# Selected Answers with Some Worked Problems for *A Mathematics Course for Political and Social Research* by Will H. Moore and David A. Siegel

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# 1 Chapter 1

## 1.1 Constants and Variables and Levels of Measurement

1. Identify whether each of the following is a constant or a variable:

    (a) Constant.

    (b)

    (c) Variable.

    (d)

    (e) Constant.

    (f)

    (g) Variable.

2. Identify whether each of the following is a variable or a value of a variable:

    (a) Value (of the variable Crisis).

    (b)

    (c) Value (of the variable Income).

    (d)

    (e) Value (of the variable Party identification, USA).

    (f)

    (g) Variable (with values No, Yes, given Bill as the unit of observation).

    (h)

    (i) Variable (with values No, Yes, given country–year as the unit of observation).

3. Identify whether each of the following indicators is measured at a nominal, ordinal, interval, or ratio level. Note also whether each is a discrete or a continuous measure:

    (a) Ordinal, discrete.

    (b)

    (c) Ratio, continuous.

    (d)

    (e) Ordinal, discrete.

    (f)

    (g) Ratio, discrete.

### 1.2 Sets, Operators, and Proofs

4.

5. Let $A = \{1, 5, 10\}$ and $B = \{1, 2, \dots, 10\}$.

   (a) $A \subset B$ because all the elements of $A$ are contained in $B$, but the reverse is not true.

   (b)

   (c) $A$ because $A$ is entirely contained within $B$, so the intersection between the two sets is just the set $A$ itself.

   (d)

   (e) $B$ because $B$ was partitioned into the sets $A$ and $C$, implying that the elements of $A$ and $C$ together make up the entirety of $B$.

   (f)

6. Anything with two components, e.g., $(0.5, 1.5)$. The Cartesian product creates a larger two-dimensional space from the two one-dimensional spaces, and all components of the space (i.e., the elements of the ordered pair) must be in the constituent sets. Thus the first element must be in $[0, 1]$ and the second in $(1, 2)$.

7.

8. Choose two even numbers $a$ and $b$. Since they are even we can write them as, respectively, $2c$ and $2d$ for integers $c, d$. Their sum is $a + b = 2c + 2d$. We can pull out the 2 on the RHS to get $2(c + d)$. As this is an even number (a multiple of 2), it must be true that the sum of any two even numbers is even.

   Now assume $a$ and $b$ are both odd. An odd number can be written as the sum of an even number plus one, so we can write $a, b$ as $2c + 1, 2d + 1$, respectively. Their sum is $a + b = 2c + 1 + 2d + 1$. Again we simplify the RHS to get $2(c + d + 1)$, which is an even number. Thus the sum of any two odd numbers is even.

   Finally, assume without loss of generality (because it doesn't matter which we choose to be even) that $a$ is even and $b$ is odd. Then we can write them as, respectively, $2c$ and $2d + 1$. Their sum is $a + b = 2c + 2d + 1$ which, after simplification, is $2(c + d) + 1$, an odd number. thus the sum of any odd and any even number is odd.

# 2 Chapter 2

## 2.1 Arithmetic Rules

1. $x$

2.

3. $x_1 + x_2 + x_3 + x_4$

4.

5. $4 \times 3 \times 2 \times 1 = 24$

6.

7. $\pm 3$

8. 3, because $3 \times 3 \times 3 = 27$

9. $\frac{-216}{125}$

## 2.2 Ratios, Proportions, Percentages

10. Represent the following as a ratio, a proportion, and a percentage:

    (a)

    (b) Ratio: 221:312.
    Proportion: $\frac{221}{221+312} = \frac{221}{533} \simeq 0.41$.
    Percentage: $100(\frac{221}{533}) \simeq 100(0.41) = 41\%$.

    (c)

11.

12.

13. $\frac{62-56}{56}$; 62-56=6, and $\frac{6}{56}$=.10714 or 10.7%.

14. 3:2 or $\frac{3}{2}$.

## 2.3 Algebra Practice

15. Simplify into one term the following expressions:

    (a) $z(x + y)$.

    (b)

    (c) $x \times y \times (z - 2)$, using the distributive property.

    (d)

16.

17. First you can group like terms, yielding $\frac{21x+12}{42x}$,
then break into two fractions to get $\frac{21x}{42x} + \frac{12}{42x}$,
then cancel the $x$s (as long as $x \neq 0$) and divide top and bottom by 21 in the first term to get $\frac{1}{2} + \frac{12}{42x}$,
then finally divide the numerator and denominator of the second term by 6 to get $\frac{1}{2} + \frac{2}{7x}$.

18.

19. $-7(\theta - 2)(\theta - 1)$.

20.

21. $(q - 9)(q - 1)$.

22.

23. $15\delta - 6\delta$ leaves $9\delta$; $9\delta + 45 = 36$.
Subtract 45 from each side; $9\delta = -9$.
Divide both sides by 9; $\delta = -1$.

24.

25. Expand $(y + 1)2$; $11 = 2y + 2 + (6y - 12y)\frac{7}{2y}$.
Subtract $12y$ from $6y$; $11 = 2y + 2 + \frac{-6y(7)}{2y}$.
Divide the top and bottom of the second term by $2y$ (as long as $y \neq 0$); $11 = 2y + 2 + (-3)(7)$.
Then multiply; $11 = 2y + 2 - 21$,
add; $11 = 2y - 19$,
add 19 to both sides; $30 = 2y$,
and finally divide both sides by 2; $y = 15$.

26.

27. Add 14 to each side; $x^2 + 14x = 14$.
$(\frac{14}{2})^2 = 49$, so add 49 to each side; $x^2 + 14x + 49 = 63$. Factor this term to complete the square; $(x + 7)^2 = 63$.
Take the square root of each side so $x + 7$ equals $\sqrt{63}$ or $-\sqrt{63}$.
Finally subtract 7 from both sides to get $x = \sqrt{63} - 7$ and $x = -\sqrt{63} - 7$.

28.

29. $x = \frac{-b \pm \sqrt{b^2 - 4(ac)}}{2(a)}$. $x = \frac{-5 \pm \sqrt{5^2 - 4(2 \times (-7))}}{2(2)}$.
$x = \frac{-5 \pm \sqrt{25 - 4(-14)}}{4}$.
$x = \frac{-5 \pm \sqrt{25 - (-56)}}{4}$.
$x = \frac{-5 \pm \sqrt{81}}{4}$.

$x = \frac{-5\pm 9}{4}$.
$x = \frac{4}{4} = 1$ or $x = \frac{-14}{4} = \frac{-7}{2} = -3.5$.

30. $x^2 + \frac{b}{a}x = -\frac{c}{a}$. $(x + \frac{b}{2a})^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$. $(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$, where we've multiplied $-\frac{c}{a}$ by $4a^2$ and eliminated an $a$ (as long as $a \neq 0$). $(x + \frac{b}{2a}) = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$. $(x + \frac{b}{2a}) = \pm\frac{\sqrt{b^2-4ac}}{2a}$. $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

31. Multiply each side by 7, so $-7\delta > \delta + 4$.
Add $-\delta$ to each side, so $-8\delta > 4$.
Divide both sides by $-8$, so $\delta < -\frac{1}{2}$. Don't forget to flip the inequality when dividing by a negative number!

# 3 Chapter 3

1. For each pair of ordered sets, state whether it represents a function or a correspondence:

   (a)

   (b) Function, since no element of the first set maps to more than one element of the second.

   (c)

2.

3. We insert $g(x)$ into the $x$ in $f(x)$ to get $h(x) = (\sqrt{x-4})^2 + 2 = x - 2$. This is not the same as the previous answer. Function composition does not in general commute.

4.

5. $x^{-2} \times x^3 = x$.

6.

7. $(qr)^{\gamma\delta}$, since $(x^n)^m = x^{nm}$.

8.

9. $\ln\left(\frac{3x}{(x+2)^2}\right)$.

10.

11.

12.

13.

14.

15.

16. $\ln(y) = \ln(\alpha) + \beta_1 \ln(x_1) + \beta_2 \ln(x_2) + \beta_3 \ln(x_3)$. Yes. Note we used two rules here: $\ln(xy) = \ln(x) + \ln(y)$ and $ln(x^a) = a \ln(x)$.

17.

18. Is this problem done correctly? Yes or no.
    Take the log of both sides of the following equation:
    $y = x_1^\beta - x_2^n + x_3^2$.
    Solution: $\log y = \beta(\log x_1) - n \log x_2 + 2 \log x_3$.
    (Note: the textbook should have included the previous lines in this problem) No, you can't distribute the log across the sum.

19.

20.

21. Answers will vary, but one common one is the quadratic loss function $u(x) = -(x - z)^2$ where $x$ is a policy (e.g., status quo or bill) and $z$ is one's ideal policy. This function is negative everywhere except at $x = z$, its maximum. We'll learn more about maxima in Chapter 8.

22.

# 4 Chapter 4

1. Your "graph" should be a series of points alternating 1 and $-1$. If you connect all the 1 points via a line, and all the $-1$ points via a line, you'll see that the two lines remain parallel forever. Or, if you connect all points, you'll see that the zigzag you've made never decreases in amplitude (i.e., it never shrinks). These indicate the lack of convergence in the sequence.

2. In 4.1.2 we showed that $\sum_{t=0}^{\infty}(\delta^t) = \frac{1}{1-\delta}$ as long as $\delta < 1$. We have $\delta^2$, but if $\delta < 1$, so is $\delta^2$ and we can use the same expression. Let $\epsilon = \delta^2$. Then our problem becomes $\sum_{t=0}^{\infty}(\epsilon^t) = \frac{1}{1-\epsilon}$, where we've made use of the fact that $(x^a)^b = x^{ab} = x^{ba} = (x^b)^a$. All that's left is to substitute $\delta$ back in to get our answer: $\frac{1}{1-\delta^2}$.

3.

4.

5. First start by factoring the numerator. We can pull out a 3 to get $3(x^2 - 4)$ and then factor the term in parentheses to get $3(x + 2)(x - 2)$. The $(x - 2)$ on top and bottom cancel, leaving $3(x + 2)$. This has a definite value at $x = 2$: 12. Thus the limit exists, and is equal to 12.

6.

7. For each of the following sets, state whether they are (a) open, closed, both, or neither; (b) bounded; (c) compact; (d) convex:

   (a) (a) closed, (b) bounded, (c) compact, (d) convex
   (b)
   (c)
   (d)
   (e) (a) closed (its complement is open), (b) not bounded (it goes off to infinity without bound, (c) not compact (because it's not bounded), (d) convex (it contains all lines between any two points)

8.

9. This function is not continuous because it has the value 22 at $x = 3$, but approaches 9 as it approaches $x = 3$ from the right. One could make it continuous by either subtracting 13 from the equation for all $x \leq 3$ (to get $x^3 - 3x - 9$), or by adding 13 to the equation for all $x > 3$ (to get $x^2 + 13$).

# 5 Chapter 5

1.

2. Use the definition of the derivative to find the derivative of $y$ with respect to $x$ for the following:

   (a)
   (b) $dy/dx = \lim_{h\to 0} \frac{3(x+h)^2-3x^2}{h} = \lim_{h\to 0} \frac{6xh+3h^2}{h} = \lim_{h\to 0} 6x + 3h = 6x.$
   (c)
   (d)
   (e) $dy/dx = \lim_{h\to 0} \frac{(x+h)^8-x^8}{h} = \lim_{h\to 0} \frac{8x^7h+O(h^2)}{h} = \lim_{h\to 0} 8x^7 + O(h) = 8x^7$, where $O(h^2)$ and $O(h)$ are polynomials of power (order) $h^2$ and greater and $h$ and greater, respectively. Since all these terms vanish in the limit, we don't need to compute them.
   (f) $dy/dx = \lim_{h\to 0} \frac{4(x+h)^3-(x+h)+1-4x^3+x-1}{h} = \lim_{h\to 0} \frac{12x^2h+12xh^2+4h^3-h}{h} = \lim_{h\to 0} 12x^2 + 12xh + 4h^2 - 1 = 12x^2 - 1.$
   (g)
   (h) $dy/dx = 25x^4 + 16x^3 + 9x^2 + 4x + 1.$
   (i)
   (j) $dy/dx = 81x^2 + 10x - 1.$

3. (a)
   (b) Sketches omitted. $f'(x) = 3x^2 - 1$, which equals 2 at $x = 1$.

4. For each of the following, find the partial derivative with respect to $x$.
   (a)
   (b) To find the partial derivative, perform the same procedure as for a full derivative, but treat all other variables as constants. So, $\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{(x+h)^2+2z^2-x^2-2z^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} 2x + h = 2x$.
   (c)

5. For each of the following, find the partial derivative with respect to $z$.
   (a)
   (b) $\frac{\partial f}{\partial z} = 6z$.
   (c)
   (d)
   (e) $\frac{\partial f}{\partial z} = \lim_{h\to 0} \frac{11(z+h)+5x^2(z+h)+7y(z+h)^2+3x^2y-11z-3x^2y-5x^2z-7z^2y}{h} = \lim_{h\to 0} \frac{11h+5x^2h+14yzh+7yh^2}{h} = \lim_{h\to 0} 11 + 5x^2 + 14zy + 7yh = 11 + 5x^2 + 14yz$.
   (f)
   (g) $\frac{\partial f}{\partial z} = 16xyz + 28x^2z$.

# 6 Chapter 6

1. Find the derivative of $y$ with respect to $x$ for the following, using the rules in this chapter:
   - $dy/dx = 0$.
   - 
   - $dy/dx = 3x^2 - 4x$.
   - 
   - $dy/dx = 8x^7$.
   - 
   - $dy/dx = 3ax^2$.
   - 
   - $dy/dx = nax^{n-1}$.
   - 
   - $dy/dx = (13x + 2x^3)(5x^4 - 4) + (13 + 6x^2)(x^5 - 4x + r)$.

- 
- $dy/dx = ((2x)(x+1) - (x^2+1)(1))/(x+1)^2$.
- 
- $dy/dx = 40x^7 + 70x^6 - 30x^5 - 25x^4 + 12x^3 + 21x^2 - 4x + 1$.
- 
- $dy/dx = 4x^3 - 3x^2 + 2x - 1$.
- 
- $dy/dx = (5x^3 + 4x^2 + 3x + 2)(35x^4 + 24x^3 + 15x^2 + 8x) + (15x^2 + 8x + 3)(7x^5 + 6x^4 + 5x^3 + 4x^2)$.
- 
- $dy/dx = 25x^4 + 9x^2 + 1 - 16x^3 - 4x$.
- 
- $dy/dx = 2(x^2 + x + 2)(2x + 1)$.
- 
- $dy/dx = \frac{(3x^2+2x+1)(x^2+x+1)-(x^3+x^2+x+1)(2x+1)}{(x^2+x+1)^2}$.

2. Using the rules in this chapter (i.e., you don't need to go back to the definition), differentiate the following:

   (a) Using the rule $\frac{dx^n}{dx} = nx^{n-1}$ for each term we get the series $f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1$. This is the series $\sum_{i=1}^{n} i a_i x^{i-1}$.

   (b)

   (c)

   (d) We first let $f(x) = h(g(x))$ where $h(x) = e^x$ and $g(x) = x - \ln(x) + 5$. Next note that $h'(x) = e^x$ and $g'(x) = 1 - \frac{1}{x}$. Finally, we use the chain rule to get $f'(x) = h'(g(x))g'(x) = e^{g(x)}g'(x) = e^{x-\ln(x)+5}(1 - \frac{1}{x})$.

   (e)

   (f) $f'(x) = \ln(a)a^x x^2 + 2a^x x - \ln(b)b^{x^2}(2x)$.

   (g)

   (h) We let $f(x) = h(g(x))$ where $h(x) = e^x$ and $g(x) = 5x^2 + x + 3$. Next note that $h'(x) = e^x$ and $g'(x) = 10x + 1$. Finally, we use the chain rule to get $f'(x) = h'(g(x))g'(x) = e^{g(x)}g'(x) = e^{5x^2+x+3}(10x + 1)$.

   (i)

   (j) $f'(x) = \frac{1}{4}e^{\frac{x}{2}}$.

   (k)

   (l) We let $f(x) = h(g(x))$ where $h(x) = e^x$ and $g(x) = 7x^3 + 5x^2 - 3x + \ln(x) - 7$. Next note that $h'(x) = e^x$ and $g'(x) = 21x^2 + 10x - 3 + \frac{1}{x}$. Finally, we use the chain rule to get $f'(x) = h'(g(x))g'(x) = e^{g(x)}g'(x) = e^{7x^3+5x^2-3x+\ln(x)-7}(21x^2 + 10x - 3 + \frac{1}{x})$.

(m)

(n) $f'(x) = 2ax(e^{\ln(x)+2x^2}) + ax^2 e^{\ln(x)+2x^2}(\frac{1}{x} + 4x) + 36x^3$.

3. Begin by noting that $a^{\log_a(x)} = x$, since the two are inverse functions of each other. Now differentiate both sides, using the chain rule on the LHS. The RHS has a derivative of 1. To get the LHS, let $g(x) = a^x$ and $h(x) = \log_a(x)$. $g'(x) = \ln(a)a^x$ and $h'(x)$ is what we want to find. The chain rule says that the derivative of the LHS is $\ln(a)a^{\log_a(x)}h'(x)$. Replacing $a^{\log_a(x)}$ with $x$ and putting this all together produce the equation $\ln(a)xh'(x) = 1$ or $\frac{d\log_a(x)}{dx} = \frac{1}{x(\ln(a))}$, which is what we were to prove.

# 7 Chapter 7

1.

2.

3. Integrate the following derivatives to find $y$:

(a)

(b) Using the rules $\int af(x)dx = a\int f(x)dx$ and $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$, we get $y = \int \frac{dy}{dx}dx = \int 3x^2 dx = 3\int x^2 dx = 3\frac{1}{3}x^3 + C = x^3 + C$.

(c)

(d) $y = -x + C$.

(e)

(f) Using the rules $\int af(x)+bg(x)dx = a\int f(x)dx + b\int g(x)dx$ and $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$, we get $y = \int \frac{dy}{dx}dx = 5\int x^4 dx - \int xdx - 4\int dx = 5(\frac{1}{5})x^5 - \frac{1}{2}x^2 - 4x + C = x^5 - \frac{1}{2}x^2 - 4x + C$.

(g)

(h)

(i)

(j) $y = \frac{3}{4}x^4 - \frac{4}{3}x^3 + \frac{5}{2}x^2 - 6x + C$.

(k)

(l)

(m) Using the rules $\int af(x)dx = a\int f(x)dx$, $\int e^x dx = e^x + C$, and substitution, we get $y = \int \frac{dy}{dx}dx = 2\int e^{5x}dx$. Letting $u = 5x$, we have that $du = 5dx$ or $dx = \frac{1}{5}du$. We can now rewrite our integral as $y = \frac{2}{5}\int e^u du = \frac{2}{5}e^u + C = \frac{2}{5}e^{5x} + C$.

(n) While we could use integration by parts here, we note that $(20x+2)$ is two times the derivative of $5x^2 + x$, so if we set $u = 5x^2 + x$ then $du = (10x+1)dx$ or $dx = \frac{1}{10x+1}du$ and via substitution our integral becomes $2\int(10x+1)e^u \frac{1}{10x+1}du = 2\int e^u du = 2e^u + C$, so that $y = 2e^{5x^2+x} + C$.

(o)

(p) This one is a bit tricky. It looks at first glance like it would be amenable to substitution with $u = x^2$, but note that there is no $2x$ multiplying the log to make substitution work. Instead try integration by parts with $f(x) = \ln(x^2)$ and $g'(x) = 1$. We find $f'(x) = \frac{1}{x^2}2x = \frac{2}{x}$ where we have used the chain rule, and $g(x) = x$. Then the integral is $x\ln(x^2) - \int \frac{2}{x}xdx$. This last integral is $\int 2dx = 2x + C$, so that our answer is $y = x\ln(x^2) - 2x + C$.

4.

5. Which of the options below best describes $\int_a^b \frac{dy}{dx}dx$?

6. Compute the following integrals:

(a)

(b) $\frac{6}{5}x^{5/2} + 8x^{-1/4} + \frac{1}{\ln(4)}4^x + C$.

(c)

(d) $\frac{1}{2}(\ln(\frac{1}{x}))^2 + C$.

(e)

(f) $\frac{2}{15}(x^3 + 15)^{5/2} + C$.

(g) Let $u = x^3 - 2x^2 + 5x$. Then $du = (3x^2 - 4x + 5)dx$ and the integral becomes $4\int \frac{1}{u}du = 4\ln|x^3 - 2x^2 + 5x| + C$.

(h)

(i) Using the first power rule, the indefinite integral is $x^3 + \frac{1}{2}x^2 + 5x + C$. The definite integral is $[x^3 + \frac{1}{2}x^2 + 5x]_2^4 = 64 + 8 + 20 - 8 - 2 - 10 = 72$.

(j)

(k) $\frac{23328}{5} + 648 + 72 + 18 + 6 - \frac{96}{5} - 8 - \frac{8}{3} - 2 - 2 = \frac{23232}{5} + 732 - \frac{8}{3}$.

(l)

# 8 Chapter 8

1. Find all extrema (local and global) of the following functions on the specified domains, and state whether each extremum is a minimum or maximum and whether each is only local or global on that domain.

(a) I will work through the method. Step one: $f'(x) = 3x^2 - 1$. Step two: $f'(x^*) = 0 \Rightarrow 3x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{3}$ or $x^* = \pm\frac{1}{\sqrt{3}}$. Only the positive value is in the domain, so it is the only stationary in this domain. Step 3: $f''(x) = 6x > 0$ in the entire domain $[0, 1]$. Since only $x^* = \frac{1}{\sqrt{3}}$ is in this domain and the second derivative is positive, $\frac{1}{\sqrt{3}}$ is a local minimum of the function. Step 5: $f(x^*) = 3^{-3/2} - 3^{-1/2} + 1 \sim 0.6$. Step 6: $f(0) = 1, f(1) = 1$. Step 7: $x^* = \frac{1}{\sqrt{3}}$ is the global minimum, while 0 and 1 are global maxima.

(b)

(c) Global minimum at $x^* = 0$, global maximum at $x^* = 2$.

(d)

(e) Global maximum at $x^* = 3$, global minimum at $x = 10$.

(f)

(g)

(h) Two stationary points are local maximum and minimum, respectively: $x^* = -\frac{1}{2}$ and $x^* = \frac{1}{3}$. Global minimum at $x = -3$, global maximum at $x = 3$.

(i) (Note: should change bounds to $[-0.8, 1]$. These will be the ones used in this answer.) Global minimum at $x = -0.8$, global maximum at $x^* = \frac{\sqrt{3}-1}{2}$.

(j) Three stationary points at $x^* = -1, 2, 3$. Global minimum at $x^* = -1$, global maximum at $x^* = 2$, local minimum at $x^* = 3$.

2. Step one: $f'(x) = -2a(x - x_0)$. Step two: $f'(x^*) = 0 \Rightarrow x^* = x_0$. Step 3: $f''(x) = -2a < 0$ for all $a > 0$, so the stationary point $x_0$ is a global maximum.

3.

# 9 Chapter 9

1. Identify the following as objective or subjective probability claims:

   (a) Objective.

   (b)

   (c) Objective.

   (d)

   (e) Objective (presumably based on polling data).

2. Identify each of the following as a *classical* (objective), *empirical* (objective), or *subjective* probability claim:

   (a) *Empirical* (objective).

   (b)

   (c) *Empirical* (objective).

3. Identify the following as simple or compound events:

   (a) Simple.

   (b)

   (c) Compound.

(d)

(e) Compound.

4. Characterize the following as independent, mutually exclusive, and/or collectively exhaustive:

   (a) Independent (well, largely: gender, race, and age are associated with income level, but the other three are independent).

   (b)

   (c) Independent (again, largely: an incumbent political party's vote share is associated with macroeconomic activity).

   (d)

   (e) Mutually exclusive and collectively exhaustive values of a variable.

5. If $a$ and $b$ are independent events, are the following true or false?

   (a) True

   (b)

   (c) True.

6. The joint probability is 1, as mutually exclusive and collectively exhaustive imply that only one of $a$ and $b$ can happen at the same time, but one of $a$ or $b$ must always happen. In other words, one and only one of $a$ and $b$ must always happen, implying that the chance of one's happening is 1.

7.

8. Mutually exclusive events cannot both occur, thus the chance that any two of these events occurs at once, let alone all four, is 0.

9.

10. In general, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and $P(A \cap B) = P(A|B)P(B)$ which, when combined, yield: $P(A \cup B) = P(A) + P(B) - P(A|B)P(B)$. If the two events are independent then $P(A|B) = P(A)$, giving $P(A \cup B) = P(A) + P(B) - P(A)P(B) = P(B)(1 - P(A)) + P(A)$. We solve for $P(B)$ to get $P(B) = \frac{P(A \cup B) - P(A)}{1 - P(A)} = \frac{0.5 - 0.3}{1 - 0.3} = \frac{2}{7}$.

11.

12. $\frac{0.6 - 0.4}{1 - 0.4} = \frac{1}{3}$.

13. Compute each of the following:

    (a)

    (b) $\frac{5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{6}$.

(c)
(d) $\frac{7!}{(5!)(2!)} = 21$.
(e)
(f) $\frac{7!}{5!} = 42$.

14.

15.

16. A committee contains fifteen legislators with ten men and five women. Find the number of ways that a delegation of six:

    (a) This is the number of ways 6 elements can be chosen from 15, or $\binom{15}{6}$.
    (b) Now we have the joint probability of two independent events: choosing 3 women from 5 and 3 men from 10. This is: $\binom{10}{3}\binom{5}{3}$.
    (c) Finally, we have the joint probability of two independent events: choosing 2 women from 5 and 4 men from 10, since there are twice as many men as women in the full group. This is: $\binom{10}{4}\binom{5}{2}$.

17.

18. $\frac{135}{512}$.

19.

20.

21.

22.

23. In a certain city, 30% of the citizens are conservatives, 30% are liberals, and 40% are independents. In a recent election, 50% of conservatives voted, 40% of liberals voted, and 30% of independents voted.

    (a) $Pr(V) = Pr(V|C)Pr(C) + Pr(V|L)Pr(L) + Pr(V|I)Pr(I) = 0.39$.
    (b) $Pr(C|V) = \frac{PR(V|C)Pr(C)}{PR(V)} = \frac{(.5)(.3)}{.39} = 0.38$.
    (c)
    (d)

24.

25. (3/1)/(1/2) = 3/0.5 = 6

26. An unemployed person is 142% more likely than an employed person, and someone with a college education is 62% less likely than a person with only a high school education, to vote for the National Front.

# 10 Chapter 10

1.

2.

3. As noted in Section 10.4, "the probability that we will randomly select any given value from a set with infinite support is zero." Since a PDF describes the distribution of a continuous random variable (with infinite support), plotting these zeros would be uninformative.

4.

5.

6.

7.

8.

9. We use a Poisson distribution with these data: $\frac{3^4}{e^3 4!} = 0.168$.

10. Bach: $EU(B) = p(B)EU(BT) + (1 - p(B))EU(BA)$, where B is Bach, T is Together and A is alone.
$EU(B) = .5(18) + .5(8) = 9 + 4 = 13$.
$EU(S) = p(B)EU(SA) + (1 - p(B))EU(ST)$, where S is Stravinsky.
$EU(S) = .5(3) + .5(13) = 1.5 + 6.5 = 8$.
So she should choose Bach because $13 > 8$. The second part of the question is the same, just with changed values for $p(B)$. $EU(B) = .3(18) + .7(8) = 5.4 + 5.6 = 11$.
$EU(S) = .3(3) + .7(13) = 0.9 + 9.1 = 10$.
So she should again choose Bach because $11 > 10$, though the decision is closer.

11. First we use Bayes' rule to figure out the posterior probability of B's being strong after A observes maneuvers. This is $Pr(S|M) = \frac{Pr(M|S)Pr(S)}{Pr(M|S)Pr(S)+Pr(M|W)Pr(W)} = \frac{(.6)(.4)}{(.6)(.4)+(.3)(.6)} = \frac{4}{7}$. Next we ask what this posterior probability must be for A to want to start a war. This is an expected utility comparison. A gets 0 for doing nothing, and expects to get $Pr(S|M) * (-1) + (1 - Pr(S|M)) * (1)$ for starting a war after observing maneuvers. This becomes $-\frac{4}{7} + \frac{3}{7} = -\frac{1}{7} < 0$, so A would prefer not to start a war after observing maneuvers.

# 11 Chapter 11

1.

2.

3. A PDF provides the relative likelihood (though not the probability) of drawing a given value from the population and the probability of drawing values in any defined region of the variable's support, while the CDF provides the probability of drawing any value less than or equal to a given value.

4.

5. One might think that any PDF that has a "bell shape" would be normal, but as the PDFs in this chapter demonstrate, some normal distributions do not look canonically bell shaped, and many other PDFs "look" more or less indistinguishable from the normal. As such, statisticians have developed specific tests that permit one to draw an inference about the likelihood that a given variable was drawn from a normal (or other) distribution(s).

6.

7.

8. $\text{Var}(X) = \int (x-\mu)^2 f(x)dx = \int x^2 f(x)dx - 2\mu \int x f(x)dx + \mu^2 \int f(x)dx = E[x^2] - 2\mu \times \mu + \mu^2 = E[x^2] - \mu^2$.

9. $Pr(X > 10) = 1 - F(10) = 1 - \frac{10}{25} = 0.6$.

10. The mean tells us the average value of the distribution, and it indicates that the average country possesses a deficit of a hundred million dollars. The standard deviation tells us how spread out the distribution is around its average, and it indicates that approximately 68% of countries have budget deficits/surpluses between −\$400 million and \$200 million.

11.

12.

13.

# 12 Chapter 12

1. Let: $\mathbf{a} = \begin{pmatrix} 10 \\ 2 \\ 5 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 15 \\ 6 \\ 8 \end{pmatrix}$, $\mathbf{c} = (2, 6, 8)$, $\mathbf{d} = (1, 15, 12)$, $\mathbf{e} = (14, 17, 11, 10)^T$, and $\mathbf{f} = (20, 4, 10, 4)^T$. Calculate each of the following, indicating that it's not possible if there is a calculation you cannot perform.

   (a)

   (b) Not possible to add because they have different dimensions.

   (c)

(d) $(30, 90, 120)$.

(e)

(f)

(g) This equals $||(3, 21, 20)|| = \sqrt{9 + 441 + 400} = 29.2$. The triangle inequality says this must be less than $||\mathbf{c}|| + ||\mathbf{d}|| = \sqrt{104} + \sqrt{370} = 29.4$. Since this is true, the inequality holds.

(h)

(i) $(10)(4) + (2)(15) + (5)(6) + (2)(8) = 116$.

(j)

2. Identify the following matrices as diagonal, identity, square, symmetric, triangular, or none of the above (note all that apply).

(a) This is a symmetric, square matrix.

(b)

(c) This matrix has the same number of rows and columns, so it is square, but that is it. (The off diagonal elements are not equal, so it is not symmetric, and it has non-zero elements both above and below the diagonal, so it is neither triangular nor diagonal (or the identity).

(d)

3. (a) $A^T = \begin{bmatrix} 0 & 1 & 5 \\ 1 & -2 & -1 \\ 5 & -1 & 2 \end{bmatrix}$.

(b)

(c) The transpose leaves the diagonal alone while flipping the row and column position of all off-diagonal elements. So $C^T = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$.

(d)

4. Given the following matrices, perform the calculations below.

$$A = \begin{bmatrix} 5 & 1 & 2 \\ 6 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 4 & 5 \\ -2 & -3 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ -5 & 3 \\ -3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

(a) Not possible to add because they have different dimensions.

(b)

(c) $\begin{bmatrix} 5 + 5(3) & 1 + 5(4) & 2 + 5(5) \\ 6 + 5(-2) & 2 + 5(-3) & 3 + 5(6) \end{bmatrix} = \begin{bmatrix} 20 & 21 & 12 \\ -4 & -13 & 33 \end{bmatrix}$.

(d)

(e)

(f) $B^T = \begin{bmatrix} 3 & -2 \\ 4 & -3 \\ 5 & 6 \end{bmatrix}$, so $B^T - C = \begin{bmatrix} 2 & -4 \\ 9 & -6 \\ 8 & 5 \end{bmatrix}$.

(g)

(h) $\begin{bmatrix} 16 & 4 & 7 \\ 38 & 10 & 17 \end{bmatrix}$.

(i)

(j) $\begin{bmatrix} (1)(2)+(2)(4) & (1)(1)+(2)(3) \\ (-5)(2)+(3)(4) & (-5)(1)+3(3) \\ (-3)(2)+(1)(4) & (-3)(1)+(1)(3) \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 2 & 4 \\ -2 & 0 \end{bmatrix}$.

(k)

(l) $\begin{bmatrix} -1 & -2 & 17 \\ -21 & -29 & -7 \\ -11 & -15 & -9 \end{bmatrix}$

5. Find the determinants of the following matrices:

(a) 6.

(b)

(c) We cannot take the determinant of a matrix that is not square.

(d)

(e) $-3$.

6. If it exists, find the inverse of the following matrices:

(a)

(b) To find the inverse of a $2 \times 2$ matrix you multiply the inverse of the determinant by the matrix formed by swapping the two diagonal elements and flipping the sign of the off-diagonal terms. The determinant of $B$ is $(1)(2) - (4)(3) = -10$, so the inverse is: $B^{-1} = \frac{1}{-10}\begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}$.

7. Let: $A = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 1 \\ 5 & -1 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 2 & -3 \\ -1 & 1 \\ 1 & 4 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $E = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$, $F = \begin{pmatrix} 1 & 1 & 5 \\ 3 & -2 & -1 \\ 2 & 4 & 2 \end{pmatrix}$, $\mathbf{g} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{h} = (1, 2, 3)$. Calculate each of the following, indicating that it's not possible if there is a calculation you cannot perform.

(a) $\begin{bmatrix} 4 & 0 & 2 \\ 5 & 3 & 5 \end{bmatrix}$.

(b)

(c)

(d) $\begin{bmatrix} 9 & -6 & 3 \\ 0 & 12 & 6 \end{bmatrix}$.

(e)

(f) We cannot perform this calculation because the columns of the first matrix don't match up with the rows of the second.

(g)

(h) $\begin{bmatrix} -7 & -8 \\ 2 & 2 \\ 13 & 18 \end{bmatrix}$.

(i)

(j) $\begin{bmatrix} 6 & 10 \\ 12 & 20 \end{bmatrix}$.

(k)

(l) $\begin{bmatrix} 13 & 9 & 9 \end{bmatrix}$.

(m)

(n) $(2, 3, 1)$.

(o)

(p) 1.

(q)

(r) 72.

(s)

(t) $E$ has determinant zero and so is singular. Its inverse is undefined.

8. We compute $\boldsymbol{\beta}$ step by step. First, we form the matrix $X = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$. Next, we compute its transpose: $X^T = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. Next, we multiply: $X^T X = \begin{pmatrix} 5 & 7 \\ 7 & 13 \end{pmatrix}$. Next, we find the inverse of this product: $(X^T X)^{-1} = \frac{1}{16} \begin{pmatrix} 13 & -7 \\ -7 & 5 \end{pmatrix}$. Next, we multiply the last two terms: $X^T \mathbf{y} = \begin{pmatrix} 11 \\ 25 \end{pmatrix}$. Finally, we multiply these last two elements together to get $\boldsymbol{\beta} = (X^T X)^{-1} X^T \mathbf{y} = \frac{1}{16} \begin{pmatrix} -32 \\ 48 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

9. True or false?

(a)

(b) False. The inverse matrix can left- or right-multiply to produce the identity.

(c)

10. Why are the following useful?

    (a)

    (b)

11. A singular matrix is not invertible. This often has implications for the set of solutions to the system of equations represented by the matrix.

12. Fill in the blanks:

    (a)

    (b)

# 13 Chapter 13

1. From the previous chapter, let $\mathbf{a} = \begin{pmatrix} 10 \\ 2 \\ 5 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 15 \\ 6 \\ 8 \end{pmatrix}$, $\mathbf{e} = (14, 17, 11, 10)^T$, and $\mathbf{f} = (20, 4, 10, 4)^T$. Answer each of the following questions, saying that it's not possible if there is a calculation you cannot perform.

    (a)

    (b) No, only two at most are. For example, one can write $\mathbf{e} = \mathbf{a} + \mathbf{b}$ and $\mathbf{f} = 2\mathbf{a}$.

    (c) These span a two-dimensional space, as there are two, and only two, linearly independent vectors. (If you wrote $\mathbf{e}$ and $\mathbf{f}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ in the previous problem, then the latter two vectors are independent and span a two-dimensional space.)

2. Solve the following systems of equations using substitution or elimination, or both:

    (a) $x = 5, y = 6, z = 2$.

    (b)

    (c) There are only two equations and three unknowns, so we know we can't solve for all the unknowns. We'll try solving in terms of $z$. First we'll add two times the second equation to the first equation to eliminate $x$. This yields the equation $4y + 6z = 6$. We can solve this equation for $y$ in terms of $z$ to get $y = \frac{3}{2} - \frac{3}{2}z$. Now we can plug this in for $y$ in the second equation to get $2x - (\frac{3}{2} - \frac{3}{2}z) + z = 1 \Rightarrow x = \frac{5}{4} - \frac{5}{4}z$. Solving for $x, y$ in terms of $z$ is as far as we can go.

3.

4. From the previous chapter, let $D = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $E = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$, $\mathbf{g} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, and $\mathbf{h} = (1, 2, 3)$.

   Calculate each of the following, indicating that it's not possible if there is a calculation you cannot perform.

   (a)

   (b) $E$ has rank 1. It is singular, so its columns (or rows) are linearly dependent. Removing one of them leaves one linearly independent column (or row), making its rank 1.

   (c) It is not possible as stated, since $\mathbf{g}$ has dimension three and $D$ is of rank two. To see an example that works, let $\mathbf{g}$ instead equal $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Then $D^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$, and $D^{-1}\mathbf{g} = \frac{1}{-2}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix}$.

   (d) Since $E$ has determinant 0 it is singular, and does not have an inverse. Thus we can't use matrix inversion. As one of the rows is a multiple of the other we only have one unique equation for two variables, indicating an infinite number of solutions to the system of equations represented by the matrix.

5. Use Cramer's rule to solve the following systems of equations:

   (a)

   (b) $\det(A) = 3(6)+2(16)-1(-10) = 60, \det(B_1) = 4(6)+2(21)-1(-6) = 72, \det(B_2) = 3(21) - 4(16) - 1(-19) = 18, \det(B_3) = 3(6) + 2(-19) + 4(-10) = -60$. Thus $x = \frac{72}{60} = \frac{6}{5}, y = \frac{18}{60} = \frac{3}{10}, z = \frac{-60}{60} = -1$.

6.

7. A singular matrix is one that does not have an inverse, and has less than full rank. This implies that the system of equations the matrix represents is likely to be under- or overdetermined, and is not likely to be uniquely determined.

# 14 Chapter 14

1.

2. First we find the eigenvalues. $A - \lambda I = \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{pmatrix}$. $|A - \lambda I| = (2-\lambda)^3 - \lambda = (2-\lambda)((2-\lambda)^2 - 1) = (2-\lambda)(3 - 4\lambda + \lambda^2) = (2-\lambda)(3-\lambda)(1-\lambda)$. Setting this equal to zero gives the three eigenvalues $\lambda = 1, 2, 3$. Next we find the eigenvectors. For each eigenvalue we solve the equation $A\mathbf{x} = \lambda\mathbf{x}$ where we let $\mathbf{x} = \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$. This produces

a system of three equations for each eigenvalue that look like: $2 + a = \lambda, 1 + 2a = \lambda a, a + 2b = \lambda b$. For $\lambda = 1$, the first two equations both yield $a = -1$, while the third yields $b = 1$. For $\lambda = 3$, the first two equations both yield $a = 1$, while the third yields $b = 1$. For $\lambda = 2$, we have some problems as the normal method doesn't work (the first two equations are contradictory). We can assign the 1 to any spot, though, so we'll try the bottom spot instead so that $\mathbf{x} = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$. This produces the system of three equations $2a + b = 2a, a + 2b = 2b, b + 2 = 2$. The first and third equations yields $b = 0$, while the second yields $a = 0$. Now we put these eigenvectors together in the matrix $Q = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. $|Q| = -1(1+1) = -2$. To find the inverse we also need the minors of the matrix. These are: $M_{11} = -1, M_{12} = -2, M_{13} = -1, M_{21} = -1, M_{22} = 0, M_{23} = 1, M_{31} = 0, M_{32} = 2, M_{33} = 0$. Putting them together produces the inverse: $Q^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & -2 \\ -1 & -1 & 0 \end{pmatrix}$. The diagonal matrix is $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 01 & 0 & 3 \end{pmatrix}$. Putting these together one can check that $A = QDA^{-1}$, as required.

3. Yes, it is ergodic because it is aperiodic and all states communicate. To find the steady state $\boldsymbol{\pi}$, we solve $M\boldsymbol{\pi} = \boldsymbol{\pi}$, where $\boldsymbol{\pi} = \begin{pmatrix} 1 \\ a \end{pmatrix}$. This matrix equation yields the system of equations $.25 + .5a = 1, .75 + .5a = a$ which has the unique solution $a = 1.5$. Thus our steady state is $\begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$ or, after normalizing it, $\begin{pmatrix} \frac{2}{5} \\ \frac{3}{5} \end{pmatrix}$.

4.

5. This transition matrix is not ergodic. Though it is aperiodic, all states do not communicate. In particular, there are two absorbing states. Once one is in the first state (i.e., $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$) one stays there forever, and the same is true with the last state (i.e., $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$). The second state is transient. Each period one-third of the probability of being in that state "bleeds off" to the first state, half of it bleeds off to the third state, and the remaining one-sixth stays in the second state. Since no transitions from state one or state three to state two occur, the chance of being in the second state rapidly approaches zero. The matrix clearly describes a history dependent process. For example, if the system starts at $\begin{pmatrix} .5 \\ 0 \\ .5 \end{pmatrix}$ it stays there, but it also stays at $\begin{pmatrix} .75 \\ 0 \\ .25 \end{pmatrix}$ or any other such combination of the first and third states. Whether or not the process is path dependent beyond initial conditions depends on how one defines path dependence.

# 15 Chapter 15

1. The following variables derive from Fair (2009), who argues that the two-party vote share in US presidential elections can be modeled from a handful of variables, including:
   $y$ = Two Party Vote Share, an integer variable for the Democratic candidate's vote share that ranges from 0 to 100.
   $x$ = Macroeconomic Growth Rate, a continuous measure of the rate of growth of GNP, which ranges from $-\infty$ to $\infty$.
   $w$ = Good News Quarters, an integer count variable of the number of quarters in which macroeconomic growth was positive during the sitting president's term, which ranges from 0 to 15.
   $z$ = Incumbent, a binary measure coded 1 if the Democratic Party currently holds the presidency.

   (a) The answers are, respectively, $\beta_1, \beta_2, \beta_3$ and they correspond to the marginal effects of, respectively, $x, w, z$, holding all other variables constant.

   (b) Now the answers are, respectively, $\beta_1 + \beta_4 w, \beta_2 + \beta_4 x, \beta_3$. They have the same interpretation, save that the first two partial derivatives depend on the values of another parameter and so change with that parameter. This changes the meaning of, for example, $\beta_1$, which now only corresponds to the marginal change of the function in $x$ at $w = 0$.

2. The following variables derive from Hafner-Burton (2005), who studies governments' human rights violations as a function of preferential trade agreements, among other things.
   $z$ = Political Terror Score, an ordinal variable that measures the extent to which a government violates its obligations to respect people's human rights.
   $x_1$ = Preferential Trade Agreements, a binary measure coded 1 if the government has signed at least one trade treaty that lowers tariffs imposed by an OECD country *if* that government respect labor rights.
   $x_2$ = Human Rights Agreements, an ordinal count of whether the government has ratified two key human rights treaties.
   $x_3$ = Democracy, a 21-point integer-level variable from the Polity IV data where pure autocracy has a value of $-10$ and pure democracy a value of 10.

   (a)

   (b)

3.

4. $\frac{\partial f}{\partial x} = 3yz$. $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{dz}{dx} = 3yz + 3xy(3y)$.

5. First compute the constituent partials. $\frac{\partial f}{\partial x} = 3y + 5x$, $\frac{\partial f}{\partial y} = 3x + 3y$, $\frac{\partial^2 f}{\partial x^2} = 5$, $\frac{\partial^2 f}{\partial y^2} = 3$, $\frac{\partial^2 f}{\partial x \partial y} = 3$. So $\nabla f = \begin{pmatrix} 3y + 5x \\ 3x + 3y \end{pmatrix}$ and $H = \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}$. We now find the eigenvalues of the

Hessian: $|H - \lambda I| = 0 \Rightarrow (5-\lambda)(3-\lambda) - 9 = 0 \Rightarrow \lambda^2 - 8\lambda + 6 = 0 \Rightarrow \lambda = 4 \pm \sqrt{10}$. Both are positive, so the Hessian is positive semi-definite at all points and the function is convex.

6.

7. First do the integral over $y$, because the bounds of the inner integral contains an $x$. So $\int_0^x (y-1)^3 dy = \frac{1}{4}(y-1)^4|_0^x = \frac{1}{4}[(x-1)^4 - 1]$. Now do the outer integral over $x$. This is $\frac{1}{4}\int_0^1 x^5[(x-1)^4 - 1]dx = \frac{1}{4}\int_0^1 x^5[x^4 - 4x^3 + 6x^2 - 4x]dx$. This equals $\frac{1}{4}\int_0^1 x^9 - 4x^8 + 6x^7 - 4x^6 dx = (\frac{1}{40}x^{10} - \frac{1}{9}x^9 + \frac{3}{16}x^8 - \frac{1}{7}x^7)|_0^1 = \frac{1}{40} - \frac{1}{9} + \frac{3}{16} - \frac{1}{7}$.

# 16 Chapter 16

1.

2. First compute the components of the gradient: $\frac{\partial f}{\partial x} = -4x + 2y + 30$ and $\frac{\partial f}{\partial y} = -5y + 2x + 5$. Next set each to zero and solve the system of equations to find the critical points. Adding twice the second equation to the first eliminates $x$ and yields $-8y = -40$, or $y = 5$. Plugging this into the first question yields $x = 10$. Next find the Hessian. Its components are $\frac{\partial^2 f}{\partial x^2} = -4$, $\frac{\partial^2 f}{\partial y^2} = -5$, and $\frac{\partial^2 f}{\partial x \partial y} = 2$, so $H = \begin{pmatrix} -4 & 2 \\ 2 & -5 \end{pmatrix}$. We now find the eigenvalues of the Hessian: $|H - \lambda I| = 0 \Rightarrow (-4-\lambda)(-5-\lambda) - 4 = 0 \Rightarrow \lambda^2 + 9\lambda + 16 = 0 \Rightarrow \lambda = -9 \pm \sqrt{17}$. Both are negative, so the Hessian is negative semi-definite at all points and the function is concave, implying that the only critical point $(10, 5)$ is a maximum.

3.

4. $x^* = \sqrt{2}$, $y^* = 1$, $\lambda^* = 1$ and $x^* = -\sqrt{2}$, $y^* = 1$, $\lambda^* = 1$.

5.

6. Solve the following constrained optimization problems:

   (a)

   (b) First write $\Lambda = x - 3y^2 - \lambda(x^2 + y^2 - 2) + \mu_1 x + \mu_2 y$. Then obtain the five equations $\partial_x \Lambda = 1 - 2x\lambda + \mu_1 = 0$, $\partial_y \Lambda = -6y - 2y\lambda + \mu_2 = 0$, $\partial_\lambda \Lambda = x^2 + y^2 - 2 = 0$, $\mu_1 x = 0$, $\mu_2 y = 0$. We have four cases to go through in turn. First, try $\mu_1 = \mu_2 = 0$, so the non-negativity constraints do not bind. From the second equation either $y = 0$ or $\lambda = -3$. But if $\lambda = -3$ then the first equation implies $x < 0$, which violates the assumption that $\mu_1 = 0$. With $y = 0$ the third equation implies $x = \sqrt{2}$, which is consistent with the first equation. $x = \sqrt{2}, y = 0$ implies $u = \sqrt{2}$. Second, try $\mu_2 \geq 0, \mu_1 = 0$, so $y = 0$ but $x \geq 0$. But this is the exact same circumstances as in the previous case, and has the same outcome. Third, try $\mu_1 \geq 0, \mu_2 = 0$, so $x = 0$ but $y \geq 0$. But the first equation then implies $\mu_1 = -1$, which contradicts the assumption that $\mu_1 \geq 0$. So this is not a stationary point. Finally, try $\mu_1, \mu_2 \geq 0$, so that $x = y = 0$. But this doesn't satisfy the budget constraint. We thus have only one stationary point, which is our maximum: $x = \sqrt{2}, y = 0$.

# 17 Chapter 17

1.

2. Let $f = x^* - e^{ax^*} = 0$. First use total differentiation. $0 = \frac{df}{da} = \frac{\partial x^*}{\partial a} - x^* e^{ax^*} - ae^{ax^*} \frac{\partial x^*}{\partial a}$. This implies $(1 - ae^{ax^*})\frac{\partial x^*}{\partial a} = x^* e^{ax^*} \Rightarrow \frac{\partial x^*}{\partial a} = \frac{x^* e^{ax^*}}{1 - ae^{ax^*}}$. Now try the implicit function theorem. It says $\frac{\partial x^*}{\partial a} = \frac{-\frac{\partial f}{\partial a}}{\frac{\partial f}{\partial x}} = \frac{x^* e^{ax^*}}{1 - ae^{ax^*}}$.

3.

4. First, with this functional form, $|J_{\mathbf{b}}| = (c_1 - c_2)(\pi(1-\pi)(1-2\pi)) + c_1 c_2$. Further, $\frac{\partial^2 \pi}{\partial b_1^2} = \frac{\partial^2 \pi}{\partial b_2^2} = -\frac{\partial^2 \pi}{\partial b_1 \partial b_2} = \pi(1-\pi)(1-2\pi)$. Putting these into the implicit function theorem we see that the sign depends on both whether or not $c_1 \geq c_2$ and whether or not $\pi \leq \frac{1}{2}$. If both are true, for example, and $c_1$ is not too big, then both $b_i$ decrease in $c_1$ and increase in $c_2$. Additional cases can be worked out in the same fashion with these partial derivatives.