Dynamic Models for Yielding and Friction Hysteresis

CEE 541. Structural Dynamics
Department of Civil and Environmental Engineering
Duke University

Henri P. Gavin
Fall 2018

In materials or elements with hysteresis, the response to a cycle reciprocating forcing depends on the forcing history — for *any* reciprocating forcing of a sufficiently large amplitude [10]. Hysteretic behavior is commonly depicted as loops in graphs (2D plots) of periodic (oscillatory) output vs. periodic input. In rate-dependent hysteresis, the size and shape of the hysteresis loop changes with the rate or frequency of the input. If the loop collapses to a function (e.g., a curved line) for any input (e.g., quasi-static), then the system is *not* hysteretic [10]. Hysteresis implies a non-linear relationship between inputs and outputs: differential equation models for hysteresis must be nonlinear; convolution models for hysteresis must be nonhomogeneous. Linear visco-elastic materials are rate-dependent but are *not hysteretic* because forces and displacements are proportional in the limit of quasi-static loading. This document describes dynamic hysteresis models that are members of the Duhem [9, 15, 17, 20] class of nonlinear ordinary differential equations

$$\dot{z}(t) = f(z(t), u(t))\ g(\dot{u}(t)) \ , \qquad z(0) = z_o \ . \tag{1}$$

Duhem models can be used to model the kinds of rate-independent hysteresis representative of material yielding and stick-slip friction.

Consider the following nonlinear ordinary differential equation

$$\dot{z}(t) = \dot{u}(t) - |\dot{u}(t)|\, z^{\eta}(t) \ , \qquad z(0) = z_o \ . \tag{2}$$

The variable $z(t)$ represents a level of force, normalized to be in the range $-1 < z(t) < 1$, so you can think of z as a force level normalized by a yield force, or friction sliding force. The variable $u(t)$ represents a level of displacement, normalized by a yield displacement, so you can think of u as a ductility ratio. In this way, equation (2) relates the rate of change of force to the rate of change of displacement (i.e., velocity). The exponent η is restricted to be odd, for the time being.

Noting that $|\dot{u}| = \dot{u}\ \mathrm{sgn}(\dot{u})$, equation (2) may be re-written as,

$$\dot{z} = (1 - z^{\eta}\ \mathrm{sgn}(\dot{u}))\ \ \dot{u} \ , \qquad z(0) = z_o \tag{3}$$

and the slope of the force-displacement relationship is

$$\frac{dz}{du} = \frac{\dot{z}}{\dot{u}} = 1 - z^{\eta}\mathrm{sgn}(\dot{u}) \tag{4}$$

From this expression it is easy to see that:

- When the force is zero ($z = 0$), the dimensionless stiffness is 1 (the dimensional stiffness is the yield force divided by the yield displacement).
- As the force approaches the yield force ($z \to 1, \dot{u} > 0$ or $z \to -1, \dot{u} < 0$) dz/du approaches zero.
- When the velocity is positive, $dz/du = 1 - z^\eta$, and when the velocity is negative, $dz/du = 1 + z^\eta$.
- $dz/du \geq 0$; $\text{sgn}(\dot{z}) = \text{sgn}(\dot{u})$; and $(\dot{z})(\dot{u}) \geq 0$.

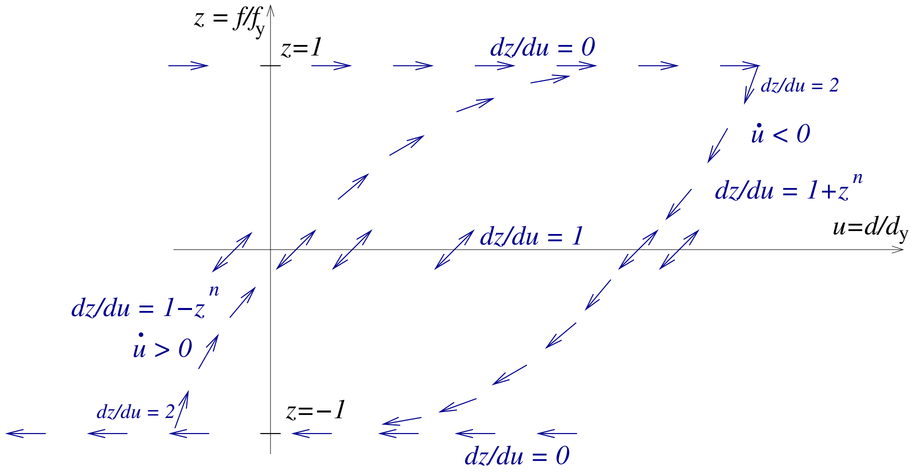

Figure 1. The vector field of dz/du for $\dot{z} = (1 - z^\eta \text{sgn}\dot{u})\dot{u}$ depends on $\text{sgn}\dot{u}$. Larger values of η result in a sharper “knee.”

Extension 1

Replacing z^η with $|z|^\eta \text{sgn}(z)$ in equation (3), and noting that $\text{sgn}(a)\text{sgn}(b) = \text{sgn}(ab)$,

$$\dot{z} = (1 - |z|^\eta \ \text{sgn}(\dot{u}z)) \ \dot{u} \ , \tag{5}$$

which allows the exponent η to be any positive value.

Extension 2

Adding parameters β and γ in equation (5), as follows,

$$\dot{z} = (1 - |z|^\eta \ (\beta \ \text{sgn}(\dot{u}z) + \gamma)) \ \dot{u} \ , \tag{6}$$

allows for a wide range of hysteretic forms, as shown in figure 2.
If $\beta + \gamma = 1$, then $-1 < z < 1$. If $\eta > 0$, $\gamma > 0$ and $-\gamma < \beta < \gamma$ the model respects the Second Law of Thermodynamics [1, 11, 14].

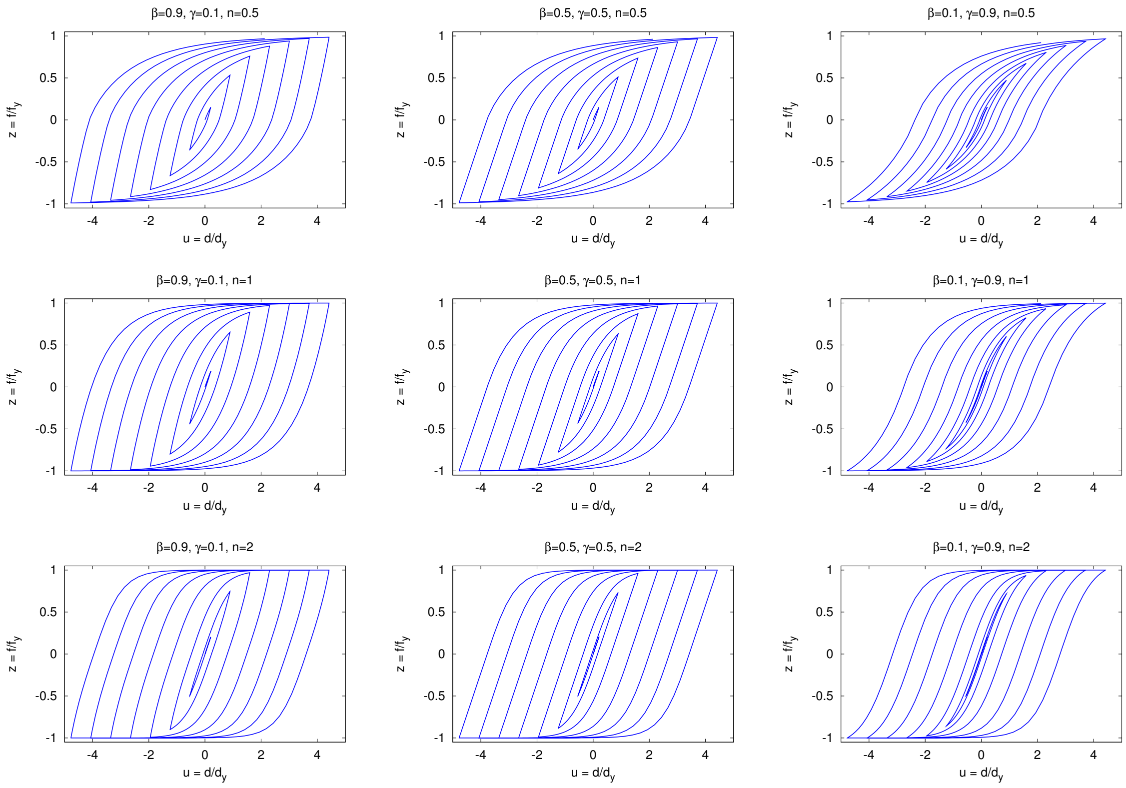

Figure 2. Dependence of hysteretic shape on β, γ, and η.

Extension 3

Adding a parameter A,

$$\dot{z} = (A - |z|^{\eta}\ (\beta\ \mathrm{sgn}(\dot{u}z) + \gamma))\ \dot{u}\ , \tag{7}$$

allows for scaling. If $\beta + \gamma = A$ then $-A < z < A$. This is the “Bouc-Wen” model for hysteresis [5, 6, 14, 26] and is an element of the Duhem class of hysteresis models [17, 25].

Extension 4

Isotropic bi-axial hysteretic behavior may be modeled in orthogonal directions x and y by coupling the hysteretic variables and velocities [12, 23].

$$\dot{z} = a(z, \dot{u})\ z\ +\ b\ \dot{u} \tag{8}$$

where $z = [z_x, z_y]$, $\dot{u} = [\dot{u}_x, \dot{u}_y]$, and

$$a(z, \dot{u}) = -\left[\beta(|z_x\dot{u}_x| + |z_y\dot{u}_y|) + \gamma(z_x\dot{u}_x + z_y\dot{u}_y)\right](z_x^2 + z_y^2)^{(\eta-2)/2} \tag{9}$$

Extension 5

Hysteretic behavior with a non-zero post yield stiffness may be simulated by combining equation (7) or (8) with

$$f(t) = f_{\mathrm{y}}((1-\kappa)z(t) + \kappa u(t))\ , \tag{10}$$

where $f_{\rm y}$ is a yield force level, u is the displacement divided by the yield displacement (the ductility), and κ is the ratio of the post yield stiffness to the pre-yield stiffness. When using (10), set $A = 1$ and $\beta + \gamma = 1$.

Stick-Slip Friction

The Dahl friction model [8],

$$\dot{z} = (\ 1 - z\ \mathrm{sgn}(\dot{u})\ )^{\eta}\ \ \dot{u} \tag{11}$$

is equivalent to the Bouc model for $\eta = 1$ (which is the usual choice for η in friction modeling). The "LuGre" friction model [7] is an extension of the Dahl model which captures the Stribeck ("stick-slip") effect [2]. The LuGre model is an element of the Duhem class of hysteresis models [17, 25].

$$\begin{aligned} \dot{z} &= (\ 1 - z\ \mathrm{sgn}(\dot{u})/g(\dot{u})\ )\ \ \dot{u} && (12)\\ k_{\rm o} g(\dot{u}) &= F_{\rm C} + (F_{\rm S} - F_{\rm C})\ e^{-(\dot{u}/v_{\rm S})^2} && (13)\\ f(t) &= k_{\rm o} z(t) + c_1 \dot{z}(t) + c_2 \dot{u}(t) && (14) \end{aligned}$$

In the LuGre model the velocity $\dot{u}$ is not normalized by a pre-slip displacement. For high values of pre-slip stiffness, $k_{\rm o}$, the LuGre model can require very small time steps for numerical stability.

Table 1. Representative parameter values for the LuGre friction model [7].

Parameter	Definition	Value	Unit
$k_{\rm o}$	pre-slip stiffness	10^4	N/m
c_1	friction rate effect	$\sqrt{10^4}$	Ns/m
c_2	viscous rate effect	0.4	Ns/m
$F_{\rm C}$	Coulomb sliding friction force	1	N
$F_{\rm S}$	Stribeck sticking friction force	1.5	N
$v_{\rm S}$	Stribeck velocity	0.001	m/s

Degrading Behavior

To model the accumulation of damage [4], strength $f_{\rm y}$ and other model parameters may be linked to a damage accumulation index $\mathcal{D}$, where $\dot{\mathcal{D}} \approx |\dot{u} - \dot{z}|$ and

$$f_{\rm y}(t) = \frac{f_{\rm y}(0)}{1 + \mathcal{D}(t)/a_{(f_{\rm y})}}\ , \tag{15}$$

where $a_{(f_{\rm y})}$ is a positive constant.

Further Extensions

Further generalizations to these equations can account for orthotropic behavior, degrading stiffness, and pinching hysteresis, as described in the references below.

References

[1] Ahmadi, G., Fan, F.C., and Noori, M., "A thermodynamically consistent model for hysteretic materials," *Iranian Journal of Science and Technology,* 1997; 21:257–278.

[2] Armstrong-Hélouvry, B., DuPont, P., Canudas de Wit, C., "A Survey of Models, Analysis Tools, and Compensation methods for the Control of Machines with Friction," *Automatica,* 1994; 30(7):1083–1138.

[3] Atalik, T.S. and Utku, S., "Stochastic Linearization of Multi-Degree-Of-Freedom Non-Linear Systems," *Earthquake Engineering and Structural Dynamics,* 1976; 4:411–420.

[4] Baber, T.T. and Wen, Y-K., "Random Vibration of Hysteretic Degrading Systems," *Journal of Engineering Mechanics,* 1982; 107(EM6):1069–1087.

[5] Bouc, R., "Influence du cycle d'hysteresis dur la resonance non lineare d'un circuit serie," *Colloques Internationeaux. Centre International de la Recherche Scientifique,* Sept 1964.

[6] Bouc, R., "Modele mathematique d'hysteresis." *Acustica,* 1971; 24:16–25.

[7] Canudas de Wit, C., Olsson, H. Astrom, K.J., and Lischinsky, P., "A New Model for Control of Systems with Friction," *IEEE Trans. Automatic Control,* 1995; 40(3):419–425.

[8] Dahl, P.R., *A Solid Friction Model,* Technical Report SAMSO-TR-77-131, May 1968.

[9] Duhem, Pierre, "Die dauerndens Aenderungen und die Thermodynamik," *I, Z. Phys. Chem.*, 22 (1897), pp. 543-589.

[10] Drinčić, Bojana, "Mechanical Models of Friction That Exhibit Hysteresis, Stick-Slip, and the Stribeck Effect," Ph.D. dissertation, University of Michigan, 2012.

[11] Erlicher, S., and Point, N., "Thermodynamic admissibility of Bouc-Wen type hysteresis models," *C R Méc,* 2004; 332(1):51–55.

[12] Harvey, P.S., Jr., and **Gavin, H.P.**, "Truly isotropic biaxial hysteresis with arbitrary knee sharpness," *Earthquake Engineering & Structural Dynamics*, 43(13):2051-2057 (2014)
doi:10.1002/eqe.2436

[13] Hassani, Vahid , Tjahjowidodo, Tegoeh, and Do, Thanh Nho, "A survey on hysteresis modeling, identification and control *Mechanical Systems and Signal Processing,* 49 (1-2) (2014) 209-233.

[14] Ismail, M., Ikhouayne, F. and Rodellar, J., "The Hysteresis Bouc-Wen Model - A Survey," *Arch. Computational Methods in Engineering,* 2009; 16:161–188.

[15] Macki, J.W., Nistri, P., and Zecca, P., "Mathematical-Models for Hysteresis," *SIAM Review*, 35(1) (1993): 94-123.

[16] Merriam-Webster Online. "hysteresis, n.," Merriam-Webster, 2016. Web. 14 October 2016.

[17] Naser, Mohammad Fuad Mohammad, and Ikhouane, Fayçal, "Consistency of the Duhem Model with Hysteresis," *Mathematical Problems in Engineering,* 2013, Article ID 586130.

[18] Oh, JinHyoung and Bernstein, Dennis S., "Semilinear Duhem model for rate-independent and rate-dependent hysteresis," *IEEE Transactions on Automatic Control*, 2005

[19] Oh, JinHyoung, Drincic, Bojana, and Bernstein, Dennis S., "Nonlinear Feedback Models of Hysteresis," *IEEE Control Systems Magazine,* February 2009; 100–119.

[20] Ouyang Ruiyue, Andrieu Vincent, and Jayawardhana, Bayu, "On the characterization of the Duhem hysteresis operator with clockwise input-output dynamic," *Systems & Control Letters* 62 (2013) 286-293.

[21] OED Online. Oxford University Press, "hysteresis, n.," September 2016. Web. 14 October 2016.

[22] Özdemir, H. *Nonlinear Transient Dynamic Analysis of Yielding Structures,* Ph.D. Dissertation, University of California, Berkeley, June 1976.

[23] Park, Y.J., Wen, Y.K., and Ang, A. H-S. "Random Vibration of Hysteretic Systems under Bi-Directional Ground Motion," *Earthquake Engineering and Structural Dynamics,* 1986; 14(4):543–557.

[24] Sain, P.M., Sain, M.K., Spencer, B.F., and Sain, J.D., "The Bouc Hysteresis Model: An Initial Study of Qualitative Characteristics," *Proc. American Control Conference,* Philadelphia, 1998: 2559–2563.

[25] Visintin, A. "Mathematical models of hysteresis" in: *The Science of Hysteresis*, (G. Bertotti, I. Mayergoyz, eds.) Elsevier (2006), cap. 1, pp. 1-123

[26] Wen, Y-K. "Method for Random Vibration of Hysteretic Systems," *Journal of Engineering Mechanics,* 1976; 102(EM2):249–263.