Bilinear Hysteresis

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Department of Civil and Environmental Engineering
Duke University

Henri P. Gavin
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1 Non degrading bilinear hysteresis

Bilinear hysteresis is a mathematically convenient model for the behavior of yielding structures. The force-displacement relationship in bilinear hysteretic systems is composed of piecewise linear and continuous relationships. Initial loading and unloading follow lines with slope k_1. These lines can intersect the x axis (zero force) at any number of residual displacements, e.g., x_C in the figure below. Post yield behavior follows one of two lines, line A if $\dot{x} > 0$ and line B if $\dot{x} < 0$. These lines intersect the x axis at points x_{A} and x_{B}. This model is parameterized by the yield force F_y, the elastic stiffness k_1, and the post-yield stiffness, k_2. The yield displacement is found from the yield force and elastic stiffness, $x_y = F_y/k_1$. At points where $\dot{x}(t)$ passes through zero, $x(t)$ is at a relative maximum or minimum. These are called a turn-around points, and are marked at coordinates (x_t, R_t). Equations for lines A,

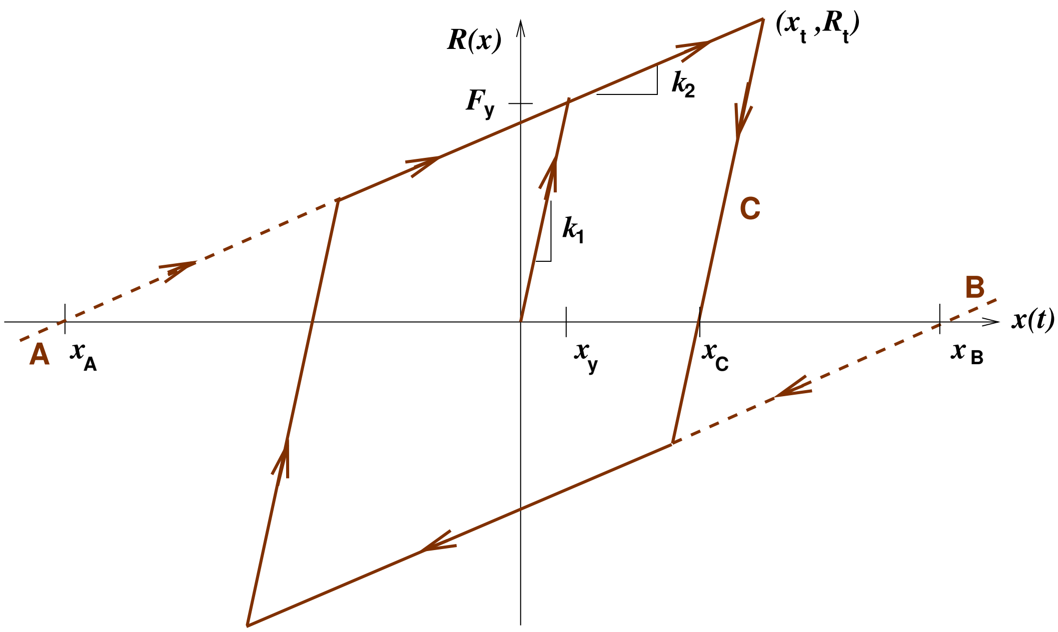


Figure 1. Nondegrading bilinear hysteresis model

B, and C and their x intercepts are at displacements can be found from the geometry of the modeled hysteretic behavior.

line A: $R(x) = k_2 x(t) + F_y(1 - k_2/k_1)$ $\quad x_{\mathsf{A}} = -(F_y/k_2)(1 - k_2/k_1)$
line B: $R(x) = k_2 x(t) - F_y(1 - k_2/k_1)$ $\quad x_{\mathsf{B}} = (F_y/k_2)(1 - k_2/k_1)$
line C: $R(x) = k_1 x(t) + R_t - k_2 x_t$ $\quad x_{\mathsf{C}} = (F_y/k_2)(1 - k_2/k_1)$

This model is called a "non-degrading" model since the stiffness of each branch of the model does not change as damage is accumulated.

The work dissipated per cycle W_D is the area of a hysteresis loop with max and min displacements $x_{\sf max}$ and $-x_{\sf max}$. The work dissipated is the product of the lengths of line segments IJ

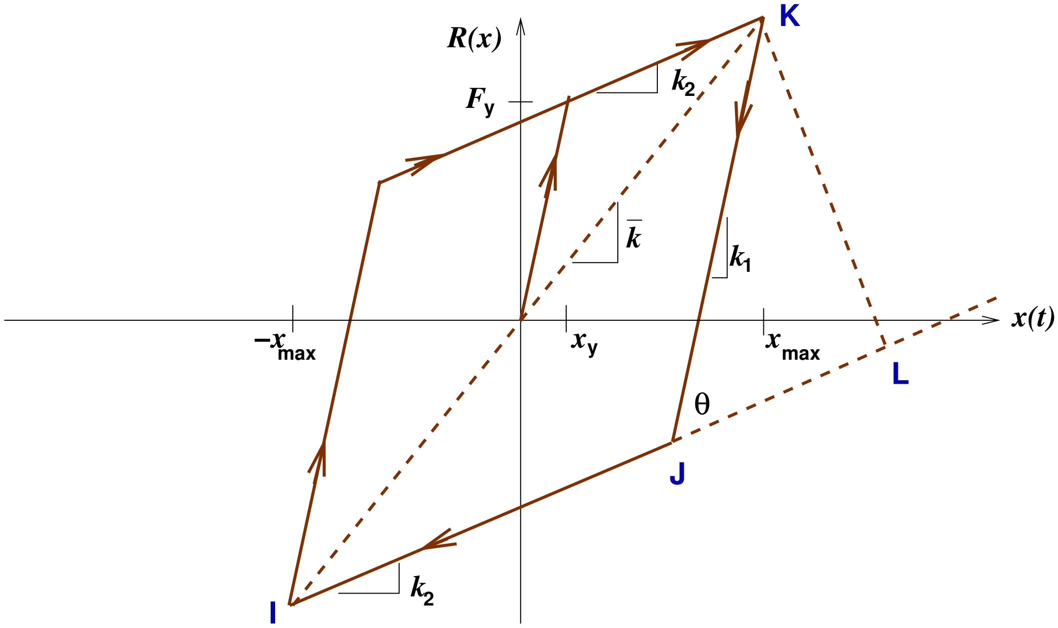


Figure 2. Geometry of the nondegrading bilinear hysteresis model

and KL. From geometry,

$$
\begin{aligned}
\bar{\sf IJ} &= 2(x_{\sf max} - x_y)\sqrt{1+k_2^2} \\
\bar{\sf JK} &= 2\sqrt{x_y^2 + F_y^2} = 2F_y\sqrt{1+1/k_1^2} = 2x_y\sqrt{1+k_1^2} \\
\theta &= \arctan k_1 - \arctan k_2 \\
\bar{\sf KL} &= \bar{\sf KJ} \sin\theta \\
W_D &= \bar{\sf IJ}\bar{\sf KL} \\
\bar{k} &= k_2 + (x_y/x_{\sf max})(k_1 - k_2)
\end{aligned}
$$

The linear viscous damping ratio that is energy-equivalent to bilinear hysteretic models can be found from equating the area of the hysteresis loop with the energy absorbed per cycle by a linear viscous damper. In hysteretic systems, the energy dissipated per cycle depends upon the amplitude of motion, $x_{\sf max}$ as well as the hysteretic parameters. This relation can be analyzed in terms of dimensionless quantities. The *ductility ratio* is the ratio of the peak displacement to the yield displacement, $x_{\sf max}/x_y$. The post-yield stiffness ratio is the ratio k_2/k_1. The equivalent viscous damping ratio also depends upon a measure of stiffness. This can be taken to be the the elastic stiffness $k = k_1$ or the secant stiffness, $k = \bar{k}$.

$$\zeta_{\sf eqv} = W_D/(2\pi\ x_{\sf max}^2\ k\ (\omega/\omega_{\rm n}))$$

For motions dominated by transients and in free response ($(\omega/\omega_{\rm n}) \approx 1$), the equivalent viscous damping ratio can be plotted with respect to the ductility ratio.

```
% bilin.m ... equivalent viscous damping of bilinear hysteresis.

xy = 1;                          % yield displacement - this value doesn't matter
Fy = 1000;                       % yield force - this value doesn't really matter


A = xy*[ 1:0.2:6];               % ductility ratio ... independent variable

k1 = Fy/xy;                      % elastic stiffnes
K2 = [0.0 0.1 0.2 0.5 ]*k1;      % strain hardening stiffness

Omega1 = 1;                      % frequency ratio
wn1    = sqrt(k1);               % natural frequency of elastic system
w      = Omega1*wn1;             % frequency of the motion

for kk = 1:length(K2)

  k2 = K2(kk);
  theta = atan(k1) - atan(k2);
  IJ    = 2*(A'-xy)*sqrt(1+k2^2);
  KJ    = 2*xy*sqrt(1+k1.^2);

  Wd    = IJ*KJ*sin(theta);      % area of hysteresis loop
  Wn(:,kk) = Wd ./ (4*Fy*A)';    % ... normalized w.r.t. frictional damping

  k_ = k2 + (k1-k2)*xy./A;       % equivalent dynamic stiffnes

  Omega_eq = w ./ sqrt(k_);      % equivalent frequency ratio

  Rp = k_ .* A;                  % peak restoring force, k_ * A

  zeq1(:,kk) =  Wd./(2*pi*k1.*A.^2.*Omega1)';   % equivalent damping ratio, k1
  zeq_(:,kk) =  Wd./(2*pi*k_.*A.^2.*Omega_eq)'; % equivalent damping ratio, k_
  K_(:,kk) = k_';

end
```

Note that:

- The equivalent viscous damping ratio ζ_{eqv} is proportional to the energy dissipated per cycle W_D.
- The equivalent viscous damping ratio ζ_{eqv} is inversely proportional to k, x^2_{max} and $(\omega/\omega_{\mathrm{n}})$.
- The equivalent viscous damping ratio ζ_{eqv} is maximized for ductility for x_{max}/x_y between 2 and 3.
- The largest equivalent viscous damping ratio ζ_{eqv} is around 0.25 for elastic-plastic systems ($k_2 = 0$).
- The natural frequency of the hysteretic system $\sqrt{\bar{k}/m}$ decreases with ductility, x_{max}/x_y.
- The secant stiffness, $\bar{k}$, decreases with ductility
- The net energy dissipation increases with ductility
- $\bar{k}/k_1 = 1 - W_D$

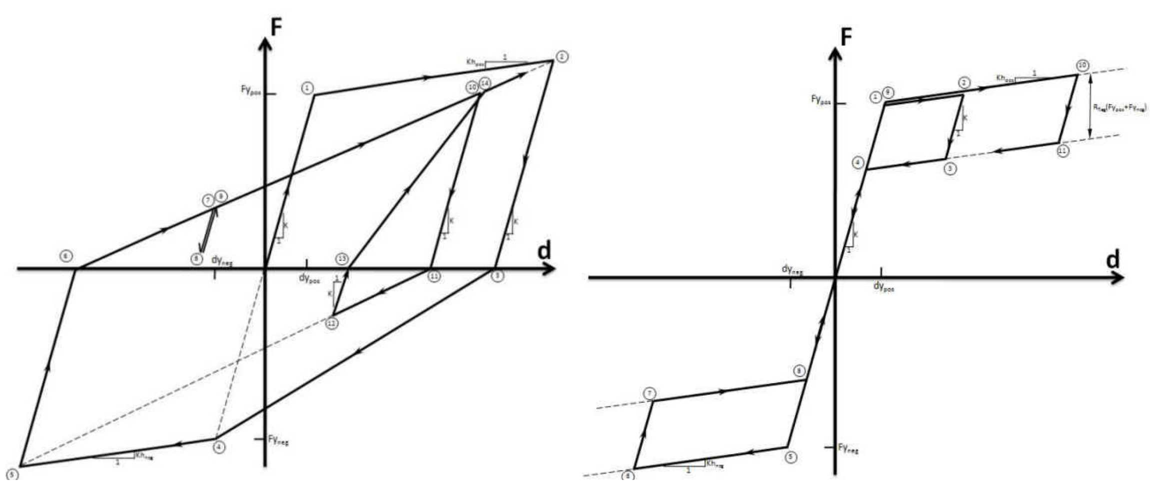
- The secant stiffness "$\bar{k}$" and frequency ratio, "$(\omega/\omega_{\rm n})$" for a hysteretic system are not precisely defined ...
- So the manner in which 'equivalent viscous damping ratio' is ductility- dependent is somewhat subjective.

Project idea ... Evaluate the equivalent viscous damping of bilinear hysteretic systems as a function of ductility and k_2/k_1 using the impulse response of the hysteretic system and the logarithmic decrement. Method for project ... Simulate free response of a SDOF bilinear hysteretic oscillator. For each cycle:

- ... call the average peak displacement, A.
- ... set $\Omega = 1$ since transient-dominated response frequency is close to the natural frequency
- ... set $k =$ (average peak force, R) / A
- ... compute $\zeta_{\sf eqv}$ from $W_D/(2\pi\ k\ A^2\ \Omega)$
- ... plot $\zeta_{\sf eqv}$ w.r.t. ductility for different values of k_2/k_1

2 Other hysteretic shapes

The bilinear hysteretic model can be extended to capture the behavior of degrading material and slipping interfaces.



Modified Johnston/Clough Degrading Hysteresis Model and the Flag Hysteresis Model.
http://www.eqsols.com/Pages/HystereticModels.aspx

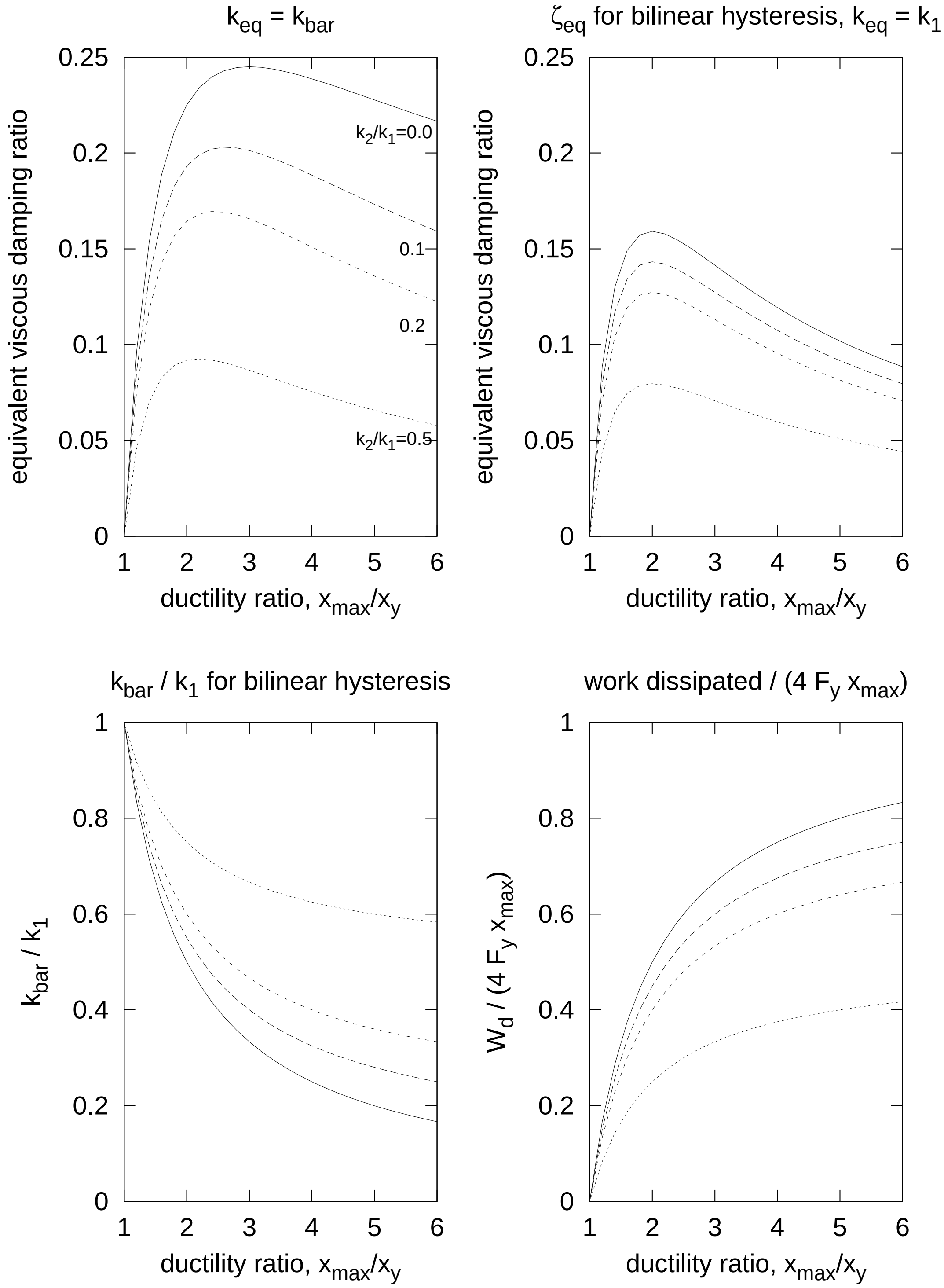


Figure 3. Equivalent linear viscous damping ratios for nondegrading bilinear hysteresis models.