Problem Set Solutions for the video course "A Mathematics Course for Political and Social Research"

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July 30, 2015

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1 Lecture 1

1. Identify whether each of the following is a constant or a variable:
   (a) Variable.
   (b) Constant.
   (c) Variable.
2. Identify whether each of the following is a variable or a value of a variable:
   (a) Value (of the variable Age).
   (b) Variable (taking the values of candidates or parties).
   (c) Variable.
   (d) Value (of the variable Education).
3. Identify whether each of the following indicators is measured at a nominal, ordinal, interval, or ratio level. Note also whether each is a discrete or a continuous measure:
   (a) Ratio, discrete.
   (b) Nominal, discrete.
   (c) Ratio, continuous.
   (d) Ordinal, discrete.
4. Let $A = (6, 8)$ and $B = [7, 9]$.
   (a) Neither.
   (b) $(6, 9]$.
   (c) $[7, 8)$.
   (d) Any (x, y) in which $x \in (6, 8), y \in [7, 9]$.
5. Simplify or evaluate each of the following:
   (a) x^3.
   (b) $a(b^3 - a)$.
   (c) $4 + 1 - 4(5) = -15$.
   (d) 120.
   (e) ± 3.
   (f) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$.
   (g) $\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{16} = \frac{1}{1024}$.
   (h) $x(1 + 4(y - 3) + z)$.

6. Ratio: 232:192.
   Proportion: $\frac{232}{232+192} = \frac{232}{424} \simeq 0.547$
   Percentage: $100\% \times (\frac{232}{424}) \simeq 54.7\%$.

7. $100\% \times (\frac{210-192}{192}) \simeq 100\% \times 0.094 = 9.4\%$.

8. $\frac{(x-4)(x+2)}{7(x+2)} = \frac{x-4}{7}$.

9. $\frac{9y-15}{20}$.

10. $16(\phi - 3)(\phi + 5)$.

11. $y = 2$.

12. $x = 4$ or $x = 3$.

13. $x = \frac{5}{4} \pm \frac{\sqrt{5}}{4}$.

14. $\theta > \frac{-1}{2}$.

2 Lecture 2

1. For each pair of ordered sets, state whether it represents a function or a correspondence:

   (a) Function, since no element of the first set maps to more than one element of the second.

   (b) Correspondence, since the element 3 in the first set maps to three values $(10, -5, 1)$ in the second.

   $f(x) = x^3 - 1$, $g(x) = \ln x$, and $h(x) = \frac{1}{x^2}$:

2. Simplify or evaluate the following expressions, using $f(x) = x^3 - 1$, $g(x) = \ln(x)$, and $h(x) = \frac{1}{x^2}$:

   (a) $(\ln(x))^3 - 1$.

   (b) $\ln(\frac{1}{x^2}) = -2\ln(x)$.

   (c) $\frac{x^3-1}{x^2} = x - \frac{1}{x^2}$ (the latter as long as $x \neq 0$).

   (d) $\frac{\ln(x)}{x^2}$.

   (e) $(x^{-2})^3 = x^{-6} = \frac{1}{x^6}$.

   (f) $\sqrt[3]{x+1}$.

   (g) $\frac{1}{|\sqrt{x}|}$ for $x > 0$.

3. Either find the following limits or show that they do not exist.

   (a) -9.

(b) 0.

(c) $\frac{4}{7}$.

(d) Does not exist (would be $\frac{1}{0}$).

(e) $\frac{-1}{4}$.

4. For each of the following sets, state whether they are (a) open, closed, both, or neither; (b) bounded; (c) compact; (d) convex:

   (a) (a) neither, (b) bounded, (c) not compact, (d) convex.

   (b) (a) closed, (b) bounded, (c) compact, (d) convex.

   (c) (a) neither, (b) bounded, (c) not compact, (d) not convex.

   (d) (a) open (it's $(2, 5)$), (b) bounded, (c) not compact, (d) convex.

5. For each of the following functions, state whether or not it is continuous on the domain provided.

   (a) Yes.

   (b) This function is not continuous because it has the value 0 at $x = 1$, but approaches 2 as it approaches $x = 1$ from the right. (One could make it continuous by either adding 2 from the equation for all $x \leq 1$ (to get $x + 1$), or by subtracting 2 to the equation for all $x > 1$ (to get $x^2 - 2$).

3 Lecture 3

1. Use the definition of the derivative to find the derivative of y with respect to x for the following:

   (a) $dy/dx = 0$.

   (b) $dy/dx = 2x + 2$.

   (c) $dy/dx = 6x^2 + 10x$.

   (d) $dy/dx = 8x^3 - 6x$.

   (e) $dy/dx = -20x^3 + 6x + 1$.

2. For each of the following, find the partial derivative with respect to both x and z.

   (a) $\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial z} = 4z$.

   (b) $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial z} = 3z^2$.

   (c) $\frac{\partial f}{\partial x} = z^2, \frac{\partial f}{\partial z} = 2xz$.

   (d) $\frac{\partial f}{\partial x} = 3x^2z^2 - 1, \frac{\partial f}{\partial z} = 2x^3z$.

4 Lecture 4

Differentiate the following:

1. $f'(x) = -3x^{-4}$.
2. $f'(x) = 4ax^3$.
3. $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$
4. $f'(x) = (1-2x)(2x^3+5x^2-7)+(x-x^2)(6x^2+10x)$.
5. $f'(x) = 5(x+1)^4$.
6. $f'(x) = ((-4x+3x^2)(x^2-x)-(-2x^2+x^3)(2x-1))/(x^2-x)^2$.
7. $f'(x) = 3(x^3-5)^2(3x^2)$.
8. $f'(x) = (4x^3-2x)\ln(x^2-5)+\frac{(x^4-x^2+2)(2x)}{x^2-5}$.
9. $f'(x) = 3(25x^4-2x)e^{5x^5-x^2}$.
10. $f'(x) = 2xe^x\ln(x)+x^2e^x\ln(x)+xe^x+15x^4$.
11. $f'(x) = \ln(c)c^{x^3-1}(3x^2)$.
12. $f'(x) = \frac{6x^2(\log_c(x))+2x^3(\frac{1}{x\ln(c)})}{(\log_c(x))^2}$.

5 Lecture 5

1. Integrate the following:
   (a) $F(x) = \frac{5}{2}x^2-3x+C$.
   (b) $F(x) = -\frac{1}{2}x^2+5x-\frac{1}{2}x^4+\frac{1}{6}x^6+C$.
   (c) $F(x) = 2\ln(|x|)+\frac{2}{11}x^{11}+C$.
   (d) $F(x) = 2e^{\frac{x}{2}}+C$.
   (e) $F(x) = e^{x^5+2}+C$.
   (f) $F(x) = \frac{1}{2}(2x\ln(2x)-2x)+C$.
2. Evaluate the following integrals:
   (a) 0.
   (b) $9-\frac{45}{2}+6-\frac{1}{3}+\frac{5}{2}-2+\frac{125}{3}-\frac{125}{2}+10-9+\frac{45}{2}-6=\frac{125}{3}-\frac{125}{2}+10-\frac{1}{3}+\frac{5}{2}-2$. Note the integrals evaluated at $x=3$ cancel. This is $\frac{124}{3}-60+8=-10\frac{2}{3}$.
   (c) $\frac{125}{3}-\frac{125}{2}+10-\frac{1}{3}+\frac{5}{2}-2=-10\frac{2}{3}$, the same as for the previous question.

(d) $(x\ln(x) - x - \frac{2}{3}x^{\frac{3}{2}})|_1^e$. This is $e - e - \frac{2}{3}e^{\frac{3}{2}} - 0 + 1 + \frac{2}{3} = 1 - \frac{2}{3}(e^{\frac{3}{2}} - 1)$.

(e) $\frac{4}{3}(\sqrt{2x^2 + 6x + 1})^3 + C$.

(f) $x^2e^x - 2(xe^x - e^x) + C = e^x(x^2 - 2x + 2) + C$.

6 Lecture 6

Find all extrema (local and global) of the following functions on the specified domains, and state whether each extremum is a minimum or maximum and whether each is only local or global on that domain.

1. Global minimum at $x^* = -5$, global maximum at the boundary $x = 10$.
2. Global minimum at $x = 0$, global maximum at $x = 1$.
3. Global minimum at $x^* = 0$, global maximum at $x^* = 1$.
4. Global maximum at $x^* = \frac{1}{2}$, global minimum at $x = 10$.
5. Global maximum at $x^* = -2$, global minimum at $x^* = 2$.
6. Three stationary points at $x^* = -2, -1, 1$. Global minimum at $x^* = 1$, global maximum at $x = 2$, local maximum at $x^* = -1$, local minimum at $x^* = -2$.

7 Lecture 7

1. Identify each of the following as a *classical* (objective), *empirical* (objective), or *subjective* probability claim:
   (a) *Empirical* (objective).
   (b) *Classical* (objective).
   (c) *Subjective* (presumably not based on a forecasting model).
2. Identify the following as simple or compound events:
   (a) Simple.
   (b) Compound.
   (c) Compound.
   (d) Simple.
3. Characterize the following as independent, mutually exclusive, and/or collectively exhaustive:
   (a) Mutually exclusive and collectively exhaustive values of a variable.
   (b) Independent.

(c) Mutually exclusive values of a variable.

4. Let $Pr(a) = 0.45$, $Pr(b) = 0.22$, and $Pr(c) = 0.31$,

   (a) .02.
   (b) 0.67.
   (c) 0.
   (d) 0.0682.
   (e) 0.37.
   (f) $\frac{.6-.45}{1-.45} = 0.273$
   (g) $\frac{.8-.45}{1-.45} = 0.636$

5. Compute each of the following:

   (a) 360.
   (b) 90.
   (c) $\frac{1}{10}$.
   (d) 35.
   (e) 28.
   (f) 210.
   (g) 20160.

6. A committee contains ten legislators with six men and four women. Find the number of ways that a delegation of five:

   (a) This is the number of ways 5 elements can be chosen from 10, or $\binom{10}{5} = 252$.
   (b) We have the joint probability of two independent events: choosing 2 women from 4 and 3 men from 6. This is: $\binom{6}{3}\binom{4}{2} = 20 * 6 = 120$.
   (c) There are two ways to do this: have 3 women or have 4 women on the committee. We add the number of ways that each can occur. For three women: $\binom{6}{2}\binom{4}{3} = 15 * 4 = 60$. For four women: $\binom{6}{1}\binom{4}{4} = 6 * 1 = 6$. So the total is 66.

7. $\frac{27}{64}$.

8. $\frac{1}{36}, \frac{1}{9}$.

9. In one legislature, 20% of the legislators are conservatives, 20% are liberals, and 60% are independents. In a recent vote to expand counterinsurgency operations, 80% of conservatives, 20% of liberals, and 50% of independents voted in favor.

   (a) $Pr(V) = Pr(V|C)Pr(C) + Pr(V|L)Pr(L) + Pr(V|I)Pr(I) = .8 * .2 + .2 * .2 + .6 * .5 = 0.5$.

(b) $Pr(C|V) = \frac{PR(V|C)Pr(C)}{PR(V)} = \frac{(.8)(.2)}{.5} = 0.32.$

(c) 0.08.

(d) 0.6.

10. 0.4/0.6 $\rightarrow$ 2:3 or .667:1.

11. $(2/1)/(1/4) = 8$.

12. An unemployed person is 118% more likely than an employed person, and someone with a college education is 55% less likely than a person with only a high school education, to vote for the National Front.

8 Lecture 8

1. We use a Poisson distribution with these data: $\frac{10^{15}}{e^{10}15!} = 0.035$.

2. We use a binomial distribution: $\binom{25}{15}(.2)^{15}(.8)^{10} = .00001$.

3. We use a negative binomial distribution:

$$\begin{aligned} P(Y = 3|2, 0.2) &= \binom{3+2-1}{3} .2^3(1-.2)^2 \\ &= \binom{4}{3} \times .008 \times .64 \\ &= \frac{4!}{(3!)(1!)} \times .00512 \\ &= 4 \times .00512 \\ &= 0.02. \end{aligned}$$

4. Bach: $EU(B) = p(B)EU(BT) + (1 - p(B))EU(BA)$, where B is Bach, T is Together and A is alone.
$EU(B) = .2(12) + .8(2) = 2.4 + 1.6 = 4$.
$EU(S) = p(B)EU(SA) + (1 - p(B))EU(ST)$, where S is Stravinsky.
$EU(S) = .2(-2) + .8(6) = -.4 + 4.8 = 4.4$.
So she should choose Stravinsky because $4.4 > 4$, even though she personally likes Bach much better.

5. First we use Bayes' rule to figure out the posterior probability of B's being strong after A observes maneuvers. This is $Pr(S|M) = \frac{Pr(M|S)Pr(S)}{Pr(M|S)Pr(S)+Pr(M|W)Pr(W)} = \frac{(.8)(.2)}{(.8)(.2)+(.1)(.8)} = \frac{2}{3}$. Next we ask what this posterior probability must be for A to want to start a war. This is an expected utility comparison. A gets 0 for doing nothing, and expects to get $Pr(S|M) * (-1) + (1 - Pr(S|M)) * (1)$ for starting a war after observing maneuvers. This becomes $-\frac{2}{3} + \frac{1}{3} = -\frac{1}{3} < 0$, so A would prefer not to start a rebellion after observing maneuvers, despite having an initially low prior belief about the strength of the state. This is because the maneuvers are very informative.

9 Lecture 9

1. This is:

$$\begin{aligned} &\int_{-\infty}^{\infty}(x-\mu)^3 f(x)dx \\ &= \int_0^1 (x^3 - 3\mu x^2 + 3\mu^2 x - \mu^3)dx \\ &= \left(\tfrac{1}{4}x^4 - \mu x^3 + \tfrac{3}{2}\mu^2 x^2 - \mu^3 x\right)|_0^1 \\ &= \left(\tfrac{1}{4} - \mu + \tfrac{3}{2}\mu^2 - \mu^3\right) \\ &= \left(\tfrac{1}{4} - \tfrac{1}{2} + \tfrac{3}{2}\tfrac{1}{4} - \tfrac{1}{8}\right) \\ &= 0. \end{aligned} \tag{1}$$

2. Assume a policy outcome, x, depends on a choice of policy, p, and an exogenous economic shock, ϵ, so that $x = p + \epsilon$. Further assume that ϵ is distributed uniformly on $[-1, 1]$.

   (a) 0.

   (b) No, still 0.

   (c) It decreases. At $p = 0$, for $[-1, 1]$: $\int_{-1}^{1} \frac{1}{2}(-\epsilon^2)d\epsilon = -\frac{1}{6}\epsilon^3|_{-1}^{1} = \frac{1}{6}(-1-1) = -\frac{1}{3}$. For $[-2, 2]$: $\int_{-2}^{2} \frac{1}{4}(-\epsilon^2)d\epsilon = -\frac{1}{12}\epsilon^3|_{-2}^{2} = \frac{1}{12}(-8-8) = -\frac{16}{12} = -\frac{4}{3}$. She is effectively risk averse around her ideal point of zero, so prefers the less uncertain case.

3. $\int_{100}^{600} \frac{1}{600-100} x dx = \frac{360000-10000}{1000} = 350.$

4. It would not change. It would still be equal to the midpoint of the distribution, which is still 350. A risk neutral person cares only about the expected value of the lottery.

5. $\int_{100}^{600} \frac{1}{600-100}\sqrt{x}dx = \frac{2}{3}\frac{600^{3/2}-100^{3/2}}{500} = 18.3.$

6. $\int_{200}^{500} \frac{1}{500-200}\sqrt{x}dx = \frac{2}{3}\frac{500^{3/2}-200^{3/2}}{300} = 18.6.$ This is greater then before; she is risk averse, and prefers the less risky lottery to the more risky lottery, given equal expected values of the lotteries.

10 Lecture 10

1. Let: $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ -5 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ 5 \\ 2 \end{pmatrix}$, $\mathbf{c} = (1, 1, 3)$, $\mathbf{d} = (6, 4, 2)$, and $\mathbf{e} = (10, 20, 30, 40)^T$.

   Calculate each of the following, indicating that it's not possible if there is a calculation you cannot perform.

   (a) $\begin{pmatrix} 6 \\ 6 \\ 0 \\ 5 \end{pmatrix}$.

(b) $\begin{pmatrix} -8 \\ -15 \\ -25 \\ -38 \end{pmatrix}$.

(c) Not possible to add because they have different dimensions.

(d) $\begin{pmatrix} 24 \\ 6 \\ -30 \\ 18 \end{pmatrix}$.

(e) $\sqrt{4+16+100+1} = 11$.

(f) $(1)(6)+(1)(4)+(3)(2) = 16$.

(g) 380.

2. Identify the following matrices as diagonal, identity, square, symmetric, triangular, or none of the above (note all that apply).

   (a) This is a square, (lower) triangular matrix.

   (b) This matrix has none of these properties.

   (c) This is a symmetric, square matrix.

3. Write down the transpose of matrices A through C from the previous problem.

   (a) $A^T = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$.

   (b) $B^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 5 & 6 \\ 7 & -3 \end{bmatrix}$.

   (c) $C = \begin{bmatrix} 10 & 5 & 2 \\ 5 & -3 & 7 \\ 2 & 7 & -9 \end{bmatrix}$.

4. Given the following matrices, perform the calculations below.

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 5 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 8 \\ -5 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 0 \\ 3 & 6 \end{bmatrix}$$

   (a) Not possible to add because they have different dimensions.

   (b) $\begin{bmatrix} 2 & 2 & -5 \\ 4 & 8 & 2 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & -2 & 6 \\ -4 & -3 & 0 \end{bmatrix}$.

(d) $\begin{bmatrix} 9 & 0 & 3 \\ 0 & 15 & 6 \end{bmatrix}$.

(e) Cannot multiply due to incompatible dimensions.

(f) $\begin{bmatrix} 1 & 14 \\ 0 & 44 \end{bmatrix}$.

(g) $\begin{bmatrix} 6 & 20 & 10 \\ 6 & 40 & 18 \\ -15 & 10 & -1 \end{bmatrix}$.

(h) $\begin{bmatrix} 1 & 0 \\ 14 & 44 \end{bmatrix}$.

(i) $\begin{bmatrix} 1 & 0 \\ 14 & 44 \end{bmatrix}$.

(j) 11.

5. Find the determinants and, if they exist, the inverses of the following matrices:

(a) $Det(A) = 26$. $A^{-1} = \frac{1}{26}\begin{bmatrix} 7 & -8 \\ -2 & 6 \end{bmatrix}$.

(b) $Det(B) = 0$. No inverse exists because the matrix is singular by virtue of its determinant of zero.

(c) $Det(C) = (3)(12-20) - (2)(4-4) + (1)(10-6) = -24 - 0 + 4 = -20$.
$M_{11} = -8, M_{12} = 0, M_{13} = 4, M_{21} = -1, M_{22} = 5, M_{23} = 13, M_{31} = 2, M_{32} = 10, M_{33} = 14$.

(d) $C^{-1} = \frac{-1}{20}\begin{bmatrix} -8 & 1 & 2 \\ 0 & 5 & -10 \\ 4 & -13 & 14 \end{bmatrix}$.

(e) We cannot take the determinant of a matrix that is not square, and so we cannot find its inverse.

11 Lecture 11

1. Let: $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ -5 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ 5 \\ 2 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 2 \\ -4 \\ -10 \\ 1 \end{pmatrix}$, and $\mathbf{d} = \begin{pmatrix} 4 \\ 10 \\ 10 \\ 4 \end{pmatrix}$. Answer each of the following questions, saying that it's not possible if there is a calculation you cannot perform.

(a) $s\mathbf{a}+t\mathbf{b}=\begin{pmatrix}4s+2t\\ s+5t\\ -5s+5t\\ 3s+2t\end{pmatrix}$

(b) No, only two at most are. For example, one can write $\mathbf{d}=2\mathbf{b}$ and $\mathbf{c}=\mathbf{a}-\mathbf{b}$.

(c) These span a two-dimensional space, as there are two, and only two, linearly independent vectors. (If you wrote $\mathbf{c}$ and $\mathbf{d}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ in the previous problem, then the latter two vectors are independent and span a two-dimensional space.)

2. Solve the following systems of equations using substitution or elimination, or both:

(a) $x=2, y=6, z=7$.

(b) $x=1, y=2, z=3$.

(c) $x=x, y=2-x, z=3-2x$. The third equation is a multiple of the first equation, so we only have two equations in three unknowns. To solve, we leave x as a parameter and solve for y and z in terms of x to get this answer.

(d) The first and second equations are contradictory, as the first implies $4x-4y-2z=9$ rather than $=10$ as the second one states. Thus these equations have no solutions.

3. Let $A=\begin{pmatrix}5&0\\3&6\end{pmatrix}$, $B=\begin{pmatrix}2&1\\4&2\end{pmatrix}$, and $\mathbf{c}=\begin{pmatrix}1\\3\end{pmatrix}$. Calculate each of the following, indicating that it's not possible if there is a calculation you cannot perform.

(a) A is full rank (2) as its determinant is non-zero and so it is non-singular.

(b) B has rank 1. It is singular, so its columns (or rows) are linearly dependent. Removing one of them leaves one linearly independent column (or row), making its rank 1.

(c) $A^{-1}=\frac{1}{30}\begin{pmatrix}6&0\\-3&5\end{pmatrix}$, and $A^{-1}\mathbf{c}=\frac{1}{30}\begin{pmatrix}6\\12\end{pmatrix}=\begin{pmatrix}\frac{1}{5}\\ \frac{2}{5}\end{pmatrix}$.

(d) Since B has determinant 0 it is singular, and does not have an inverse. Thus we can't use matrix inversion. As one of the rows is a multiple of the other we only have one unique equation for two variables, indicating an infinite number of solutions to the system of equations represented by the matrix.

4. $\det(A)=-12, \det(B_1)=-48, \det(B_2)=-24, \det(B_3)=-12 \Rightarrow x=4, y=2, z=1$.

12 Lecture 12

1. $\lambda=3,-1$, associated eigenvectors $\begin{pmatrix}1\\1\end{pmatrix}$, $\begin{pmatrix}1\\-1\end{pmatrix}$. Let $Q=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$, so $Q^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$. Now $A=QDQ^{-1}$ so $D=Q^{-1}AQ=\begin{pmatrix}3&0\\0&-1\end{pmatrix}$, as required.

2. First we find the eigenvalues. $A-\lambda I = \begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 0-\lambda \end{pmatrix}$. $|A-\lambda I| = (2-\lambda)(1-\lambda)(-\lambda)$. Setting this equal to zero gives the three eigenvalues $\lambda = 0, 1, 2$. Next we find the eigenvectors. For each eigenvalue we solve the equation $A\mathbf{x} = \lambda\mathbf{x}$ where we let $\mathbf{x} = \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$, moving around the location of the 1 if we get a contradiction. This produces a system of three equations for each eigenvalue that look like: $2+a = \lambda, a = \lambda a, a = \lambda b$. For $\lambda = 2$, the first equation yields $a = 0$, the second $a = 2a$, and the third $a = 2b$. The solution to these equations set $a = b = 0$, so the corresponding eigenvector is $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. For $\lambda = 1$, the first equation yields $a = -1$, the second $a = a$, and the third $a = b$. The solution to these equations set $a = b = -1$, so the corresponding eigenvector is $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. For $\lambda = 0$, the first two equations are contradictory, so we try a different location for the 1 in the eigenvector. We'll try $\mathbf{x} = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$. Now the system of three equations is $2a + b = 0, b = 0, b = 0$. The solution to this set of equations is $a = b = 0$, so the corresponding eigenvector to 0 is $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

3. Yes, it is ergodic because it is aperiodic and all states communicate. To find the steady state $\boldsymbol{\pi}$, we solve $M\boldsymbol{\pi} = \boldsymbol{\pi}$, where $\boldsymbol{\pi} = \begin{pmatrix} 1 \\ a \end{pmatrix}$. This matrix equation yields the system of equations $.75 + .8a = 1, .25 + .2a = a$ which has the unique solution $a = \frac{5}{16}$. Thus our steady state is $\begin{pmatrix} 1 \\ \frac{5}{16} \end{pmatrix}$ or, after normalizing it, $\begin{pmatrix} \frac{16}{21} \\ \frac{5}{21} \end{pmatrix}$.

4. Yes, it is ergodic because it is aperiodic and the first state is a distinguished state, reachable with positive probability from all other states. To find the steady state $\boldsymbol{\pi}$, we solve $M\boldsymbol{\pi} = \boldsymbol{\pi}$, where $\boldsymbol{\pi} = \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$. This matrix equation yields the system of equations $1 + .75a + .5b = 1, .25a = a, and .5b = b$ which has the unique solution $a = 0, b = 0$. Thus our steady state is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, as suggested by the form of the transition matrix, which basically "transfers" all probability to the first, distinguished state, which is an absorbing state.

13 Lecture 13

1. The partial derivatives are, respectively, $\beta_1 + \beta_3 x_2^2$ and $\beta_2 + 2\beta_3 x_1 x_2$. They correspond to the marginal effects of, respectively, x_1 and x_2, holding all other variables constant. Note that the model is non-linear in both variables. The marginal effect of x_1 depends on the value of x_2 in a substantial way, increasing rapidly as x_2 moves away from zero in either direction. The marginal effect of x_2 depends on the values of both x_2 and x_1, implying an even more complex effect.

2. $\frac{\partial f}{\partial x_1} = 3x_2 e^{x_1 x_2} - \frac{3x_3}{x_1} + \frac{2x_1 e^{x_3}}{\ln x_2}$.
$\frac{\partial f}{\partial x_2} = 3x1 e^{x_1 x_2} - \frac{x_1^2 e^{x_3}}{x_2(\ln x_2)^2}$.
$\frac{\partial f}{\partial x_3} = -\ln x_1^3 + \frac{x_1^2 e^{x_3}}{\ln x_2}$.

3. $\frac{\partial f}{\partial x_1} = 14x_1^6 - \ln(x_3) x_3^{x_1}$.
$\frac{\partial f}{\partial x_2} = x_3$.
$\frac{\partial f}{\partial x_3} = -x_1 x_3^{x_1 - 1} + x_2$.

4. First do the integral over x, because the bounds of the inner integral contains an y. So $\int_1^y x^2 dy = \frac{1}{3}x^3|_1^y = \frac{1}{3}[y^3 - 1]$. Now do the outer integral over y. This is $\frac{1}{3}\int_0^2 y^2[y^3 - 1]dy = \frac{1}{3}\int_0^2 (y^5 - y^2)dy$. This equals $\frac{1}{3}(\frac{1}{6}(2^6 - 0) - \frac{1}{3}(2^3 - 0)) = \frac{8}{3}$.

14 Lecture 14

1. $\frac{\partial f}{\partial x} = 2xz$. $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{dz}{dx} = 2xz + x^2(\frac{3}{x} + e^y)$.

2. $\nabla f = \begin{pmatrix} 16xe^{yz} \\ 8x^2 z e^{yz} \\ 8x^2 y e^{yz} \end{pmatrix}$.

3. $H = \begin{pmatrix} 16e^{yz} & 16xze^{yz} & 16xye^{yz} \\ 16xze^{yz} & 8x^2z^2e^{yz} & 8x^2e^{yz}(1+yz) \\ 16xye^{yz} & 8x^2e^{yz}(1+yz) & 8x^2y^2e^{yz} \end{pmatrix}$. At the point $(1,1,1)$ this is $H = \begin{pmatrix} 16e & 16e & 16e \\ 16 & 8e & 8e \\ 16e & 8e & 8e \end{pmatrix}$ or $8e\begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$. The eigenvalues of this matrix are $0, 2(1+\sqrt{2}), 2(1-\sqrt{2})$. Since these signs change, the matrix is indeterminate and we have a saddle point.

4. $J = \begin{pmatrix} 2z & 0 & 2x \\ 0 & z & y \\ y^3 & 3xy^2 & 0 \end{pmatrix}$.

15 Lecture 15

1. $(x^*, y^*) = (5, 1)$ is the only critical point, and it is a minumum.

2. $x^* = 1$, $y^* = 4$, $\lambda^* = 4$.
3. $x^* = 1$, $y^* = \frac{1}{2}$, $\lambda^* = 2$.
4. $x^* = 1$, $y^* = 2$.

16 Lecture 16

1. This is $\frac{dx^*(b)}{b} = 4b^3 + 2b$.
2. Write $f = (\ln(bx^*(b) + 1))^2 - x^*(b) = 0$, so the implicit function theorem says that $\frac{\partial x^*(b)}{\partial b} = \frac{-\frac{\partial f}{\partial b}}{\frac{\partial f}{\partial x}}$. $\frac{\partial f}{\partial b} = 2(\ln(bx^*(b) + 1))\frac{x^*(b)}{bx^*(b)+1}$, $\frac{\partial f}{\partial x} = 2(\ln(bx^*(b) + 1))\frac{x^*(b)}{bx^*(b)+1} - 1$, so $\frac{\partial x^*(b)}{\partial b} = -\frac{2(\ln(bx^*(b)+1))\frac{x^*(b)}{bx^*(b)+1}}{2(\ln(bx^*(b)+1))\frac{x^*(b)}{bx^*(b)+1} - 1}$.
3. So

$$f_1 = \frac{\partial u_1}{\partial x_1} = x_2 - 2c_1x_1 = 0$$
$$f_2 = \frac{\partial u_2}{\partial x_2} = x_1 - 2c_2x_2 = 0.$$

Now we can apply the implicit function theorem in more than one dimension. We need to compute two Jacobians. The first is with respect to **b**: $J_{\mathbf{x}}$. This is

$$\begin{pmatrix} -2c_1 & 1 \\ 1 & -2c_2 \end{pmatrix}.$$

We need the inverse of this matrix, so we first compute the determinant: $4c_1c_2 - 1$.

Now we can find the inverse:

$$J_{\mathbf{b}}^{-1} = \frac{1}{4c_1c_2} \begin{pmatrix} -2c_2 & -1 \\ -1 - 2c_1 & \end{pmatrix}.$$

Finally, we need to find the second Jacobian, $J_{\mathbf{c}} = \begin{pmatrix} -2x_1 & 0 \\ 0 & -2x_2 \end{pmatrix}$.

We can put these together to get:

$J_{\mathbf{x}^*(\mathbf{c})} = \frac{-1}{4c_1c_2 - 1} \begin{pmatrix} 4c_2x_1 2x_2 & \\ 2x_1 & 4c_1x_2 \end{pmatrix}$. If the costs are sufficiently high then each person's choice decreases in both people's costs. However, if they are sufficiently low ($4c_1c_2 < 1$), then each person's choice increases in both costs. This kind of result indicates the power of considering both choices at the same time.