Complex-Stiffness Damping

CEE 541. Structural Dynamics
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1 Time Domain Analysis

Dynamic test results on structural solids support damping mechanisms in which the energy dissipated per cycle is frequency-independent. *Hysteretic* systems have input-output relationships that are, by definition, frequency independent. The output of a hysteretic system plotted with respect to the input creates loops of the same size and shape regardless of the frequency of oscillation of the input (and output). Coulomb friction,

$$f_D(r, \dot{r}) = F_{\mathrm{f}} \ \mathrm{sgn}[\dot{r}(t)] \ , \tag{1}$$

is one such mechanism. Another is a mechanism which provides forces that are proportional to displacement, $r(t)$, and are in phase with the velocity, $\dot{r}(t)$,

$$f_{\mathrm{D}}(r, \dot{r}) = (\xi k) \ |r(t)| \ \ (\dot{r}(t)/|\dot{r}(t)|) \ \ . \tag{2}$$

A damping force which is proportional to both displacement and velocity, and is in phase with velocity,

$$f_{\mathrm{D}}(r, \dot{r}) = (\xi k) \ |r(t)| \ \dot{r}(t) \ , \tag{3}$$

and a damping force which is proportional to velocity and the square of displacement

$$f_{\mathrm{D}}(r, \dot{r}) = (\xi k) \ r(t)^2 \ \dot{r}(t) \ , \tag{4}$$

are frequency-dependent, and therefore not hysteretic.

To examine the damping behavior resulting from these expressions, consider sinusoidal input, $r(t) = \bar{r}\cos(\omega t)$ (and $\dot{r}(t) = -\omega\bar{r}\sin(\omega t)$). Figure 1 show the force-displacement and force-velocity relationships for equations (2) - (4). Of these relations, only relation (2) models a damping force is proportional to displacement, in-phase with velocity, and independent of frequency, ω.

These damping expression are non-linear in terms displacements and velocities. Oscillators damped with the mechanisms of equation (2),

$$m\ddot{r}(t) + (\xi k)|r(t)|\dot{r}(t)/|\dot{r}(t)| + kr(t) = f(t) \; , \qquad r(0) = 0 \; , \quad \dot{r}(0) = 0 \; , \qquad (5)$$

are homogeneous. A doubling of the external forcing in equation (5) always results in a doubling of the the response. Oscillators damped with the mechanisms of equations (1), (3), and (4), on the other hand, are not homogeneous. They exhibit amplitude-dependent behavior.

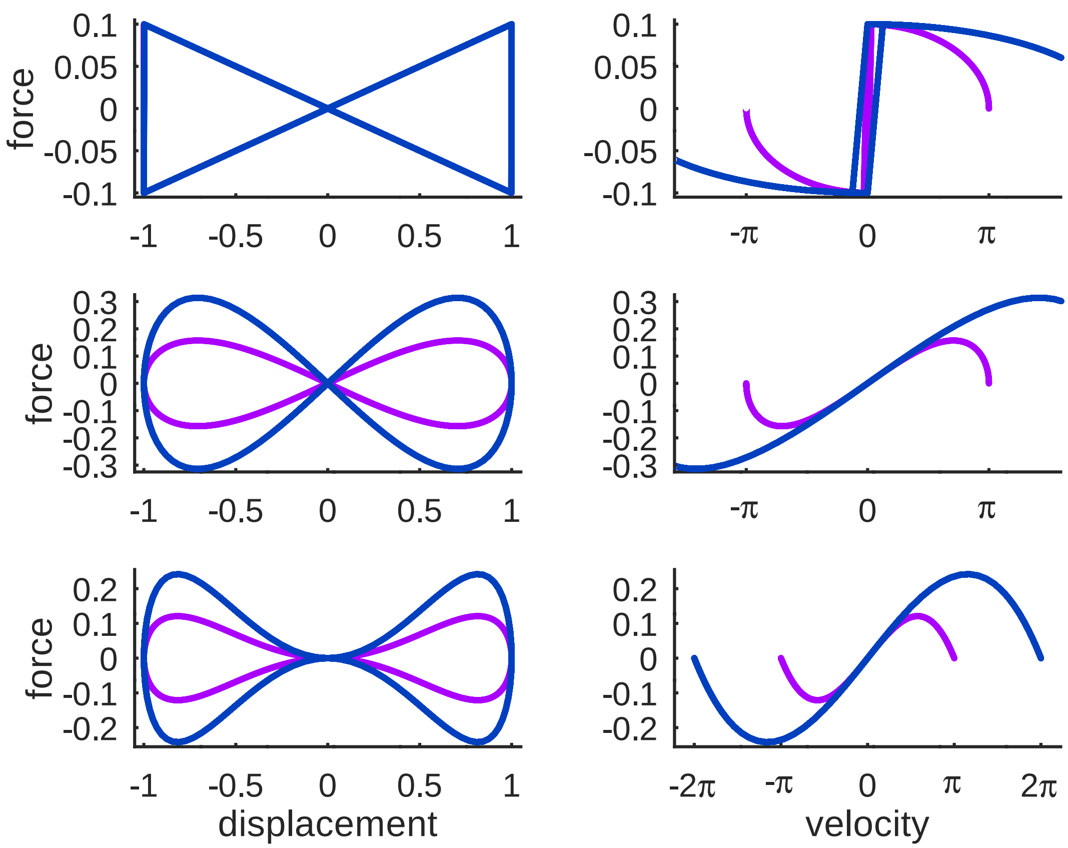

Figure 1. Force-displacement and force-velocity relationship for complex-stiffness damping (equations (2), (3), and (4), sometimes called “butterfly damping”) for $\omega = \pi$ (violet) and $\omega = 2\pi$ (blue) and $\xi = 0.1$.

2 Equivalent Linear Viscous Damping

The equivalent linear viscous damping is conventionally defined in terms of equal energy dissipation per cycle of motion, W_D. For any dissipative structural component deforming dynamically with period T,

$$W_D = \int_0^T f_D(t)\dot{r}(t)\ dt \tag{6}$$

and is geometrically equal to the area within the hysteresis loop of one closed cycle. It is not hard to see from figure 1 that for $r(t) = \bar{r}\cos\omega t$,

$$W_D = (4)\left(\frac{1}{2}\bar{r}\cdot\xi k r\right) = 2\xi k\bar{r}^2\ , \tag{7}$$

and is independent of the frequency of oscillation, ω. The equivalent linear viscous damping rate is proportional to W_D, and the equivalent viscous damping rate is amplitude-independent, but inversely proportional to the frequency of motion,

$$c_{\mathsf{eqv}}(\omega) = \frac{W_D}{\pi\omega\bar{r}^2} = \frac{2\xi k\bar{r}^2}{\pi\omega\bar{r}^2} = \frac{2\xi k}{\pi\omega}\ . \tag{8}$$

The equivalent viscous damping *ratio* depends on the frequency ratio, $\Omega = \omega/\omega_{\rm n}$,

$$\zeta_{\mathsf{eqv}}(\Omega) = \frac{W_D}{2k\pi\Omega\bar{r}^2} = \frac{2\xi k\bar{r}^2}{2k\pi\Omega\bar{r}^2} = \frac{\xi k}{\pi\Omega}\ . \tag{9}$$

3 Homogeneity and Linearity

- A function $f(r, \dot{r})$ is *homogeneous* if and only if

$$f(\alpha r, \alpha \dot{r}) = \alpha f(r, \dot{r}) \quad \forall \quad \alpha > 0$$

The relation (2) is homogeneous since for any $\alpha > 0$, $|\alpha x| = \alpha |x|$

- A function $f(r, \dot{r})$ is *linear* if and only if, for any values α and β,

$$f(\alpha r_1 + \beta r_2, \alpha \dot{r}_1 + \beta \dot{r}_2) = \alpha f(r_1, \dot{r}_1) + \beta f(r_2, \dot{r}_2)$$

None of the damping relationships on page 1 are linear. Consider the relation in equation (2).

$$\begin{aligned} f(\alpha r_1 + \beta r_2, \alpha \dot{r}_1 + \beta \dot{r}_2) &= (\zeta k)|\alpha r_1 + \beta r_2|(\alpha \dot{r}_1 + \beta \dot{r}_2)/|(\alpha \dot{r}_1 + \beta \dot{r}_2| \\ \alpha f(r_1, \dot{r}_1) &= \alpha(\zeta k)\ |r_1|\ (\dot{r}_1/|\dot{r}_1|) \\ \beta f(r_2, \dot{r}_2) &= \beta(\zeta k)\ |r_2|\ (\dot{r}_2/|\dot{r}_2|) \end{aligned}$$

In general, the sum of the second two expressions above does not equal the first because

$$|ax| = a|x| \quad \Leftarrow \quad a \geq 0 \tag{10}$$

and because

$$|x + y| = |x| + |y| \quad \Leftarrow \quad x \geq 0 \quad \text{and} \quad y \geq 0 \tag{11}$$

Since the equalities in (10) and (11) are only conditionally valid, the damping relationship in equation (2) is *nonlinear*.

Viscous damping, $f_D = c\dot{r}$ is homogeneous and linear.

4 Frequency Domain Analysis and Transmissibility

Consider an inertially forced simple oscillator with linear viscous damping

$$\ddot{r}(t) + 2\zeta\omega_{\rm n}\dot{r}(t) + {\omega_{\rm n}}^2 r(t) = -\ddot{z}(t) \tag{12}$$

and an inertially forced simple oscillator with complex stiffness damping

$$\ddot{r}(t) + \xi{\omega_{\rm n}}^2 |r(t)| (\dot{r}(t)/|\dot{r}(t)|) + {\omega_{\rm n}}^2 r(t) = -\ddot{z}(t) \tag{13}$$

Substituting sinusoidal inputs, $z(t) = Z(\omega)e^{\lambda t}$ *and* responses, $r(t) = Re^{\lambda t}$, into the differential equation of the system with linear-viscous damping,

$$\begin{aligned}
(\lambda^2 + 2\zeta\omega_{\rm n}\lambda + {\omega_{\rm n}}^2)\ R\ e^{\lambda t} &= -\lambda^2\ Z\ e^{\lambda t} \\
(\lambda^2 + 2\zeta\omega_{\rm n}\lambda + {\omega_{\rm n}}^2)\ R\ e^{\lambda t}|_{\lambda = i\omega} &= -\lambda^2\ Z\ e^{\lambda t}|_{\lambda = i\omega} \\
(-\omega^2 + 2\zeta\omega_{\rm n} i\omega + {\omega_{\rm n}}^2)\ R\ e^{i\omega t} &= \omega^2\ Z\ e^{i\omega t} \\
(-\Omega^2 + 2\zeta i\Omega + 1)\ R\ e^{i\omega t} &= \Omega^2\ Z\ e^{i\omega t}
\end{aligned} \tag{14}$$

where Ω is the *frequency ratio* $\omega/\omega_{\rm n}$, $\omega_{\rm n}$ is the natural frequency, and ζ is the damping ratio. Substituting these solutions into the differential equation of the system with complex-stiffness damping,

$$\begin{aligned}
(\lambda^2 R + (\xi{\omega_{\rm n}}^2)\ |R|\ \lambda R\ /\ |\lambda R| + {\omega_{\rm n}}^2 R)\ e^{\lambda t} &= -\lambda^2\ Z\ e^{\lambda t} \\
(\lambda^2 R + (\xi{\omega_{\rm n}}^2)\ |R|\ \lambda R\ /\ (|\lambda||R|) + {\omega_{\rm n}}^2 R)\ e^{\lambda t} &= -\lambda^2\ Z\ e^{\lambda t} \\
(\lambda^2 R + (\xi{\omega_{\rm n}}^2)\ R\ \lambda/|\lambda| + {\omega_{\rm n}}^2 R)\ e^{\lambda t}|_{\lambda = i\omega} &= -\lambda^2\ Z\ e^{\lambda t}|_{\lambda = i\omega} \\
(-\omega^2 + (\xi{\omega_{\rm n}}^2)\ i\omega/|\omega| + {\omega_{\rm n}}^2)\ R\ e^{i\omega t} &= \omega^2\ Z\ e^{i\omega t} \\
(-\omega^2 + {\omega_{\rm n}}^2(i\xi {\rm sgn}(\omega) + 1))\ R\ e^{i\omega t} &= \omega^2\ Z\ e^{i\omega t} \\
(-\Omega^2 + (i\xi {\rm sgn}(\Omega) + 1))\ R\ e^{i\omega t} &= \Omega^2\ Z\ e^{i\omega t}
\end{aligned} \tag{15}$$

In arriving at equation (15) the term $(\omega/|\omega|)$ defines $\mathrm{sgn}(\omega)$. The term $(i\xi\mathrm{sgn}(\Omega) + 1)$ illustrates why this form of damping is called "complex-stiffness" damping. This derivation presumes sinusoidal forcing $Ze^{\lambda t}$ generates sinusoidal responses $Re^{\lambda t}$. This is a gross assumption considering the nonlinear behavior exhibited by the Butterfly hysteresis model shown in Figure 1.

5 Causality

The unit impulse response function $h(t)$ describes the response of a system to a unit impulse at time $= 0$. A system is *causal* if and only if the unit impulse response is zero for all times prior to the impulse. That is, $h(t) = 0 \;\; \forall \;\; t < 0$.

The unit impulse function $h(t)$ and the frequency response function $H(\omega)$ are related via Fourier transforms

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} \, dt \quad \text{and} \quad h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} \, d\omega \tag{16}$$

If the unit impulse response computed from a frequency response is non-zero for any $t < 0$, the corresponding system is not causal.

In the time domain, the transient response $r(t)$ of a system is related to transient forcing $z(t)$ through the convolution integral,

$$r(t) = \int_0^t h(\tau) z(t - \tau) \, d\tau$$

In the frequency domain, convolution is a multiplication

$$R(\omega) = H(\omega) Z(\omega)$$

where the frequency response function $H(f)$ is the Fourier transform of the unit impulse response function $h(t)$.

The frequency domain criteria for causality relates the real and imaginary parts of a frequency response function (see [1] section 13.1.4). This criterion is

$$\mathsf{Im}(H(f) = \int_{-\infty}^{\infty} \frac{\mathsf{Re}(H(g))}{\pi(f - g)} \, dg$$

where the integral on the right is known as the *Hilbert transform* of the real part of the frequency response function.

6 Linearity and causality of simple oscillators with linear viscous damping and complex stiffness damping

We consider the linearity and causality of inertially forced simple oscillators with linear viscous damping (12) and with complex stiffness damping (13) in terms of:

- two frequency response functions (frf's), $H(\omega) = R/(\Omega^2 Z)$ (compliance) and $H(\omega) = (R+Z)/Z$ (transmissibility), and
- their associated unit impulse response functions $h(t)$.

The transmissibility ratio $\mathrm{Tr}(\omega)$ is the ratio of the total response, $(R+Z)$, to the forcing, Z. $\mathrm{Tr}(\omega) = (R+Z)/Z$.

$H(\Omega)$	linear viscous damping	complex stiffness damping
$\dfrac{R}{\Omega^2 Z}$	$\dfrac{1}{-\Omega^2 + 2i\zeta\Omega + 1}$	$\dfrac{1}{-\Omega^2 + i\xi \mathrm{sgn}(\Omega) + 1}$
$\dfrac{R+Z}{Z}$	$\dfrac{2i\zeta\Omega + 1}{-\Omega^2 + 2i\zeta\Omega + 1}$	$\dfrac{i\xi \mathrm{sgn}(\Omega) + 1}{-\Omega^2 + i\xi \mathrm{sgn}(\Omega) + 1}$

The magnitudes, real and imaginary parts, and impulse response functions of these four frequency response functions are plotted in Figures 2 - 5.

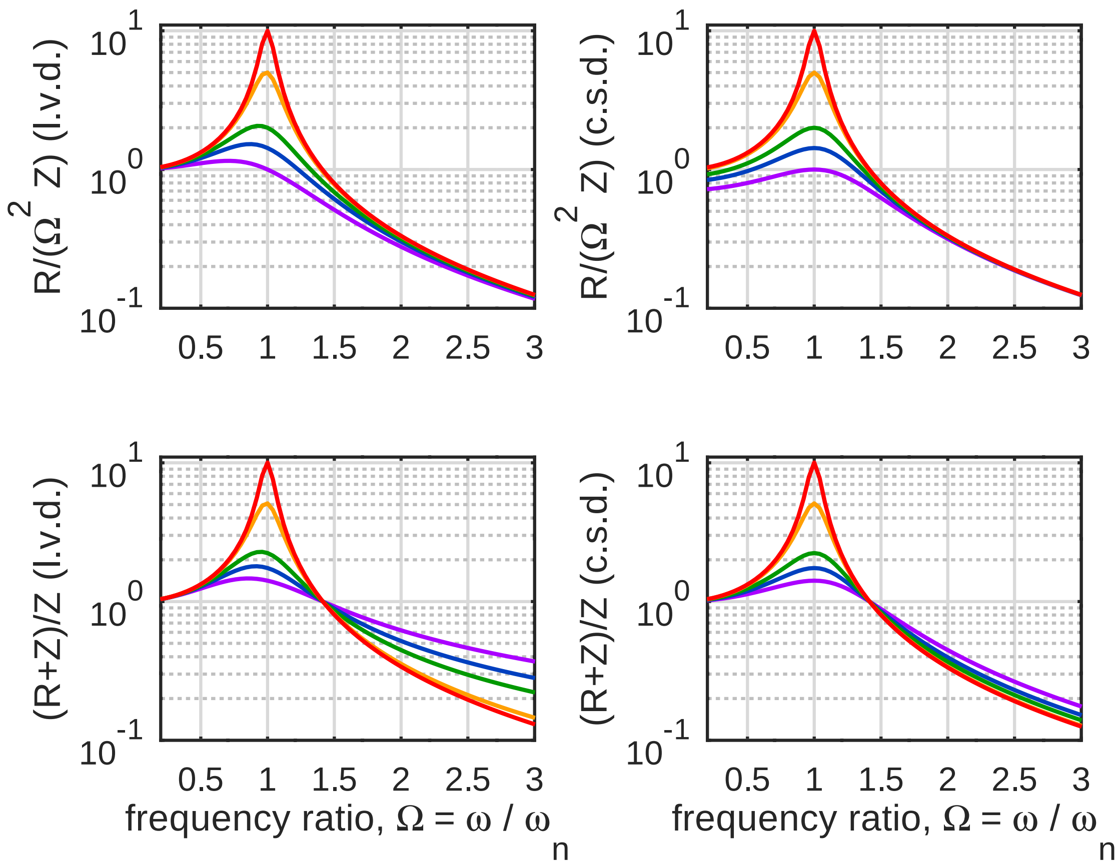

Figure 2. Magnitude of Compliance (top) and Transmissibility (bottom) of inertially-driven oscillators with linear viscous damping (left) and complex-stiffness damping (right). $\xi = 0.1, 0.2, 0.5, 0.7, 1.0$ and $\zeta = 0.05, 0.10, 0.25, 0.35, 0.50$. For comparable suppression of resonant responses, the complex stiffness damping model attenuates the high frequency transmissibility more than the linear viscous damping does.

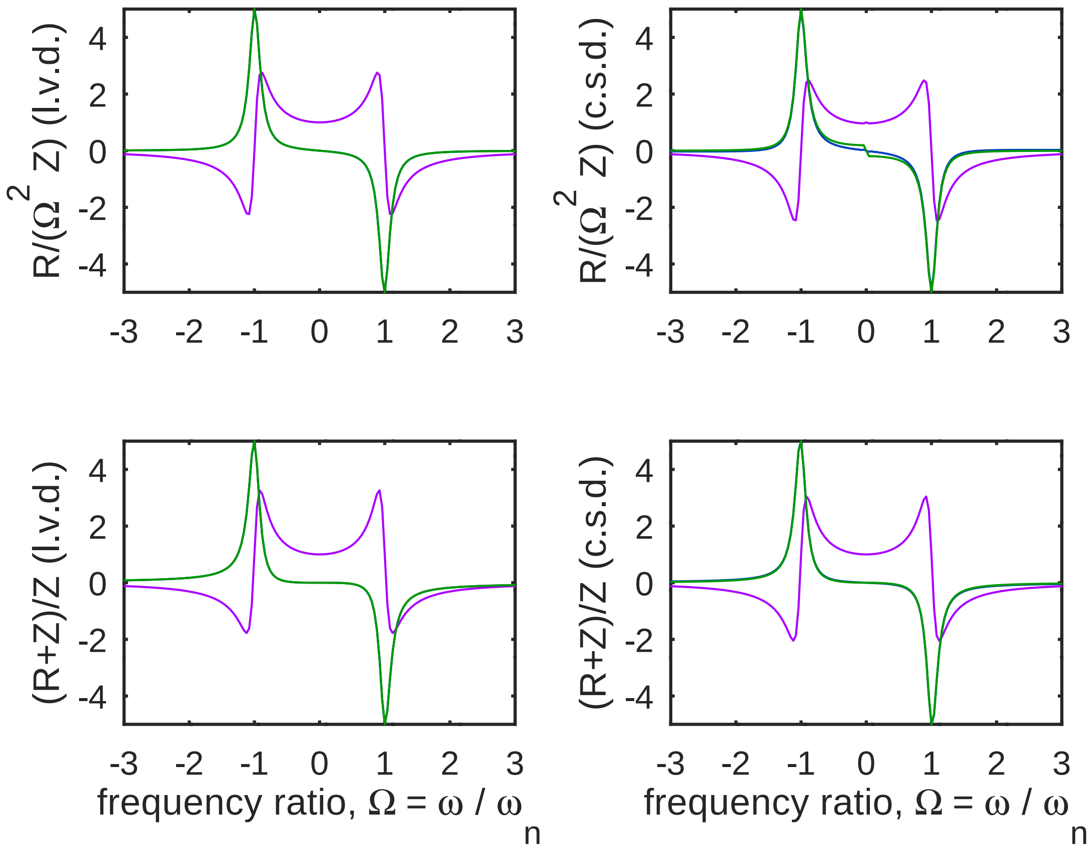

Figure 3. Real and Imaginary parts of of Compliance (top) and Transmissibility (bottom) of inertially-driven oscillators with linear viscous damping (left) and complex stiffness damping (right). $\xi = 0.1, 0.2, 0.5, 0.7, 1.0$ and $\zeta = 0.05, 0.10, 0.25, 0.35, 0.50$ Violet: Real part, Green: Imaginary part, Blue: Hilbert transform of Real part. Both the linear viscous damping and complex stiffness damping frequency response functions are Hermetian symmetric, that is, $(H(\Omega) = H^*(-\Omega))$. All systems with real-valued unit impulse response functions have Hermetian symmetric frequency response functions. For the linear viscous damping case, the imaginary part is exactly the Hilbert transform of the real part, indicating the system is causal. For the complex stiffness damping case, the imaginary part is *not* the Hilbert transform of the real part. This is apparent at low frequencies and for the compliance (top right), though this is also the case at low frequencies with the transmissibility (bottom right), though it is not apparent in this figure. The complex stiffness damping system is *slightly* non-causal.

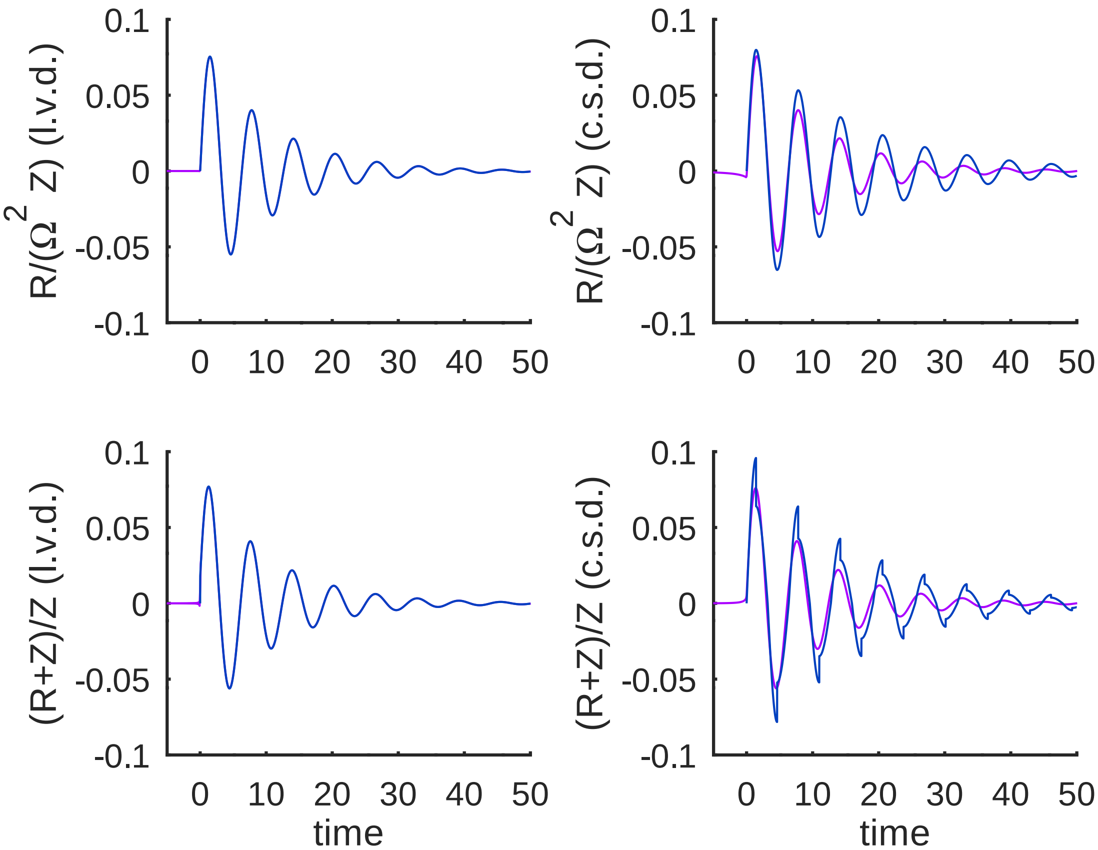

Figure 4. Unit Impulse Response functions of Compliance (top) and Transmissibility (bottom) of inertially-driven oscillators with linear viscous damping (left) and complex stiffness damping (right). $\xi = 0.2$ and $\zeta = 0.10$ Blue: time-domain solution of the ordinary differential equations via numerical Runge-Kutta integration. Violet: frequency-domain solution from the inverse Fourier transform of the compliance and transmissibility functions. With linear viscous damping, the time-domain and frequency-domain computations of the unit impulse response functions are identical. With (discontinuous) complex stiffness damping, the time-domain and frequency-domain computation of the unit impulse response functions are not equivalent. Discontinuities in the time-domain computation of the (acceleration) transmissibility (blue) is due to the abrupt sign change in the Butterfly-Hysteresis damping model. The Fourier series-based frequency-domain model can not capture the nonlinear damping model presented in equation (2) because all Fourier series methods are based upon superposition, which presumes linear behavior. Modeling complex stiffness damping in the frequency-domain (violet, right) corresponds to a non-causal unit impulse response (It is not zero for all times less than zero.) The violet line of the complex stiffness damping compliance (violet, top, right) is seen to drift slightly downwards at times before $t = 0$. Thus, complex stiffness damping is non-linear (the time domain and frequency domain calculations of the unit impulse response do not match) and *slightly* non-causal (the unit impulse response function is not zero for all times less than zero).

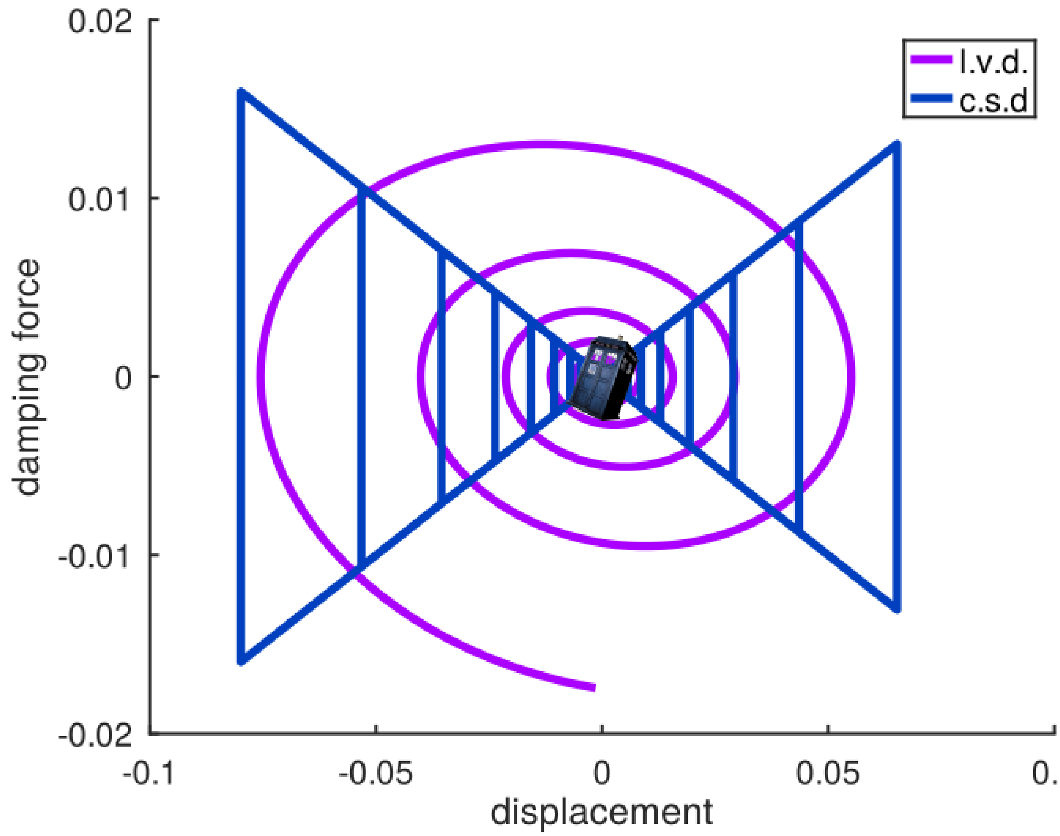

Figure 5. Force displacement relationship for linear viscous damping (violet) and complex-stiffness damping (blue) for a unit impulse response. $\xi = 0.2$ and $\zeta = 0.1$. The response evolves in a clockwise direction. The linear viscous damping forces evolve smoothly. The complex stiffness damping forces evolve with sharp discontinuities, a downward jump at positive-displacement turnarounds and an upward jump at negative-displacement turnarounds. These step-like changes in damping forces correspond to step-like changes in the response accelerations. Complex Stiffness Damping does present issues with causality, but not like the tardis does.

7 Summary

- In the low-frequency range, linear viscous damping and complex-stiffness damping give similar compliance transmissibilities (Figure 2), whereas in the high frequency (short period) range, the transmissibility for complex stiffness damping is less sensitive to increases in damping than is linear viscous damping.
- The frequency response functions of both the linear viscous damping and the complex stiffness damping compliance and transmissibilities are *Hermetian* symmetric, that is, $H(-\Omega) = H^*(\Omega)$. (Figure 3)
- The linear viscous compliance and transmissibility functions are causal as demonstrated by relationships between the real and imaginary parts of the frequency response, and that $h(t) = 0 \ \forall t < 0$. The complex stiffness transmissibility nearly satisfy these causality criteria. (Figure 3)
- Since complex-stiffness damped oscillators (equation (5)) are *not* linear, the transmissibility is valid for a particular amplitude-dependent value of ξ. That is, the model parameter ξ must be calibrated to the amplitude of the motion.
- Butterfly Hysteresis Damping is a causal, homogeneous, but nonlinear damping model that results in Complex Stiffness Damping in the Frequency Domain. This frequency-domain representation of the nonlinear damping mechanism is *slightly* non-causal.
- Complex Stiffness Damping does present issues with causality, but not like the tardis does.

References

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