Dynamic Models for Yielding and Friction Hysteresis

CEE 541. Structural Dynamics
Department of Civil and Environmental Engineering
Duke University
Henri P. Gavin
Fall 2018

In materials or elements with hysteresis, for any monotonic or reciprocating deformation of a sufficiently large amplitude, response of stresses (forces) depend on the deformation history [10]. Such hysteretic behavior is commonly depicted as loops in graphs (2D plots) of periodic (oscillatory) output vs. periodic input. In rate-dependent hysteresis, the size and shape of the hysteresis loop changes with the rate or frequency of the input. If the loop collapses to a function (e.g., a curved line) for any input (e.g., quasi-static), then the system is not hysteretic [10]. Hysteresis implies a non-linear relationship between inputs and outputs: differential equation models for hysteresis must be nonlinear and convolution models for hysteresis must be nonhomogeneous. Linear visco-elastic materials are rate-dependent but are not hysteretic because forces and displacements are proportional in the limit of quasi-static loading. This document describes dynamic hysteresis models that are members of the Duhem [9, 15, 17, 20] class of nonlinear ordinary differential equations

\[
\dot{z}(t) = f(z(t), \dot{u}(t)) g(\dot{u}(t)) , \quad z(0) = z_0 ,
\] (1)

where \(z(t)\) is a force, \(u(t)\) is a displacement, and \(f(.)\) and \(g(.)\) are functions. Such differential equations relate the force, displacement, and velocity to the rate of change of force. Duhem models can be used to model the kinds of rate-independent hysteresis representative of material yielding and stick-slip friction. The following is a basic example of a Duhem ordinary differential equation.

\[
\dot{z}(t) = \dot{u}(t) - |\dot{u}(t)| z^n(t) , \quad z(0) = z_0 , \quad -1 < z_0 < 1 .
\] (2)

If \(n\) is an odd positive integer then \(z(t)\) represents the force normalized by the fully-plastic plastic force or friction sliding force \((-1 < z(t) < 1 \ \forall \ t)\) and \(u(t)\) represents a ductility ratio (the displacement divided by the yield displacement, defined here as the plastic force divided by the stiffness).

Noting that \(|\dot{u}| = \dot{u} \ \text{sgn}(\dot{u})\), equation (2) may be re-written as,

\[
\dot{z} = (1 - z^n \ \text{sgn}(\dot{u})) \ \dot{u} , \quad z(0) = z_0
\] (3)

and the slope of the force-displacement relationship is

\[
\frac{dz}{du} = \frac{\dot{z}}{\dot{u}} = 1 - z^n \text{sgn}(\dot{u})
\] (4)

From this expression it is easy to see that:
• When the force is zero \((z = 0)\), the dimensionless stiffness is 1 (the dimensional stiffness is the plastic force divided by the yield displacement).

• As the force approaches the plastic force \((z \to 1, \dot{u} > 0 \text{ or } z \to -1, \dot{u} < 0)\) \(dz/du\) approaches zero.

• When the velocity is positive, \(dz/du = 1 - z^\eta\), and when the velocity is negative, \(dz/du = 1 + z^\eta\).

• \(dz/du \geq 0\); \(\text{sgn}(\dot{z}) = \text{sgn}(\dot{u})\); and \((\dot{z})(\dot{u}) \geq 0\).

\[ z = \frac{f}{f_p} \]
\[ dz/du = 1 + z^\eta \]
\[ u > 0 \]
\[ dz/du = 1 \]
\[ u < 0 \]
\[ dz/du = 0 \]

\[ dz/du = 1 - z^\eta \]

\[ u > 0 \]
\[ dz/du = 2 \]

\[ u = 0 \]
\[ dz/du = 0 \]

\[ u < 0 \]
\[ dz/du = 1 + z^\eta \]

\[ 0 < z < 1 \]

\[ \dot{u} = 0 \]

\[ \dot{z} = (1 - |z|^\eta \text{sgn}(\dot{uz})) \dot{u}, \]

\[ \text{Extension 1} \]

Replacing \(z^\eta\) with \(|z|^\eta\text{sgn}(z)\) in equation (3), and noting that \(\text{sgn}(a)\text{sgn}(b) = \text{sgn}(ab)\),

\[ \dot{z} = (1 - |z|^\eta \text{sgn}(\dot{uz})) \dot{u}, \]

which allows the exponent \(\eta\) to be any positive value.

\[ \text{Extension 2} \]

Adding parameters \(\beta\) and \(\gamma\) in equation (5), as follows,

\[ \dot{z} = (1 - |z|^\eta (\beta \text{sgn}(\dot{uz}) + \gamma)) \dot{u}, \]

allows for a wide range of hysteretic forms, as shown in figure 2. If \(\beta + \gamma = 1\), then \(-1 < z < 1\). If \(\eta > 0\), \(\gamma > 0\) and \(-\gamma < \beta < \gamma\) the model respects the Second Law of Thermodynamics [1, 11, 14].
Extension 3

Adding a parameter $A,$

$$\dot{z} = (A - |z|^{\eta} (\beta \text{sgn}(\dot{u}z) + \gamma)) \dot{u},$$

allows for scaling. If $\beta + \gamma = A$ then $-A < z < A$. This is the “Bouc-Wen” model for hysteresis [5, 6, 14, 26] and is a member of the Duhem class of hysteresis models [17, 25].

Extension 4

Isotropic bi-axial hysteretic behavior may be modeled in orthogonal directions $x$ and $y$ by coupling the hysteretic variables and velocities [12, 23].

$$\dot{z} = a(z, \dot{u}) z + b \dot{u}$$

where $z = [z_x, z_y]$, $\dot{u} = [\dot{u}_x, \dot{u}_y]$, and

$$a(z, \dot{u}) = -[\beta(|z_x\dot{u}_x| + |z_y\dot{u}_y|) + \gamma(z_x\dot{u}_x + z_y\dot{u}_y)](z_x^2 + z_y^2)^{(-2)/2}$$

Extension 5

Hysteretic behavior with material hardening (a non-zero post-yield stiffness) may be modeled by combining equation (7) or (8) with

$$f(t) = f_p((1 - \kappa)z(t) + \kappa u(t)),$$
where \( f_p \) is a plastic force level, \( u \) is the displacement divided by the yield displacement (the ductility), and \( \kappa \) is the ratio of the post yield stiffness to the pre-yield stiffness. When using (10), set \( A = 1 \) and \( \beta + \gamma = 1 \).

**Stick-Slip Friction**

The Dahl friction model \[8\],

\[
\dot{z} = (1 - z \, \text{sgn} (\dot{u}) \, g(\dot{u})) \, \dot{u}
\]

is equivalent to the Bouc model with \( \eta = 1 \) (which is the usual choice for \( \eta \) in friction modeling). The “LuGre” friction model \[7\] is an extension of the Dahl model which captures the Stribeck (“stick-slip”) effect \[2\]. The LuGre model is a member of the Duhem class of hysteresis models \[17, 25\].

\[
k_o g(\dot{u}) = F_C + (F_S - F_C) \, e^{-(\dot{u}/v_S)^2}
\]

\[
f(t) = k_o z(t) + c_1 \dot{z}(t) + c_2 \dot{u}(t)
\]

In the LuGre model the velocity \( \dot{u} \) is not normalized. (It carries units, like mm.) For high values of pre-slip stiffness, \( k_o \), the LuGre model can require very small time steps for numerical stability.

**Table 1. Representative parameter values for the LuGre friction model [7].**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_o )</td>
<td>pre-slip stiffness</td>
<td>( 10^4 )</td>
<td>N/m</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>friction rate effect</td>
<td>( \sqrt{10^4} )</td>
<td>Ns/m</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>viscous rate effect</td>
<td>0.4</td>
<td>Ns/m</td>
</tr>
<tr>
<td>( F_C )</td>
<td>Coulomb sliding friction force</td>
<td>1</td>
<td>N</td>
</tr>
<tr>
<td>( F_S )</td>
<td>Stribeck sticking friction force</td>
<td>1.5</td>
<td>N</td>
</tr>
<tr>
<td>( v_S )</td>
<td>Stribeck velocity</td>
<td>0.001</td>
<td>m/s</td>
</tr>
</tbody>
</table>

**Degrading Behavior**

To model the accumulation of damage \[4\], strength \( f_p \) and other model parameters may be linked to a damage accumulation index \( \mathcal{D} \), where \( \dot{\mathcal{D}} \approx |\dot{u} - \dot{z}| \) and

\[
f_p(t) = \frac{f_p(0)}{1 + \mathcal{D}(t)/a_{(f_p)}}
\]

where \( a_{(f_p)} \) is a positive constant.

**Further Extensions**

Further generalizations to these equations can account for orthotropic behavior, degrading stiffness, and pinching hysteresis, as described in the references below.
References


doi:10.1002/eqe.2436


