Classical Damping, Non-Classical Damping and Complex Modes

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1 Classical Damping

The equations of motion of an un-forced N degree of freedom elastic structure with viscous damping are

$$\boldsymbol{M}\ddot{\boldsymbol{r}}(t) + \boldsymbol{C}\dot{\boldsymbol{r}}(t) + \boldsymbol{K}\boldsymbol{r}(t) = \boldsymbol{0}, \quad (1)$$

with initial conditions $\boldsymbol{r}(0) = \boldsymbol{d}_{\mathrm{o}}$ and $\dot{\boldsymbol{r}}(0) = \boldsymbol{v}_{\mathrm{o}}$. If the system is un-damped ($\boldsymbol{C} = \boldsymbol{0}_{N\times N}$), the free response of the system will not decay with time, and a suitable trial solution to the differential equation (1) is $\boldsymbol{r}(t) = \bar{\boldsymbol{r}}\sin(\omega_{\mathrm{n}}t)$, where $\bar{\boldsymbol{r}}$ is a constant vector of dimension N. Differentiating $\boldsymbol{r}(t)$ twice, $\ddot{\boldsymbol{r}}(t) = -{\omega_{\mathrm{n}}}^2\bar{\boldsymbol{r}}\sin(\omega_{\mathrm{n}}t)$, and substituting the trial solution into equation (1) we obtain

$$-{\omega_{\mathrm{n}}}^2\boldsymbol{M}\bar{\boldsymbol{r}}\sin(\omega_{\mathrm{n}}t) + \boldsymbol{K}\bar{\boldsymbol{r}}\sin(\omega_{\mathrm{n}}t) = 0. \quad (2)$$

For the assumed trial solution to be true for all time,

$$[\boldsymbol{K} - \omega_{\mathrm{n}j}^2\boldsymbol{M}]\bar{\boldsymbol{r}}_j = \boldsymbol{0}, \quad (3)$$

which is a *general eigen-value problem*, in which eigen-values are squared natural frequencies, ${\omega_{\mathrm{n}}}_j^2$, and the eigen-vectors are mode-shape vectors, $\bar{\boldsymbol{r}}_j$. If the structure is modeled with N degrees of freedom, then there will be N natural frequencies and N modal vectors. The modal matrix $\bar{\boldsymbol{R}}$ is the column-wise concatenation of the N mode-shape vectors, $\bar{\boldsymbol{R}} = [\bar{\boldsymbol{r}}_1\ \bar{\boldsymbol{r}}_2 \cdots \bar{\boldsymbol{r}}_N]$. The modal matrix $\bar{\boldsymbol{R}}$ diagonalizes both the mass and stiffness matrices. The Rayleigh quotient is the ratio of the diagonalized stiffness matrix to the diagonalized mass matrix.

$$\frac{\bar{\boldsymbol{R}}^{\mathsf{T}}\boldsymbol{K}\bar{\boldsymbol{R}}}{\bar{\boldsymbol{R}}^{\mathsf{T}}\boldsymbol{M}\bar{\boldsymbol{R}}} = \begin{bmatrix} k_1^*/m_1^* & & \\ & \ddots & \\ & & k_N^*/m_N^* \end{bmatrix} = \begin{bmatrix} \omega_{\mathrm{n}1}^2 & & \\ & \ddots & \\ & & \omega_{\mathrm{n}N}^2 \end{bmatrix} = \boldsymbol{\Omega}^2. \quad (4)$$

For *mass-normalized modal vectors* $\bar{\boldsymbol{R}}^{\mathsf{T}}\boldsymbol{M}\bar{\boldsymbol{R}} = \boldsymbol{I}_N$ and $\bar{\boldsymbol{R}}^{\mathsf{T}}\boldsymbol{K}\bar{\boldsymbol{R}} = \boldsymbol{\Omega}^2$.

A damping matrix that is diagonalizeable by $\bar{\boldsymbol{R}}$ is called a *classical damping matrix*.

$$\frac{\bar{\boldsymbol{R}}^{\mathsf{T}}\boldsymbol{C}\bar{\boldsymbol{R}}}{\bar{\boldsymbol{R}}^{\mathsf{T}}\boldsymbol{M}\bar{\boldsymbol{R}}} = \begin{bmatrix} c_1^*/m_1^* & & \\ & \ddots & \\ & & c_N^*/m_N^* \end{bmatrix} = \begin{bmatrix} 2\zeta_1\omega_{\mathrm{n}1} & & \\ & \ddots & \\ & & 2\zeta_N\omega_{\mathrm{n}N} \end{bmatrix}. \quad (5)$$

where ζ_j is the damping ratio of the i-th mode, and $\omega_{\mathrm{n}i}$ is the un-damped natural frequency of the i-th mode. Systems with classical damping are *triple diagonalizeable*. The modal vectors of triple diagonalizeable systems depend only on $\boldsymbol{M}$ and $\boldsymbol{K}$, and are independent of $\boldsymbol{C}$, regardless of how heavily the system is damped. There are many ways to compute a classical damping matrix from mass and stiffness matrices.

A *Rayleigh* damping matrix is proportional to the mass and stiffness matrices [6],

$$\boldsymbol{C} = \alpha \boldsymbol{M} + \beta \boldsymbol{K}. \tag{6}$$

where α and β are related to damping ratios and frequencies by

$$\zeta_k = \alpha \frac{1}{2\omega_k} + \beta \frac{\omega_k}{2} \tag{7}$$

Mass proportional damping ratios decrease inversely with ω and stiffness proportional damping ratios increase linearly with ω.

Rayleigh damping can be extended. It can be shown that the damping matrix

$$\boldsymbol{C} = \alpha \boldsymbol{M} + \beta \boldsymbol{K} + \gamma \boldsymbol{M} \boldsymbol{K}^{-1} \boldsymbol{M} + \delta \boldsymbol{K} \boldsymbol{M}^{-1} \boldsymbol{K} \tag{8}$$

is a classical damping matrix. An extended Rayleigh damping matrix, called Caughey damping [1, 2], can be computed from

$$\boldsymbol{C} = \boldsymbol{M} \sum_{j=n_1}^{j=n_2} \alpha_j (\boldsymbol{M}^{-1}\boldsymbol{K})^j \tag{9}$$

where n_1 and n_2 can be positive or negative, as long as $n_1 < n_2$. The coefficients α_j are related to the damping ratios, ζ_k, by

$$\zeta_k = \frac{1}{2} \frac{1}{\omega_k} \sum_{j=n_1}^{j=n_2} \alpha_j \omega_k^{2j} \tag{10}$$

Alternatively, a classical damping matrix can be computed for a specified set of modal damping ratios ζ_j from the mass matrix and *all* N modal vectors and natural frequencies.

$$\boldsymbol{C} = \boldsymbol{M} \bar{\boldsymbol{R}} \begin{bmatrix} 2\zeta_1 \omega_{\mathrm{n}1}/m_1^* & & \\ & \ddots & \\ & & 2\zeta_N \omega_{\mathrm{n}N}/m_N^* \end{bmatrix} \bar{\boldsymbol{R}}^{\mathsf{T}} \boldsymbol{M}. \tag{11}$$

The displacements $\boldsymbol{r}(t)$ of triple-diagonalizeable systems can always be expressed as a linear combination of real-valued *modal coordinates*, $\boldsymbol{q}(t)$,

$$\boldsymbol{r}(t) = \bar{\boldsymbol{r}}_1 q_1(t) + \bar{\boldsymbol{r}}_2 q_2(t) + \cdots + \bar{\boldsymbol{r}}_N q_N(t) = \bar{\boldsymbol{R}} \boldsymbol{q}(t). \tag{12}$$

Substituting equation (12) into equation (1) and pre-multiplying by $\bar{\boldsymbol{R}}^{\mathsf{T}}$ gives

$$\bar{\boldsymbol{R}}^{\mathsf{T}} \boldsymbol{M} \bar{\boldsymbol{R}} \ddot{\boldsymbol{q}}(t) + \bar{\boldsymbol{R}}^{\mathsf{T}} \boldsymbol{C} \bar{\boldsymbol{R}} \dot{\boldsymbol{q}}(t) + \bar{\boldsymbol{R}}^{\mathsf{T}} \boldsymbol{K} \bar{\boldsymbol{R}} \boldsymbol{q}(t) = \boldsymbol{0}, \tag{13}$$

or, for each mode, i, $1 \le i \le N$,

$$\ddot{q}_j(t) + 2\zeta_j \omega_{\mathrm{n}j} \dot{q}_j(t) + \omega_{\mathrm{n}j}^2 q_j(t) = 0, \tag{14}$$

which are the N uncoupled equations of motion in modal coordinates. The damped free response of each modal coordinate decays exponentially with time

$$q_j(t) = e^{-\zeta_j \omega_{\mathrm{n}j} t} (\bar{q}_{\mathrm{c}j} \cos \omega_{\mathrm{d}j} t + \bar{q}_{\mathrm{s}j} \sin \omega_{\mathrm{d}j} t), \tag{15}$$

where $\omega_{\mathrm{d}j}$ is the j-th *damped natural frequency*, is related to the j-th un-damped natural frequency and damping ratio by $\omega_{\mathrm{d}j} = \omega_{\mathrm{n}j} \sqrt{|1 - \zeta_j^2|}$, and the coefficients $\bar{q}_{\mathrm{c}j}$, $\bar{q}_{\mathrm{s}j}$ depend on the initial conditions, the modal vectors, and the mass matrix.

2 Non-Classical Damping

In general, the damping is *not* classical, $\bar{\boldsymbol{R}}^{\mathsf{T}}\boldsymbol{C}\bar{\boldsymbol{R}}$ is not a diagonal matrix, and the natural frequencies, damping ratios, and modal vectors depend on the mass, stiffness, *and* damping matrices of the structural system. To determine the mode-shape vectors, natural frequencies, and damping ratios from $\boldsymbol{M}$, $\boldsymbol{C}$, and $\boldsymbol{K}$ it is necessary to write the 2nd order differential equation (1) as two sets of first order differential equations. Defining the velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t)$, so that $\ddot{\boldsymbol{r}}(t) = \dot{\boldsymbol{v}}(t)$, and solving equation (1) for $\ddot{\boldsymbol{r}}(t)$,

$$\frac{d}{dt}\boldsymbol{v}(t) \equiv \ddot{\boldsymbol{r}}(t) = -\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{r}(t) - \boldsymbol{M}^{-1}\boldsymbol{C}\dot{\boldsymbol{r}}(t). \tag{16}$$

Re-writing these two sets of first order differential equations in matrix form,

$$\frac{d}{dt}\left\{\begin{array}{c}\boldsymbol{r}(t)\\ \boldsymbol{v}(t)\end{array}\right\} = \left[\begin{array}{cc}\boldsymbol{0}_{N\times N} & \boldsymbol{I}_N\\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C}\end{array}\right]\left\{\begin{array}{c}\boldsymbol{r}(t)\\ \boldsymbol{v}(t)\end{array}\right\}. \tag{17}$$

The $2N$-by-$2N$ matrix in the square brackets is called the *dynamics matrix.* Note that it is not symmetric.

For any damped system (classically or non-classically damped) we must assume that the free-vibration response decays with time,

$$\boldsymbol{r}(t) = 2\bar{\boldsymbol{r}}_{\mathrm{r}}e^{\sigma t}\cos(\omega_{\mathrm{d}}t) - 2\bar{\boldsymbol{r}}_{\mathrm{i}}e^{\sigma t}\sin(\omega_{\mathrm{d}}t). \tag{18}$$

All of the terms in equation (18) are real valued, however, it will be convenient to express this equation in terms of *complex* values. We now introduce a complex mode shape vector $\bar{\boldsymbol{r}} = \bar{\boldsymbol{r}}_{\mathrm{r}} + i\bar{\boldsymbol{r}}_{\mathrm{i}}$ and a complex modal coordinate.

$$q(t) = q_{\mathrm{r}}(t) + iq_{\mathrm{i}}(t) = e^{\sigma t}(\cos(\omega_{\mathrm{d}}t) + i\sin(\omega_{\mathrm{d}}t)), \tag{19}$$

where $\bar{\boldsymbol{r}}_{\mathrm{r}}$ and $\bar{\boldsymbol{r}}_{\mathrm{i}}$ are the real and imaginary parts of $\bar{\boldsymbol{r}}$ and $q_{\mathrm{r}}(t)$ and $q_{\mathrm{i}}(t)$ are the real and imaginary parts of $q(t)$. With these new definitions, the trial function may be written compactly as

$$\boldsymbol{r}(t) = \bar{\boldsymbol{r}}q(t) + \bar{\boldsymbol{r}}^*q^*(t).$$

Note here that the subscripts "r" and "i" indicate *real* and *imaginary* and are not indices. Note also that

$$e^{\sigma t}(\cos(\omega_{\mathrm{d}}t) + i\sin(\omega_{\mathrm{d}}t)) = e^{\lambda t} \tag{20}$$

where $\lambda = \sigma + i\omega_{\mathrm{d}}$. So, the complex modal coordinate, $q(t)$, can be written $q(t) = e^{\lambda t}$. The real part of λ equals $-\zeta\omega_{\mathrm{n}}$, the imaginary part of λ equals $\omega_{\mathrm{d}} = \omega_{\mathrm{n}}\sqrt{|\zeta^2 - 1|}$, and $\lambda\lambda^* = \omega_{\mathrm{n}}^2$.

Re-writing and differentiating equation (18) to solve the first order differential equations (17),

$$\boldsymbol{r}(t) = \bar{\boldsymbol{r}}e^{\lambda t} + \bar{\boldsymbol{r}}^*e^{\lambda^* t} \tag{21}$$

$$\boldsymbol{v}(t) = \lambda\bar{\boldsymbol{r}}e^{\lambda t} + \lambda^*\bar{\boldsymbol{r}}^*e^{\lambda^* t}, \tag{22}$$

or

$$\left\{\begin{array}{c}\boldsymbol{r}(t)\\ \boldsymbol{v}(t)\end{array}\right\} = \left[\begin{array}{cc}\bar{\boldsymbol{r}} & \bar{\boldsymbol{r}}^*\\ \lambda\bar{\boldsymbol{r}} & \lambda^*\bar{\boldsymbol{r}}^*\end{array}\right]\left\{\begin{array}{c}e^{\lambda t}\\ e^{\lambda^* t}\end{array}\right\}, \tag{23}$$

and

$$\frac{d}{dt}\left\{\begin{array}{c}\boldsymbol{r}(t)\\ \boldsymbol{v}(t)\end{array}\right\} = \left[\begin{array}{cc}\bar{\boldsymbol{r}} & \bar{\boldsymbol{r}}^*\\ \lambda\bar{\boldsymbol{r}} & \lambda^*\bar{\boldsymbol{r}}^*\end{array}\right]\left[\begin{array}{cc}\lambda & 0\\ 0 & \lambda^*\end{array}\right]\left\{\begin{array}{c}e^{\lambda t}\\ e^{\lambda^* t}\end{array}\right\}. \tag{24}$$

Substituting equations (23) and (24) into the differential equations (17),

$$\begin{bmatrix} \bar{\boldsymbol{r}} & \bar{\boldsymbol{r}}^* \\ \lambda\bar{\boldsymbol{r}} & \lambda^*\bar{\boldsymbol{r}}^* \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^* \end{bmatrix} \begin{Bmatrix} e^{\lambda t} \\ e^{\lambda^* t} \end{Bmatrix} = \begin{bmatrix} \mathbf{0}_{N\times N} & \boldsymbol{I}_N \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{r}} & \bar{\boldsymbol{r}}^* \\ \lambda\bar{\boldsymbol{r}} & \lambda^*\bar{\boldsymbol{r}}^* \end{bmatrix} \begin{Bmatrix} e^{\lambda t} \\ e^{\lambda^* t} \end{Bmatrix}, \tag{25}$$

For this equation to be true for all time,

$$\begin{bmatrix} \bar{\boldsymbol{r}} & \bar{\boldsymbol{r}}^* \\ \lambda\bar{\boldsymbol{r}} & \lambda^*\bar{\boldsymbol{r}}^* \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^* \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{N\times N} & \boldsymbol{I}_N \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{r}} & \bar{\boldsymbol{r}}^* \\ \lambda\bar{\boldsymbol{r}} & \lambda^*\bar{\boldsymbol{r}}^* \end{bmatrix}, \tag{26}$$

which represents a complex-conjugate pair of *standard eigen-value problems*:

$$\begin{bmatrix} \mathbf{0}_{N\times N} & \boldsymbol{I}_N \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \begin{Bmatrix} \bar{\boldsymbol{r}} \\ \lambda\bar{\boldsymbol{r}} \end{Bmatrix} = \begin{Bmatrix} \bar{\boldsymbol{r}} \\ \lambda\bar{\boldsymbol{r}} \end{Bmatrix} \lambda \tag{27}$$

and

$$\begin{bmatrix} \mathbf{0}_{N\times N} & \boldsymbol{I}_N \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \begin{Bmatrix} \bar{\boldsymbol{r}}^* \\ \lambda^*\bar{\boldsymbol{r}}^* \end{Bmatrix} = \begin{Bmatrix} \bar{\boldsymbol{r}}^* \\ \lambda^*\bar{\boldsymbol{r}}^* \end{Bmatrix} \lambda^*. \tag{28}$$

The solution to one of these two standard eigen-value problems implies the solution to the other.

A relationship between the modal vectors found by solving the general eigen-value problem (3) and the standard eigen-value problem (27) can be found by solving equation (27) for the un-damped case ($\boldsymbol{C} = \mathbf{0}_{N\times N}$):

$$\det\left(\begin{bmatrix} -\lambda\boldsymbol{I}_N & \boldsymbol{I}_N \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\lambda\boldsymbol{I}_N \end{bmatrix}\right) = \det\left(\lambda^2\boldsymbol{I}_N + \boldsymbol{M}^{-1}\boldsymbol{K}\right) = 0 \tag{29}$$

Comparing this characteristic equation to the general eigen-value problem, it can be seen that $\lambda^2 = -{\omega_{\rm n}}^2$, or that $\lambda = \pm i\omega_{\rm n}$. The eigen-vectors of this standard eigen-value problem for the un-damped system, $[\bar{\boldsymbol{r}}^{\mathsf{T}} \;\; i\omega_{\rm n}\bar{\boldsymbol{r}}^{\mathsf{T}}]^{\mathsf{T}}$, are directly related to the solution of the general eigen-value problem. Recall that eigen-vectors may be arbitrarily scaled, and it is not uncommon for numerical solutions to (27) to be scaled so that $\bar{\boldsymbol{r}}$ is imaginary and $i\omega_{\rm n}\bar{\boldsymbol{r}}$ is real. For the un-damped case, the eigen-vectors can be more-intuitively scaled so that $\bar{\boldsymbol{r}}$ is purely real and $i\omega_{\rm n}\bar{\boldsymbol{r}}$ is purely imaginary.

The real modes arising from systems with zero or classical damping have *nodes*, which are stationary points at which the structure has zero displacement. In contrast, for a complex modal vector, $\bar{\boldsymbol{r}} = \bar{\boldsymbol{r}}_{\rm r} + i\bar{\boldsymbol{r}}_{\rm i}$, there is not always a point on the structure at which the modal displacement is zero at all times within a periodic cycle.

3 Numerical Examples

The MATLAB programs Cmodes3run.m, Cmodes3analysis.m, and N_dof_anim.m, may be used to explore the modal characteristics of non-classically damped structures. These programs make plots of the real and imaginary parts of the displacement modal vector, $\bar{\boldsymbol{r}}$, the modal phasors for each degree of freedom, the real and imaginary parts of the displacement modal coordinates, $\boldsymbol{q}(t)$, and the displacement responses of the coordinates of a three-degree-of-freedom building model, for which,

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \quad \boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

Values for the floor masses, m_i, inter-story viscous damping rates, c_i, inter-story stiffnesses, k_i, and displacement initial conditions, $\boldsymbol{r}(0)$, are specified in `Cmodes3run.m`. Running `Cmodes3run.m` results in plots and an animation of the free response to the specified initial conditions.

In the .m-function `Cmodes3analysis.m`, each complex mode vector $\bar{\boldsymbol{r}}_j$ is scaled by a rotation θ_j in the complex plane (via multiplication by the complex scalar $e^{-i\theta_j}$) so that the real part of the displacement the mode shape, $\mathsf{Re}(\bar{\boldsymbol{r}})$, is maximized (and the imaginary part is minimized). For this rotation, $\tan\theta_j = \mathsf{Im}(\bar{r}_{jk})/\mathsf{Re}(\bar{r}_{jk})$, where $\bar{r}_{jk} = \max|\bar{\boldsymbol{r}}_j|$. The magnitude of each mode is then scaled so that the displacement parts of the modes are mass-normalized by dividing the real and imaginary parts of $\bar{\boldsymbol{r}}_j$ and $i\omega_{\mathrm{n}}\bar{\boldsymbol{r}}_j$ by $\bar{\boldsymbol{r}}_j^{T*}\boldsymbol{M}\bar{\boldsymbol{r}}_j = \boldsymbol{I}_N$.

When running `Cmodes3run.m`, you may try to:

1. Run a simulation with the as-provided default values for m_i, c_i, k_i, and r_o ($m_i = 1$ tonne, $c_i = [0, 3, 0]$ N/mm/s, $k_i = 1000$ N/mm, $r_{oi} = [1, -2, 3]$ mm). Observe how the real part of mode j has $j-1$ zero-crossings; how the free response of each modal displacement $q_j(t)$ contains only a single frequency, the damped natural frequency, $\omega_{\mathrm{d}j}$; how all three modes are damped even if there is damping in one story only; and how the free response of a higher-frequency mode decays faster (in less time) than that of a lower-frequency mode, even if the higher-frequency mode has slightly less damping.
2. Confirm that if $\boldsymbol{C} = \boldsymbol{0}$ the modes are purely real (with the normalization implemented as described above.)
3. Examine modal characteristics for systems with a Rayleigh damping matrix. For example by setting $k_i = 1000$ N/mm and $c_i = 2.0$ N/mm/s, $\boldsymbol{C}$ is stiffness-proportional ($\boldsymbol{C} = 0.002\boldsymbol{K}$). Is $\bar{\boldsymbol{R}}$ real or complex in this case?
4. Determine values of c_i that will give approximately 5 percent damping in all three modes, for $m_i = 1$ tonne and $k_i = 1000$ N/mm. This will involve some trial-and-error iteration on the three values of c_i. (hint: $c_1 > c_2 > c_3$; $11 < c_1 < 13$ N/mm/s; and $2 < c_2 < 4$ kN/mm/s) Are the resulting modes real or complex? Is there anything unusual or surprising about any of the values of c_i required to meet this goal? Does this finding imply a fallacy in the concept of "damped real normal modes" with arbitrary modal damping ratios?
5. Set the initial displacement, $\boldsymbol{r}_o = \boldsymbol{r}(0)$, proportional to each of the three mode shape vectors, and observe that the free response consists almost entirely of that mode. In `Cmodes3run`, if you set $r_{\mathrm{o}i} = j$, where $j \in [1, 2, 3]$, $\boldsymbol{r}_{\mathrm{o}}$ will be set to $\bar{\boldsymbol{r}}_j$. Next select some other set of initial displacements and observe that the free response contains all three modes.

6. The *phasor matrix*, $\mathbf{\Phi}$, of a complex modal matrix, $\bar{\boldsymbol{R}}$, is given by $\Phi_{ij} = \arctan(\bar{R}_{\mathrm{i}ij}/\bar{R}_{\mathrm{r}ij})$ $(-\pi/2 < \Phi_{ij} < +\pi/2)$. How does multiplying a modal vector by $\sqrt{-1}$ affect the associated column of $\mathbf{\Phi}$? For a complex-valued mode, are values in the associated column of $\mathbf{\Phi}$ equal to one another? Why, or why not? The "complexity" of modal vector $\bar{\boldsymbol{r}}_j$ can be characterized by $\mathcal{C}_j = \max_i |\Phi_{ij} - \Phi_{(i-1)j}|$ Using the phasor plots generated by `Cmodes3run.m` with $m_i = 1$ tonne and $k_i = 1000$ N/mm, find values of c_1, c_2, c_3 that give a mode with a complexity greater than about 30 degrees.

7. Explore the effects of changing the values of mass, damping, and stiffness. When changing a value of m_i, c_i, k_i, and r_{oi}, try to predict the effect of the change on the natural frequencies, damping ratios, mode-shapes, modal responses, and floor responses; then use `Cmodes3run.m` to check yourself.

   (a) What happens if you increase a value of c_i so that the damping of one of the modes approaches 100 percent?

   (b) What happens if a single value of c_i is negative?

   (c) What happens if a value of c_i is so negative that one of the modal damping ratios becomes slightly negative ($\approx -0.50\%$)?

   (d) What happens if one of the stiffness coefficients is much much larger than the other coefficients?

   (e) What happens if one of the stiffness coefficients is slightly *negative*?

   (f) What happens if one of the mass coefficients is very *negative*?

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