rate s_t . Therefore, they know when and by how much the central bank is changing the money supply in response to the exchange rate. A non-explosive solution of the model is:

$$(9) \quad s_t = s_{t-1} + \frac{\phi}{1+\phi} A + \frac{1}{(1+\phi+\beta)} (u_t - u_{t-1}) .$$

Note that when $\phi=0$, (9) reduces to (7).

The two money supply rules in (6) and (8) are well known in the literature. In both cases, the rational expectation exchange rates are linear stochastic processes. There are no interesting nonlinear dynamics. I now proceed to modify the central bank's intervention rule to generate nonlinear dynamics.

II. Stochastic Intervention Rule

In this section, the central bank follows a stochastic intervention rule, switching randomly between a neutral and a leaning against the wind money

supply rule. At this stage, I can offer little motivation for this particular intervention rule, except to say that it is an intermediate step to the more interesting conditional intervention rule in Section 4. However, I find it instructive to first solve the rational expectations equilibrium for the stochastic intervention rule, because it is much simpler and it employs the same solution algorithm.

At time t , the probability of intervention is θ , which is given exogenously and fixed over time. If the central bank intervenes, it sets the money supply as:

$$(10) \quad h_t^i = h_{t-1} - \phi (s_t - s_{t-1} - A) ,$$

where A is an arbitrary constant. (For this section, A can be set to zero. But Section 4 will use a non-zero value for A .) If the central bank does not intervene, it keeps the money supply unchanged:

$$(11) \quad h_t^n = h_{t-1} .$$

Thus, the observed money supply h_t can be written as:

$$(12) \quad h_t = h_{t-1} - \phi (s_t - s_{t-1} - A) z_t ,$$

where $z_t = 1$, with probability θ , and $z_t = 0$, with probability $(1-\theta)$.

I shall continue to assume that agents know the intervention rule (12), and the current values of the money supply and exchange rates. Since all current values of all variables are known, it is no surprise to find that the current value of the exchange rate depends only upon the current value of the money supply and the disturbance. This is in fact shown in (42) of Appendix A.

This does not give any information concerning the univariate time series property of s_t . To do so, I must relate s_t to past information, s_{t-1} , h_{t-1} , and an innovation. This is done using the method of undetermined coefficients to solve for the rational expectations exchange rate. I define two

hypothetical exchange rate processes, s_t^i and s_t^n . The former represents the observed exchange rate when intervention takes place (i.e. $z_t=1$), and the latter represents the observed exchange rate when intervention does not occur (i.e. $z_t=0$). In other words, the observed exchange rate s_t is given by:

$$(13) \quad s_t = s_t^i, \quad \text{if } z_t = 1, \\ = s_t^n, \quad \text{if } z_t = 0.$$

Hypothetically, s_t^i and s_t^n can be described by the following equations:

$$(14) \quad s_t^i = a_0 + a_1 h_{t-1} + a_2 s_{t-1} + a_3 u_t, \\ s_t^n = b_0 + b_1 h_{t-1} + b_2 s_{t-1} + b_3 u_t.$$

Hence the expected exchange rate at time $t+1$ given information at time t is:

$$(15) \quad s_{t+1|t} = \theta E_t[s_{t+1}^i | z_{t+1}=1] + (1-\theta) E_t[s_{t+1}^n | z_{t+1}=0].$$

The details of the solution are contained in Appendix A. The solution is:

$$(16) \quad s_t^i = \frac{1}{1+\phi} \left[\frac{1+\phi+\beta\phi}{1+\phi} \phi A + h_{t-1} + \phi s_{t-1} + \frac{1+\phi}{1+\phi+\beta+\beta\phi-\beta\phi\theta} u_t \right], \\ s_t^n = \left[\frac{\beta\theta}{1+\phi} \phi A + h_{t-1} + \frac{1+\phi}{1+\phi+\beta+\beta\phi-\beta\phi\theta} u_t \right].$$

Further calculations (in Appendix A) reduce these expressions to:

$$(17) \quad s_t^i = s_{t-1} + \frac{1}{1+\phi} \left[\phi A + \frac{1+\phi}{1+\phi+\beta+\beta\phi-\beta\phi\theta} (u_t - u_{t-1}) \right], \\ s_t^n = s_{t-1} + \frac{1+\phi}{1+\phi+\beta+\beta\phi-\beta\phi\theta} (u_t - u_{t-1}).$$

Thus, the exchange rate switches between two linear stochastic processes. When the central bank is intervening, the rate of change of the exchange rate is a first order moving average with a non-zero mean. When the central bank is not intervening, the rate of change of the exchange rate is a first order moving average with a zero mean and a higher variance. Note that when the probability of intervention is unity, i.e., $\theta=1$, (17) reduces to the rational expectations solution under the deterministic intervention rule in (9). When the probability of intervention is not unity, however, the exchange rate cannot be represented as in (9).

Equation (17) describes an exchange rate process which meets some but not all of my requirements. The observed rate of change of the exchange rate has little serial correlation [being a moving average of order 1], and its variance changes over time. However, there is no nonlinear dependence in this process, because the probability of switching between the two linear processes is constant and does not depend on past exchange rates. The next section remedies this situation with a model which can produce nonlinear dependencies by a time-varying probability of intervention.

III. Conditional Intervention Rule

This section considers the following conditional intervention rule:

$$(18) \quad \begin{aligned} h_t^i &= h_{t-1} - \phi (s_t - s_{t-1} - A) , & \text{if } [s_t^n - s_{t-1}] > A, \\ h_t^n &= h_{t-1} , & \text{if } [s_t^n - s_{t-1}] \leq A. \end{aligned}$$

This intervention rule is much more complex than the previous one. Here, the central bank intervenes only if the rate of depreciation of the exchange rate would exceed A if no intervention takes place.

Equation (18) describes a one-sided intervention rule, in the sense that the central bank only intervenes if the rate of depreciation is large. This one-sided rule is simpler to work with than the two-sided rule of the typical "managed float," in which intervention occurs whenever the rate of appreciation or depreciation is large. I have solved the equilibrium exchange rate for the two-sided rule. The results are analogous, and available upon request.

I shall continue to assume that agents know this intervention rule, as well as the current values of the money supply and the exchange rate. This means that the current value of the exchange rate depends only on the current

money supply and the disturbance term, as pointed out earlier. (See equation 58 in Appendix B.) However, to obtain the univariate process of the exchange rate, I must again rely on the method of undetermined coefficients.

In general, the rational expectations solution of switching models are difficult to find. However, in the present case, there exists an explicit solution. Furthermore, the probability of intervention depends on the past, which produces interesting nonlinearity. To show this, I shall only consider the "stationary" solution of the model, by which I mean that whenever a given u_t is realized, I observe the same values of $(s_t - s_{t-1})$, $(h_t - h_{t-1})$, and the same probability of intervention at time $t+1$, $\text{Prob}(s_{t+1}^n - s_t^n > A)$, holding fixed the past history of the economy. The solution is obtained by using the method in Aiyagari, Eckstein, and Eichenbaum (1985), by hypothesizing that the stationary solution is indexed by the state variable u_t :

$$(19) \quad \begin{aligned} s_t^i &= a_1(u_t) h_{t-1} + a_2(u_t) s_{t-1} + a_3(u_t) , \\ s_t^n &= b_1(u_t) h_{t-1} + b_2(u_t) s_{t-1} + b_3(u_t) . \end{aligned}$$

Following the methods used in Section 3, I can show that the form of the solution is:

$$(20) \quad \begin{aligned} s_t^i &= \frac{1}{1+\phi} [h_{t-1} + \phi s_{t-1} + \phi A + b_3(u_t)] , \\ s_t^n &= h_{t-1} + b_3(u_t) , \end{aligned}$$

where $b_3(u_t)$ must satisfy the following nonlinear stochastic difference equation:

$$(21) \quad \begin{aligned} [1 + \beta - \beta \frac{\phi}{1+\phi} \theta(u_{t+1})] b_3(u_t) &- \beta \theta(u_{t+1}) \frac{1}{1+\phi} E_{t,i} [b_3(u_{t+1})] \\ - \beta [1 - \theta(u_{t+1})] E_{t,n} [b_3(u_{t+1})] &- \beta \theta(u_{t+1}) \frac{\phi}{1+\phi} A = u_t , \end{aligned}$$

and $\theta(u_{t+1}) = \text{Prob}(b_3(u_{t+1}) > A + b_3(u_t))$.

With further manipulations, (20) can be reduced to:

$$(22) \quad s_t^i = s_{t-1} + \frac{1}{1+\phi} [\phi A + b_3(u_t) - b_3(u_{t-1})] ,$$

$$s_t^n = s_{t-1} + b_3(u_t) - b_3(u_{t-1}) .$$

The details are provided in Appendix B.

To obtain an explicit solution, I must solve $b_3(u_t)$ in (21). There appears to be no general solution, and I am not even sure of the existence or uniqueness of a solution for an arbitrary u_t process. I do, however, have an explicit solution for a specific example.

Let w_t be iid with an exponential distribution, i.e., its density is given by $f(w) = \lambda e^{-\lambda w}$ for $0 < w < \infty$. Define u_t as follows:

$$(23) \quad u_t = (1+\beta)w_t - \beta \frac{1}{\lambda} - \frac{1}{\lambda} \alpha \frac{\phi}{1+\phi} e^{-\lambda(A+w_t)} .$$

Since w_t is iid, so is u_t .

It is straight forward to show that u_t as defined in (23) satisfies (21), by noting that $w_t = b_3(u_t)$, and rewriting (21) as:

$$(24) \quad (1+\beta)w_t - \beta \frac{\phi}{1+\phi} (A+w_t) \int_{A+w_t}^{\infty} f(w)dw - \beta \int_{-\infty}^{\infty} wf(w)dw - \beta \frac{\phi}{1+\phi} \int_{A+w_t}^{\infty} wf(w)dw = u_t .$$

In this case, the equilibrium exchange rate is given by:

$$(25) \quad \begin{aligned} s_t^i &= s_{t-1} + \frac{1}{1+\phi} [\phi A + w_t - w_{t-1}] , & \text{if } w_t > A+w_{t-1} , \\ s_t^n &= s_{t-1} + w_t - w_{t-1} , & \text{if } w_t \leq A+w_{t-1} , \end{aligned}$$

so that the probability of intervention next period is given by:

$$(26) \quad \theta(u_{t+1}) = \text{Prob}(s_{t+1} - s_t > A) = e^{-\lambda(A+w_t)} ,$$

which varies over time. This is motivated by Flood and Garber (1984), who use this exponential density to obtain explicit solutions for a collapsing fixed exchange rate regime. A similar result obtains when the double exponential density is used. This is discussed in Appendix B.

Equations (25) and (26) describe a switching process. The behavior of the rate of change of the exchange rate depends on whether there is central bank intervention or not. Furthermore, the switching probability at time $t+1$

depends on w_t , which is nonlinearly related to s_t as described in (23). This is how nonlinear dependence enters the model. The time variation in this probability will induce time variation in the conditional means and variances of the rate of change of the exchange rate. Simulation results, obtained by arbitrarily picking values of A , β , ϕ , and λ , show that the switching process can generate exchange rate changes which have a negative first order serial correlation while the squared exchange rate changes have a positive first order serial correlation, which is consistent with the observed data.^{7/}

IV. Conclusion

This paper constructs a theoretical model of a nonlinear stochastic rational expectations exchange rate model which has an explicit solution. Even though shocks are independent and identically distributed (and hence linear), the actions of the central bank and economic agents produce a rational expectations equilibrium in which the exchange rate switches between two linear stochastic processes, depending on the presence or absence of intervention. Nonlinearity enters through the probability of switching between these two regimes, which is time varying and depends on past outcomes. This model is consistent with observations that exchange rate movements have low autocorrelation, time-varying variance and nonlinear dependence. This result does not come from asymmetric information in this model. The agents know the current money supply, and hence they know whether (and how much) the central bank is intervening in the foreign exchange market.

Appendix A

This appendix deals with the steps in solving for the rational expectations exchange rate in Section 3. The important equations are reproduced here:

$$(4) \quad (1+\beta) s_t - \beta s_{t+1|t} = h_t + u_t,$$

$$(12) \quad h_t = h_{t-1} - \phi (s_t - s_{t-1} - A) z_t ,$$

where $z_t = 1$, with probability θ , and $z_t = 0$, with probability $(1-\theta)$.

$$(14) \quad s_t^i = a_0 + a_1 h_{t-1} + a_2 s_{t-1} + a_{3t} ,$$

$$s_t^n = b_0 + b_1 h_{t-1} + b_2 s_{t-1} + b_{3t} .$$

$$(15) \quad s_{t+1|t} = \theta E_t[s_{t+1}^i | z_{t+1}=1] + (1-\theta) E_t[s_{t+1}^n | z_{t+1}=0]$$

Substitute (14) into (15), I obtain:

$$(27) \quad s_{t+1|t} = \theta [a_0 + a_1 h_t + a_2 s_t + a_{3,t+1|t,i}] \\ + (1-\theta) [b_0 + b_1 h_t + b_2 s_t + b_{3,t+1|t,n}] ,$$

where $a_{3,t+1|t,i} = E_t[a_{3,t+1} | z_{t+1}=1]$, and

$$b_{3,t+1|t,n} = E_t[b_{3,t+1} | z_{t+1}=0] .$$

Now, put (27) into (4) and obtain:

$$(28) \quad (1+\beta) s_t - \beta \theta [a_0 + a_1 h_t + a_2 s_t + a_{3,t+1|t,i}] \\ - \beta (1-\theta) [b_0 + b_1 h_t + b_2 s_t + b_{3,t+1|t,n}] = h_t + u_t .$$

Collecting terms, I have:

$$(29) \quad \Phi h_t = \Gamma s_t - \beta \theta a_0 - \beta (1-\theta) b_0 - \beta \theta a_{3,t+1|t,i} - \beta (1-\theta) b_{3,t+1|t,n} - u_t ,$$

where $\Gamma = 1+\beta-\beta\theta a_2-\beta(1-\theta)b_2$, and $\Phi = 1+\beta\theta a_1+\beta(1-\theta)b_1$.

Suppose $z_t=1$. Then use (12) to substitute for h_t in (29) to yield :

$$(30) \quad \Phi [h_{t-1} - \phi s_t + \phi s_{t-1} + \phi A] = \\ \Gamma s_t - \beta \theta a_0 - \beta (1-\theta) b_0 - \beta \theta a_{3,t+1|t,i} - \beta (1-\theta) b_{3,t+1|t,n} - u_t .$$

Now use (14) to substitute for s_t :

$$(31) \quad \begin{aligned} & \Phi(1-\phi a_1) h_{t-1} + \Phi\phi(1-a_2) s_{t-1} - \Phi\phi a_0 + \Phi\phi A - \Phi\phi a_{3t} = \\ & \Gamma a_1 h_{t-1} + \Gamma a_2 s_{t-1} + (\Gamma-\beta\theta) a_0 - \beta(1-\theta) b_0 + \Gamma a_{3t} \\ & - \beta\theta a_{3,t+1|t,i} - \beta(1-\theta) b_{3t+1|t,n} - u_t . \end{aligned}$$

Thus, the following set of equations are obtained:

$$(32) \quad \begin{aligned} (\Gamma-\beta\phi)a_0 - \beta(1-\theta)b_0 &= -\Phi\phi a_0 + \Phi\phi A, \\ \Gamma a_1 &= \Phi(1-\phi a_1), \\ \Gamma a_2 &= \Phi\phi(1-a_2), \\ \Gamma a_{3t} - \beta\theta a_{3,t+1|t,i} - \beta(1-\theta)b_{3,t+1|t,n} - u_t &= -\Phi\phi a_{3t} . \end{aligned}$$

Suppose $z_t=0$. Use (18) to substitute for h_t and (22) to substitute for s_t in (29) to obtain:

$$(33) \quad \begin{aligned} & \Phi h_{t-1} \\ & = \Gamma b_1 h_{t-1} + \Gamma b_2 s_{t-1} - \beta\theta a_0 + [\Gamma-\beta(1-\theta)] b_0 \\ & \quad + \Gamma b_{3t} - \beta\theta a_{3,t+1|t,i} - \beta(1-\theta) b_{3t+1|t,n} - u_t . \end{aligned}$$

Thus, the following set of equations are obtained:

$$(34) \quad \begin{aligned} -\beta\phi a_0 + [\Gamma-\beta(1-\theta)] b_0 &= 0 , \\ \Gamma b_1 &= \Phi , \\ \Gamma b_2 &= 0 , \\ \Gamma b_{3t} - \beta\theta a_{3,t+1|t,i} - \beta(1-\theta) b_{3,t+1|t,n} - u_t &= 0 . \end{aligned}$$

A solution for (32) and (34) is:

$$(35) \quad \begin{aligned} a_0 &= \frac{1+\phi+\beta\theta}{(1+\phi)^2} \phi A , \\ a_1 &= \frac{1}{(1+\phi)} , \\ a_2 &= \frac{\phi}{(1+\phi)} , \\ a_{3t} &= \frac{1}{1+\phi+\beta+\beta\phi-\beta\phi\theta} \sum_{k=0}^{\infty} \psi^k u_{t+k|t} , \quad \psi = \frac{\beta+\beta\phi-\beta\phi\theta}{1+\phi+\beta+\beta\phi-\beta\phi\theta} , \\ b_0 &= \frac{\beta\theta}{(1+\phi)} \phi A , \end{aligned}$$

$$\begin{aligned}
b_1 &= 1 \quad , \\
b_2 &= 0 \quad , \\
b_{3t} &= \frac{1+\phi}{1+\phi+\beta+\beta\phi-\beta\phi\theta} \sum_{k=0}^{\infty} \psi^k u_{t+k|t} .
\end{aligned}$$

Since u_t is white noise with zero mean, I have:

$$\begin{aligned}
(36) \quad a_{3t} &= \frac{1}{1+\phi+\beta+\beta\phi-\beta\phi\theta} u_t , \\
b_{3t} &= \frac{1+\phi}{1+\phi+\beta+\beta\phi-\beta\phi\theta} u_t .
\end{aligned}$$

For this case, if there is intervention, i.e. $z_t=1$, I have:

$$(37) \quad h_t^i = h_{t-1} - \phi (s_t^i - s_{t-1} - A) .$$

This can be rewritten as:

$$(38) \quad h_t^i + \phi s_t^i = h_{t-1} + \phi s_{t-1} + \phi A .$$

Thus, the solution is:

$$(39) \quad s_t^i = \frac{1}{1+\phi} \left[\frac{1+\phi+\beta\theta}{1+\phi} \phi A + h_t^i + \phi s_t^i - \phi A + \Psi u_t \right] ,$$

where $\Psi = \frac{1+\phi}{1+\phi+\beta+\beta\phi-\beta\phi\theta}$, which can be expressed as:

$$(40) \quad s_t^i = \frac{\beta\theta}{1+\phi} \phi A + h_t^i + \Psi u_t .$$

When there is no intervention, i.e. $z_t=0$,

$$(41) \quad s_t^n = \frac{\beta\theta}{1+\phi} \phi A + h_t^n + \Psi u_t .$$

Thus, the observed values (whether there is intervention or not) satisfy the relation:

$$(42) \quad s_t = \frac{\beta\theta}{1+\phi} \phi A + h_t + \Psi u_t .$$

Taking first differences of (40), I have

$$(43) \quad s_t^i - s_{t-1} = h_t^i - h_{t-1} + \Psi (u_t - u_{t-1}) .$$

Substituting out $(h_t^i - h_{t-1})$ with the intervention rule, we have:

$$(44) \quad s_t^i - s_{t-1} = -\phi (s_t^i - s_{t-1} - A) + \Psi (u_t - u_{t-1}) .$$

Thus

$$(45) \quad s_t^i - s_{t-1} = \frac{1}{1+\phi} \left[\phi A + \frac{1+\phi}{1+\phi+\beta+\beta\phi-\beta\phi\theta} (u_t - u_{t-1}) \right] .$$

When there is no intervention, I can take the first difference of

(41):

$$(46) \quad s_t^n - s_{t-1} = h_t^n - h_{t-1} + \Psi (u_t - u_{t-1}) .$$

But $h_t^n = h_{t-1}$. So we have:

$$(47) \quad s_t^n - s_{t-1} = \frac{1+\phi}{1+\phi+\beta+\beta\phi-\beta\phi\theta} (u_t - u_{t-1}) .$$

Q.E.D.

Appendix B

This appendix shows the steps used in solving for the rational expectations exchange rate in Section 4. The relevant equations are:

$$(4) \quad (1+\beta) s_t - \beta s_{t+1|t} = h_t + u_t .$$

$$(18) \quad \begin{aligned} h_t^i &= h_{t-1} - \phi (s_t - s_{t-1} - A) , & \text{if } [s_t^n - s_{t-1}] > A, \\ h_t^n &= h_{t-1} , & \text{if } [s_t^n - s_{t-1}] \leq A. \end{aligned}$$

$$(19) \quad \begin{aligned} s_t^i &= a_1(u_t) h_{t-1} + a_2(u_t) s_{t-1} + a_3(u_t) , \\ s_t^n &= b_1(u_t) h_{t-1} + b_2(u_t) s_{t-1} + b_3(u_t) . \end{aligned}$$

Let $\theta(u_t) = \text{Prob}([s_t^n - s_{t-1}] > A)$ be the probability of intervention at time t . Let $z_{t+1} = 1$ if $[s_{t+1}^n - s_t] > A$, and $z_{t+1} = 0$ if $[s_{t+1}^n - s_t] \leq A$.

Then

$$(48) \quad s_{t+1|t} = \theta(u_{t+1}) E_{t,i} [a_1(u_{t+1}) h_t + a_2(u_{t+1}) s_t + a_3(u_{t+1})] + [1 - \theta(u_{t+1})] E_{t,n} [b_1(u_{t+1}) h_t + b_2(u_{t+1}) s_t + b_3(u_{t+1})] .$$

This yields the following equations:

$$(49) \quad \begin{aligned} \Gamma(u_{t+1}) a_1(u_t) &= \Phi(u_{t+1}) [1 - \phi a_1(u_t)] , \\ \Gamma(u_{t+1}) a_2(u_t) &= \Phi(u_{t+1}) \phi [1 - a_2(u_t)] , \\ \Gamma(u_{t+1}) a_3(u_t) &= \Phi(u_{t+1}) [\phi A - \phi a_3(u_t)] + u_t \\ &+ \beta \theta(u_{t+1}) E_{t,i} [a_3(u_{t+1})] + \beta [1 - \theta(u_{t+1})] E_{t,n} [b_3(u_{t+1})] , \\ \Gamma(u_{t+1}) b_1(u_t) &= \Phi(u_{t+1}) , \\ \Gamma(u_{t+1}) b_2(u_t) &= 0 , \\ \Gamma(u_{t+1}) b_3(u_t) &= u_t \\ &+ \beta \theta(u_{t+1}) E_{t,i} [a_3(u_{t+1})] + \beta [1 - \theta(u_{t+1})] E_{t,n} [b_3(u_{t+1})] , \end{aligned}$$

where

$$\begin{aligned} \Gamma(u_{t+1}) &= 1 + \beta - \beta \theta(u_{t+1}) E_{t,i} [a_2(u_{t+1})] \\ &\quad - \beta [1 - \theta(u_{t+1})] E_{t,n} [b_2(u_{t+1})] , \\ \Phi(u_{t+1}) &= 1 + \beta \theta(u_{t+1}) E_{t,i} [a_1(u_{t+1})] \end{aligned}$$

$$+ \beta [1-\theta(u_{t+1})] E_{t,n}[b_1(u_{t+1})] .$$

Four equations are exactly the same as four in the previous model. So the solution is:

$$(50) \quad \begin{aligned} a_1(u_t) &= \frac{1}{(1+\phi)} , \\ a_2(u_t) &= \frac{\phi}{(1+\phi)} , \\ b_1(u_t) &= 1 , \\ b_2(u_t) &= 0 . \end{aligned}$$

So:

$$(51) \quad \begin{aligned} \Gamma(u_{t+1}) &= (1+\beta-\beta\theta(u_{t+1})\phi/[1+\phi]) , \text{ and} \\ \Phi(u_{t+1}) &= (1+\beta\theta(u_{t+1})/(1+\phi) + \beta(1-\theta(u_{t+1}))) = \Gamma(u_{t+1}) . \end{aligned}$$

There are two remaining equations:

$$(52) \quad \begin{aligned} \Gamma(u_{t+1}) a_3(u_t) &= \Phi(u_{t+1})[\phi A - \phi a_3(u_t)] \\ &+ u_t + \beta\theta(u_{t+1})E_{t,i}[a_3(u_{t+1})] + \beta[1-\theta(u_{t+1})]E_{t,n}[b_3(u_{t+1})] , \end{aligned}$$

$$(53) \quad \begin{aligned} \Gamma(u_{t+1}) b_3(u_t) &= \\ &u_t + \beta\theta(u_{t+1})E_{t,i}[a_3(u_{t+1})] + \beta[1-\theta(u_{t+1})]E_{t,n}[b_3(u_{t+1})] . \end{aligned}$$

One of these equation can be written as:

$$(54) \quad a_3(u_t) = \frac{1}{1+\phi} [\phi A + b_3(u_t)] .$$

Substituting for $a_3(u_{t+1})$, we have

$$(55) \quad \begin{aligned} [1+\beta-\beta\frac{\phi}{1+\phi} \theta(u_{t+1})] b_3(u_t) - \beta\theta(u_{t+1}) \frac{1}{1+\phi} E_{t,i}[b_3(u_{t+1})] \\ - \beta[1-\theta(u_{t+1})]E_{t,n}[b_3(u_{t+1})] - \beta\theta(u_{t+1}) \frac{\phi}{1+\phi} A = u_t , \end{aligned}$$

$$\begin{aligned} \text{where } \theta(u_{t+1}) &= \text{Prob}(s_{t+1}^n - s_t > A) , \\ &= \text{Prob}(b_3(u_{t+1}) > A + s_t - h_t) . \end{aligned}$$

The form of the solution is:

$$(56) \quad \begin{aligned} s_t^i &= \frac{1}{1+\phi} [h_{t-1} + \phi s_{t-1} + \phi A + b_3(u_t)] , \\ s_t^n &= h_{t-1} + b_3(u_t) . \end{aligned}$$

If there is intervention, i.e. $s_t^n - s_{t-1} > A$,

$$(57) \quad h_t^i + \phi s_t^i = h_{t-1} + \phi s_{t-1} + \phi A .$$

Substituting into (56), I obtain:

$$(58) \quad s_t^i = h_t^i + b_3(u_t) .$$

If there is no intervention, i.e. $s_t^n - s_{t-1} < A$, $h_t^n = h_{t-1}$.

Substituting into (56), we obtain:

$$(57) \quad s_t^n = h_t^n + b_3(u_t) .$$

Hence, for observed values:

$$(58) \quad s_t = h_t + b_3(u_t) .$$

Lagging this once, and subtract from the two solutions, we have

$$(59) \quad s_t^i = s_{t-1} + \frac{1}{1+\phi} [\phi A + b_3(u_t) - b_3(u_{t-1})] ,$$

$$(60) \quad s_t^n = s_{t-1} + b_3(u_t) - b_3(u_{t-1}) .$$

Also using (58), we have:

$$(61) \quad \begin{aligned} \theta(u_{t+1}) &= \text{Prob}(b_3(u_{t+1}) > A + s_t - h_t) \\ &= \text{Prob}(b_3(u_{t+1}) > A + b_3(u_t)) . \end{aligned}$$

The solution involves finding $b_3(u_t)$, such that:

$$(62) \quad \begin{aligned} [1 + \beta - \beta \frac{\phi}{1+\phi} \theta(u_{t+1})] b_3(u_t) - \beta \theta(u_{t+1}) \frac{1}{1+\phi} E_{t,i}[b_3(u_{t+1})] \\ - \beta [1 - \theta(u_{t+1})] E_{t,n}[b_3(u_{t+1})] - \beta \theta(u_{t+1}) \frac{\phi}{1+\phi} A = u_t . \end{aligned}$$

Since u_t is iid, then $b_3(u_t)$ is also iid. Let $F(\cdot)$ be the distribution of $b_3(u_t)$, i.e., $F(x) = \text{Prob}(b_3(u_t) < x)$.

Then we can write:

$$(63) \quad \begin{aligned} \theta(u_{t+1}) &= \text{Prob}(b_3(u_{t+1}) > A + b_3(u_t)) , \\ &= 1 - F(A + b_3(u_t)) . \end{aligned}$$

Then (62) can be rewritten as:

$$(64) \quad \begin{aligned} [1 + \beta] b_3(u_t) \\ - \beta \frac{\phi}{1+\phi} [1 - F(A + b_3(u_t))] [A + b_3(u_t)] \end{aligned}$$

$$\begin{aligned}
& - \beta E_t[b_3(u_{t+1})] \\
& + \beta \frac{\phi}{1+\phi} [1 - F(A+b_3(u_t))] E_t[b_3(u_{t+1}) | b_3(u_{t+1}) > A + b_3(u_t)] \\
& = u_t.
\end{aligned}$$

We can reverse the procedure. Instead of finding a $b_3(u_t)$ which satisfies this equation for a given iid u_t , we can begin with an iid $b_3(u_t) = w_t$, and then we define u_t as:

$$\begin{aligned}
(65) \quad u_t & = [1+\beta] w_t \\
& - \beta \frac{\phi}{1+\phi} [1 - F(A+w_t)] [A + w_t] \\
& - \beta E_t[w_{t+1}] \\
& + \beta \frac{\phi}{1+\phi} [1 - F(A+w_t)] E_t[w_{t+1} | w_{t+1} > A + w_t] .
\end{aligned}$$

This u_t will satisfy the equilibrium conditions.

As an example, let $b_3(u_t) = w_t$, where w_t is iid, with distribution $f(w) = \lambda e^{-\lambda w}$ for $0 < w < \infty$. Then:

$$\begin{aligned}
(66) \quad F(A+w_t) & = 1 - e^{-\lambda(A+w_t)} , \\
E_t[w_{t+1}] & = \frac{1}{\lambda} , \\
E_t[w_{t+1} | w_{t+1} > A + w_t] & = [A + w_t + \frac{1}{\lambda}] .
\end{aligned}$$

Substituting into the equation (64), we obtain:

$$(67) \quad (1+\beta)w_t - \beta \frac{1}{\lambda} + \frac{1}{\lambda} \beta \frac{\phi}{1+\phi} e^{-\lambda(A+w_t)} = u_t .$$

Hence if u_t is given by (67), $b_3(u_t) = w_t$ will satisfy (64), and

$$(68) \quad s_t^i = s_{t-1} + \frac{1}{1+\phi} [\phi A + w_t - w_{t-1}] ,$$

$$(69) \quad s_t^n = s_{t-1} + w_t - w_{t-1} .$$

Another distribution which yields a closed form solution is the double exponential distribution: $f(w) = .5 \lambda e^{-\lambda|w|}$, $-\infty < w < \infty$. The solution can be written as:

$$(70) \quad u_t = (1+\beta)w_t + .5 \beta \frac{\phi}{1+\phi} \frac{1}{\lambda} e^{-\lambda|A+w_t|} + .5 \beta \frac{\phi}{1+\phi} [|A+w_t| - (A+w_t)].$$

Notes

- 1/ See Burt, Kaen, and Booth (1977), Cornell (1977), Friedman and Vandersteel (1982), Giddy and Dufey (1975), Hsieh (1988), Logue and Sweeney (1977), Logue, Sweeney, and Willett (1978), Manas-Anton (1986), and, Rogalski and Vinso (1978).
- 2/ See Hsieh (1989a).
- 3/ See Manas-Anton (1986).
- 4/ See Hsieh (1988, 1989a). Cumby and Obstfeld (1984) also found conditional heteroscedasticity in inflation and exchange rate forecast errors.
- 5/ See Bollerslev (1986, 1987), Engle and Bollerslev (1986), Domowitz and Hakkio (1985), Hsieh (1988, 1989b), Manas-Anton (1986), and Milhøj (1986). See also the survey paper by Bollerslev et al (1990).
- 6/ See Froot and Obstfeld (1991) for a generalization of stochastic regime switching.
- 7/ This simple model fails to fit the observed exchange rate changes in two aspects. One, the model cannot simultaneously generate a small serial correlation of exchange rate changes and a large serial correlation of squared exchange rate changes. Two, the model cannot generate large serial correlation of squared exchange rates beyond the first lag. A more complicated model of intervention may be required to achieve these results.

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