Games of Incomplete Information

The Harsanyi Setup

- \( n \): the number of players.
- \( D_i \): the set of possible actions or decisions available to the \( i \)th player, \( i = 1, \ldots, n \).
- \( T_i \): the set of possible types for the \( i \)th player. Each \( t_i \in T_i \) is a complete description of one possible state of the \( i \)th player’s private information and beliefs about any uncertain factors relevant to the game.
- \( T_{-i} = T_1 \times \cdots \times T_{i-1} \times T_{i+1} \times \cdots \times T_n \): the set of possible combinations of types for all players other than \( i \).
- \( p_i(t_{-i} | t_i) \): the subjective probability that the \( i \)th player assigns to the event that \( t_{-i} \in T_{-i} \) is the combination of other players’ types given that \( i \)'s type is \( t_i \).
- \( u_i(d, t) \): the (von Neumann) utility of the \( i \)th player when \( d = (d_1, \ldots, d_n) \) is the combination of decisions and \( t = (t_1, \ldots, t_n) \) is the combination of types.
- \( \Gamma = (D_1, \ldots, D_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n) \): the Bayesian game.
- \( \sigma_i \): \( T_i \rightarrow D_i \) the strategy of the \( i \)th player.
- \( \sigma = (\sigma_1, \ldots, \sigma_n) \): a Bayesian (Nash) equilibrium of \( \Gamma \) provided that for every player \( i \) and every \( t_i \in T_i \),
  \[
  \sigma_i(t_i) \in \arg \max_{d_i \in D_i} \sum_{t_{-i} \in T_{-i}} p(t_{-i} | t_i) u_i((\sigma_{-i}(t_{-i}), d_i), t)
  \]
  i.e. \( d_i = \sigma_i(t_i) \) maximizes the expected utility of the \( i \)th player if the other players choose their actions according to \( \sigma_{-i} \) — no type of any player has an incentive to deviate from \( \sigma \).

Illustration: The “Average-Value” Auction

Suppose that each of \( n \) risk-neutral bidders is (privately) informed of the value of an independent drawing from the uniform distribution on \([0,1]\). A sum of money equal to the average of these \( n \) values is then (privately) placed in an urn. Each bidder now has an opportunity to submit a sealed bid. The high bidder will win the contents of the earn unless there is a tie in which case the winner will be selected at random from the bidders who submitted the highest bid. The winning bidder will pay an amount equal to the highest bid submitted by any of the losing bidders. What is the Nash equilibrium bidding strategies in this auction?

Think of yourself as the first bidder. Now let \( e(x,y) \) denote the expected value of winning given that your \( v \) is equal to \( x \) and the highest of the remaining \( v \)'s is equal to \( y \). It follows that it is a symmetric Nash equilibrium for a bidder with valuation \( v \) to submit the bid \( s(v) = e(v,v) \). Why? Suppose everyone else uses \( s(v) \), that your \( v \) is equal to \( x \) and that \( y \) is the highest of the other \( v \)'s. Then the highest bid from the other bidders will be \( s(y) = e(y,y) \). Suppose \( x > y \). Then you would win by submitting \( s(x) \), you would pay \( s(y) \) and your expected payoff would equal \( e(x,y) - s(y) = e(x,y) - e(y,y) > 0 \). Changing your bid would not affect your payoff unless you lowered it enough to lose in which case your payoff would be zero. Similarly, if \( x < y \) then you lose if you bid \( s(x) \) and...
would otherwise pay $s(y)$ and expect a (negative) net payoff of $e(x, y') - s(y) = e(x, y') - e(y, y') < 0$. All that remains then is to note that since

$$e(x, y) = \frac{x + y + (n - 2)\frac{y}{n}}{n} = \frac{x}{n} + \frac{y}{2}$$

it follows that

$$s(v) = e(v, v) = \frac{v}{n} + \frac{v}{2}$$