

Fourier Series, Fourier Transforms, and Periodic Response to Periodic Forcing

CEE 541. Structural Dynamics

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This document describes methods to analyze the steady-state forced-response of a simple oscillator to general periodic loading. The analysis is carried out using Fourier series approximations to the periodic external forcing and the resulting periodic steady-state response.

1 Fourier Series for Real-Valued Functions

Any real-valued function, $f(t)$, that is:

- periodic, with period T ,

$$\dots = f(t - 2T) = f(t - T) = f(t) = f(t + T) = f(t + 2T) = \dots$$

- square-integrable

$$\int_0^T f^2(t) dt < \infty.$$

may be represented as a series expansion of sines and cosines, in a Fourier series,

$$\hat{f}(t; \mathbf{a}, \mathbf{b}) = \frac{1}{2}a_0 + \sum_{q=1}^{\infty} a_q \cos \frac{2\pi qt}{T} + \sum_{q=1}^{\infty} b_q \sin \frac{2\pi qt}{T}, \quad (1)$$

where the Fourier coefficients, a_q and b_q are given by the Fourier integrals,

$$a_q = \frac{2}{T} \int_0^T f(t) \cos \frac{2\pi qt}{T} dt, \quad q = 1, 2, \dots \quad (2)$$

$$b_q = \frac{2}{T} \int_0^T f(t) \sin \frac{2\pi qt}{T} dt, \quad q = 1, 2, \dots \quad (3)$$

The Fourier series $\hat{f}(t; \mathbf{a}, \mathbf{b})$ is a least-squares fit to the function $f(t)$. This may be shown by first defining the error function, $e(t) = f(t) - \hat{f}(t; \mathbf{a}, \mathbf{b})$, the quadratic error criterion, $J(\mathbf{a}, \mathbf{b}) = \int_0^T e^2(t) dt$, and finding the Fourier coefficients by solving the linear equations resulting from minimizing $J(\mathbf{a}, \mathbf{b})$ with respect to the coefficients \mathbf{a} and \mathbf{b} . (by setting $\partial J(\mathbf{a}, \mathbf{b})/\partial \mathbf{a}$ and $\partial J(\mathbf{a}, \mathbf{b})/\partial \mathbf{b}$ equal to zero). So doing,

$$\begin{aligned} J(\mathbf{a}, \mathbf{b}) &= \int_0^T (f(t) - \hat{f}(t; \mathbf{a}, \mathbf{b}))^2 dt \\ &= \int_0^T (f^2(t) - 2f(t)\hat{f}(t; \mathbf{a}, \mathbf{b}) + \hat{f}^2(t; \mathbf{a}, \mathbf{b})) dt. \end{aligned} \quad (4)$$

Setting $\partial J/\partial a_0$ equal to zero results in

$$\begin{aligned}
0 &= \int_0^T \frac{\partial}{\partial a_0} \left[f^2(t) - 2f(t)\hat{f}(t; \mathbf{a}, \mathbf{b}) + \hat{f}^2(t; \mathbf{a}, \mathbf{b}) \right] dt \\
&= -2 \int_0^T f(t) \frac{\partial}{\partial a_0} \hat{f}(t; \mathbf{a}, \mathbf{b}) dt + 2 \int_0^T \hat{f}(t; \mathbf{a}, \mathbf{b}) \frac{\partial}{\partial a_0} \hat{f}(t; \mathbf{a}, \mathbf{b}) dt \\
&= -2 \int_0^T f(t) \cdot \frac{1}{2} dt + 2 \int_0^T \left[\frac{1}{2} a_0 + \sum_{q=1}^{\infty} a_q \cos \frac{2\pi qt}{T} + \sum_{q=1}^{\infty} b_q \sin \frac{2\pi qt}{T} \right] \cdot \frac{1}{2} dt \\
&= - \int_0^T f(t) dt + \int_0^T \frac{1}{2} a_0 dt + \sum_{q=1}^{\infty} a_q \int_0^T \cos \frac{2\pi qt}{T} dt + \sum_{q=1}^{\infty} b_q \int_0^T \sin \frac{2\pi qt}{T} dt \\
&= - \int_0^T f(t) dt + \frac{1}{2} a_0 T + \sum_{q=1}^{\infty} a_q \cdot 0 + \sum_{q=1}^{\infty} b_q \cdot 0
\end{aligned}$$

From which,

$$a_0 = \frac{2}{T} \int_0^T f(t) dt. \quad (5)$$

Note here that $(a_0/2)$ is the average value of $f(t)$.

Proceeding with the rest of the a coefficients, setting $\partial J(\mathbf{a}, \mathbf{b})/\partial a_k$ equal to zero results in

$$\begin{aligned}
0 &= \int_0^T \frac{\partial}{\partial a_k} \left[f^2(t) - 2f(t)\hat{f}(t; \mathbf{a}, \mathbf{b}) + \hat{f}^2(t; \mathbf{a}, \mathbf{b}) \right] dt \\
&= -2 \int_0^T f(t) \frac{\partial}{\partial a_k} \hat{f}(t; \mathbf{a}, \mathbf{b}) dt + 2 \int_0^T \hat{f}(t; \mathbf{a}, \mathbf{b}) \frac{\partial}{\partial a_k} \hat{f}(t; \mathbf{a}, \mathbf{b}) dt \\
&= - \int_0^T f(t) \cos \frac{2\pi kt}{T} dt + \int_0^T \left[\frac{a_0}{2} + \sum_{q=1}^{\infty} a_q \cos \frac{2\pi qt}{T} + \sum_{q=1}^{\infty} b_q \sin \frac{2\pi qt}{T} \right] \cos \frac{2\pi kt}{T} dt \\
&= - \int_0^T f(t) \cos \frac{2\pi kt}{T} dt + \\
&\quad \frac{a_0}{2} \int_0^T \cos \frac{2\pi qt}{T} dt + \sum_{q=1}^{\infty} a_q \int_0^T \cos \frac{2\pi qt}{T} \cos \frac{2\pi kt}{T} dt + \sum_{q=1}^{\infty} b_q \int_0^T \sin \frac{2\pi qt}{T} \cos \frac{2\pi kt}{T} dt \\
&= - \int_0^T f(t) \cos \frac{2\pi kt}{T} dt + \frac{a_0}{2} \cdot 0 + a_k \int_0^T \cos^2 \frac{2\pi kt}{T} dt + \sum_{q=1}^{\infty} b_q \cdot 0 \\
&= - \int_0^T f(t) \cos \frac{2\pi kt}{T} dt + 0 + a_k \cdot \frac{T}{2} + 0
\end{aligned}$$

From which,

$$a_k = \frac{2}{T} \int_0^T f(t) \cos \frac{2\pi kt}{T} dt. \quad (6)$$

In a completely similar fashion, $\partial J(\mathbf{a}, \mathbf{b})/\partial b_k = 0$ results in

$$b_k = \frac{2}{T} \int_0^T f(t) \sin \frac{2\pi kt}{T} dt. \quad (7)$$

The derivation of the Fourier integrals (equations (5), (6), and (7)) makes use of orthogonality properties of sine and cosine functions.

$$\begin{aligned} \int_0^T \sin^2\left(\frac{n\pi t}{T}\right) dt &= \frac{T}{2}, \quad n = 1, 2, \dots \\ \int_0^T \cos^2\left(\frac{n\pi t}{T}\right) dt &= \frac{T}{2}, \quad n = 1, 2, \dots \\ \int_0^T \sin\left(\frac{n\pi t}{T}\right) \sin\left(\frac{m\pi t}{T}\right) dt &= 0, \quad n \neq m, \quad n, m = 1, 2, \dots \\ \int_0^T \cos\left(\frac{n\pi t}{T}\right) \cos\left(\frac{m\pi t}{T}\right) dt &= 0, \quad n \neq m, \quad n, m = 1, 2, \dots \\ \int_0^T \sin\left(\frac{n\pi t}{T}\right) \cos\left(\frac{m\pi t}{T}\right) dt &= 0, \quad n \neq m, \quad n, m = 1, 2, \dots \end{aligned}$$

1.1 The complex exponential form of Fourier series

Recall the trigonometric identities for complex exponentials, $e^{i\theta} = \cos\theta + i\sin\theta$. Defining the complex scalar F as $F = \frac{1}{2}(a - ib)$, and its complex conjugate, F^* as $F^* = \frac{1}{2}(a + ib)$, it is not hard to show that

$$F e^{i\theta} + F^* e^{-i\theta} = a \cos\theta + b \sin\theta. \quad (8)$$

Note that the left hand side of equation (8) is the sum of two complex conjugate values, and that the right hand side is the sum of two real values. So the imaginary parts on the left cancel out and we can see that the sum of complex conjugates can be used to express the real Fourier series given in equation (1). Substituting equation (8) into equation (1),

$$\begin{aligned} \hat{f}(t; \mathbf{a}, \mathbf{b}) &= \frac{1}{2}a_0 + \sum_{q=1}^{\infty} \left[a_q \cos \frac{2\pi q t}{T} + b_q \sin \frac{2\pi q t}{T} \right] \\ &= \frac{1}{2}a_0 + \sum_{q=1}^{\infty} \left[\frac{1}{2}(a_q - ib_q) \exp \left[i \frac{2\pi q}{T} t \right] + \frac{1}{2}(a_q + ib_q) \exp \left[-i \frac{2\pi q}{T} t \right] \right] \\ &= \frac{1}{2}a_0 + \sum_{q=1}^{\infty} F_q \exp \left[i \frac{2\pi q}{T} t \right] + \sum_{q=1}^{\infty} F_q^* \exp \left[-i \frac{2\pi q}{T} t \right] \\ &= \frac{1}{2}a_0 + \sum_{q=1}^{\infty} F_q \exp \left[i \frac{2\pi q}{T} t \right] + \sum_{q=-\infty}^{-1} F_{-q}^* \exp \left[i \frac{2\pi q}{T} t \right] \\ &= \sum_{q=-\infty}^{\infty} F_q \exp \left[i \frac{2\pi q}{T} t \right] = \sum_{q=-\infty}^{\infty} F_q e^{i\omega_q t} \end{aligned} \quad (9)$$

where the Fourier frequency ω_q is $2\pi q/T$, $F_0 = \frac{1}{2}a_0$, $F_q = \frac{1}{2}(a_q - ib_q)$, and $F_{-q}^* = F_q$. The condition $F_{-q}^* = F_q$ holds for the Fourier series of any *real-valued* function that is periodic

and integrable. Going the other way,

$$\begin{aligned}
\hat{f}(t; \mathbf{F}) &= \sum_{q=-\infty}^{q=+\infty} F_q e^{i\omega_q t} \\
&= F_0 + \sum_{q=-\infty}^{q=-1} F_q e^{i\omega_q t} + \sum_{q=1}^{q=+\infty} F_q e^{i\omega_q t} \\
&= F_0 + \sum_{q=1}^{q=+\infty} F_{-q} e^{-i\omega_q t} + \sum_{q=1}^{q=+\infty} F_q e^{i\omega_q t} \\
&= F_0 + \sum_{q=1}^{\infty} [F_q e^{i\omega_q t} + F_{-q} e^{-i\omega_q t}] \\
&= F_0 + \sum_{q=1}^{\infty} [(F'_q + iF''_q) (\cos \omega_q t + i \sin \omega_q t) + (F'_q - iF''_q) (\cos \omega_q t - i \sin \omega_q t)] \\
&= F_0 + \sum_{q=1}^{\infty} [2F'_q \cos \omega_q t - 2F''_q \sin \omega_q t]
\end{aligned}$$

So, the real part of F_q , F'_q , is half of a_q , the imaginary part of F_q , F''_q , is half of $-b_q$, and $F_q = F_{-q}^*$. The *real* Fourier coefficients, a_q , are *even* about $q = 0$ and the *imaginary* Fourier coefficients, b_q , are *odd* about $q = 0$. In other words, the complex Fourier coefficients of a real valued function are *Hermetian symmetric*.

Just as the Fourier expansion may be expressed in terms of complex exponentials, the coefficients F_q may also be written in this form.

$$\begin{aligned}
F_q &= \frac{1}{2} (a_q - ib_q) = \frac{1}{T} \int_0^T f(t) \left[\cos \frac{2\pi q}{T} t - i \sin \frac{2\pi q}{T} t \right] \\
&= \frac{1}{T} \int_0^T f(t) \exp \left[-i \frac{2\pi q}{T} t \right] dt
\end{aligned} \tag{10}$$

Since the integrand, $f(t)e^{-i\omega_q t}$, of the Fourier integral is periodic in T , the limits of integration can be shifted arbitrarily without affecting the resulting Fourier coefficients.

$$\int_0^T f(t)e^{-i\omega_q t} dt = \int_0^\tau f(t)e^{-i\omega_q t} dt + \int_\tau^{T+\tau} f(t)e^{-i\omega_q t} dt - \int_T^{T+\tau} f(t)e^{-i\omega_q t} dt.$$

But, since the integrand is periodic in T ,

$$\int_0^\tau f(t)e^{-i\omega_q t} dt = \int_T^{T+\tau} f(t)e^{-i\omega_q t} dt.$$

So, the interval of integration can be shifted arbitrarily.

$$\int_0^T f(t)e^{-i\omega_q t} dt = \int_{0+\tau}^{T+\tau} f(t)e^{-i\omega_q t} dt. \tag{11}$$

On the other hand, shifting the function in time, affects the relative values of a_q and b_q (i.e., the phase the the complex coefficient F_q), but does not affect the magnitude, $|F_q| = \frac{1}{2}\sqrt{a_q^2 + b_q^2}$. If $\hat{f}(t) = \sum F_q e^{-i\omega_q t}$, then $\hat{f}(t + \tau) = \sum F_q e^{-i\omega_q(t+\tau)} = \sum (F_q e^{-i\omega_q \tau}) e^{-i\omega_q t}$, and $|\hat{F}_q| = |F_q e^{-i\omega_q \tau}|$.

2 Fourier Integrals in Maple

The Fourier integrals for real valued functions (equations (6) and (7)) can be evaluated using symbolic math software, such as Maple or Mathematica.

2.1 a periodic square wave function: $f(t) = \text{sgn}(t - \pi)$ on $0 < t < 2\pi$ and $f(t) = f(t + n(2\pi))$

```
> assume (k::integer);
> f := signum(t-Pi);
      f := signum(t - Pi)
> ak := (2/(2*Pi)) * int(f*cos(2*Pi*k*t/(2*Pi)),t=0..2*Pi);
      ak := 0
> bk := (2/(2*Pi)) * int(f*sin(2*Pi*k*t/(2*Pi)),t=0..2*Pi);
      bk := 
$$\frac{2 \left( (-1)^{\tilde{k}} - 1 \right)}{\text{Pi } \tilde{k}}$$

```

2.2 a periodic sawtooth function: $f(t) = t - \pi$ on $0 < t < 2\pi$ and $f(t) = f(t + n(2\pi))$

```
> assume (k::integer);
> f := t-Pi;
      f := t - Pi
> ak := (2/(2*Pi)) * int(f*cos(2*Pi*k*t/(2*Pi)),t=0..2*Pi);
      ak := 0
> bk := (2/(2*Pi)) * int(f*sin(2*Pi*k*t/(2*Pi)),t=0..2*Pi);
      bk := 
$$-\frac{2}{\tilde{k}}$$

```

2.3 a periodic triangle function: $f(t) = \pi/2 - (t - \pi) \text{sgn}(t - \pi)$ on $0 < t < 2\pi$ and $f(t) = f(t + n(2\pi))$

```
> assume (k::integer);
> f := Pi/2 - (t-Pi) * signum(t-Pi);
      f := (Pi/2) - (t - Pi) signum(t - Pi)
> ak := (2/(2*Pi)) * int(f*cos(2*Pi*k*t/(2*Pi)),t=0..2*Pi);
      ak := 
$$\frac{2 \left( (-1)^{\tilde{k}} - 1 \right)}{\text{Pi } \tilde{k}}$$

> bk := (2/(2*Pi)) * int(f*sin(2*Pi*k*t/(2*Pi)),t=0..2*Pi);
      bk := 0
```

3 Fourier Transforms

Recall for periodic functions of period, T , the Fourier series expansion may be written

$$f(t) = \sum_{q=-\infty}^{q=\infty} F_q e^{i\omega_q t} , \quad (12)$$

where the Fourier coefficients, F_q , have the same units as $f(t)$, and are given by the Fourier integral,

$$F_q = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-i\omega_q t} dt , \quad (13)$$

in which the limits of integration have been shifted by $\tau = -T/2$.

Now, consider a change of variables, by introducing the definition of a frequency increment, $\Delta\omega$, and a *scaled amplitude*, $F(\omega_q)$.

$$\Delta\omega \triangleq \omega_1 = \frac{2\pi}{T} \quad (\omega_q = q \Delta\omega) \quad (14)$$

$$F(\omega_q) \triangleq T F_q = \frac{2\pi}{\Delta\omega} F_q \quad (15)$$

Where the scaled amplitude, $F(\omega_q)$, has units of $f(t) \cdot t$ or $f(t)/\omega$.

Using these new variables,

$$f(t) = \frac{1}{2\pi} \sum_{q=-\infty}^{q=\infty} F(\omega_q) e^{i\omega_q t} \Delta\omega , \quad (16)$$

$$F(\omega_q) = \int_{-T/2}^{T/2} f(t) e^{-i\omega_q t} dt . \quad (17)$$

Finally, taking the limit as $T \rightarrow \infty$, implies $\Delta\omega \rightarrow d\omega$ and $\sum \rightarrow \int$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega , \quad (18)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt . \quad (19)$$

These expressions are the famous Fourier transform pair. Equation (18) is commonly called the *inverse* Fourier transform and equation (19) is commonly called the *forward* Fourier transform. They differ only by the sign of the exponent and the factor of 2π .

By convention, the forward fast Fourier transform (FFT) of an N -point time series of duration T ($x_k = x((k-1)\Delta t)$, $k = 1, \dots, N$) scales the N , complex-valued, Fourier amplitudes/coefficients as follows: $\text{FFT}(x) = X(\omega_q)/\Delta t = NX_q$, where the Fourier-transform frequencies, ω_q , are given by $\omega_q = 2\pi q/T$, and are sorted as follows: $q = 0, \dots, N/2, -N/2 + 1, \dots, -1$.

4 Fourier Approximation

Any periodic function may be *approximated* as a *truncated* series expansion of Q sines and Q cosines, as a Fourier series,

$$\tilde{f}(t) = \frac{1}{2}a_0 + \sum_{q=1}^Q a_q \cos \frac{2\pi qt}{T} + \sum_{q=1}^Q b_q \sin \frac{2\pi qt}{T}, \quad (20)$$

where the Fourier coefficients, a_q and b_q may be found by solving the Fourier integrals,

$$a_q = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \frac{2\pi qt}{T} dt, \quad q = 1, 2, \dots, Q \quad (21)$$

$$b_q = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \frac{2\pi qt}{T} dt, \quad q = 1, 2, \dots, Q \quad (22)$$

The Fourier approximation (20) may also be represented using complex exponential notation.

$$\tilde{f}(t) = \sum_{q=-Q}^{q=Q} F_q e^{i\omega_q t} = \frac{1}{T} \sum_{q=-Q}^{q=Q} F(\omega_q) e^{i\omega_q t} \quad (23)$$

where $e^{i\omega_q t} = \cos \omega_q t + i \sin \omega_q t$, $\omega_q = 2\pi q/T$, $F_q = \frac{1}{2}(a_q - ib_q)$, $F_q = F_{-q}^*$, and

$$F_q = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-i\omega_q t} dt. \quad (24)$$

The accuracy of Fourier approximations of non-sinusoidal functions increases with the number of terms, Q , in the series. Especially for time series that are discontinuous in time, such as square-waves or saw-tooth waves, Fourier approximations, $\tilde{f}(t)$, will contain a degree of over-shoot at the discontinuity and will oscillate about the approximated function $f(t)$. This is called *Gibb's phenomenon*. Examples in Section 7 demonstrate this effect for square waves and triangle waves.

The response of a system described by a frequency response function $H(\omega)$ to arbitrary periodic forces described by a Fourier series may be found in the frequency domain,

$$X_q = H(\omega_q) F_q, \quad (25)$$

or in the time domain,

$$x(t) = \sum_{q=-Q}^{q=Q} H(\omega_q) F_q e^{i\omega_q t} = \frac{1}{T} \sum_{q=-Q}^{q=Q} H(\omega_q) F(\omega_q) e^{i\omega_q t}. \quad (26)$$

5 Discrete Fourier Transform

If the function $f(t)$ is represented by N uniformly-spaced points in time, $t_n = (n)(\Delta t)$, ($n = 0, \dots, N-1$), the N Fourier coefficients are computed by the discrete Fourier transform (DFT),

$$F_q = \frac{1}{N} \sum_{n=0}^{N-1} f(t_n) \cdot \exp \left[-i \frac{2\pi qn}{N} \right], \quad (27)$$

The frequency increment Δf in the DFT is $1/(N\Delta t)$ Hertz, and the highest frequency is $1/(2\Delta t)$ Hertz, the so-called *Nyquist* frequency.

Note that the exponentials in the DFT, (27), are periodic in N points.

$$\exp \left[-i \frac{2\pi qn}{N} \right] = \exp \left[-i \frac{2\pi qn}{N} + 2\pi \right] = \exp \left[-i \frac{2\pi (q - N)n}{N} \right]$$

Further, referring to equation (11), as long as the set of frequency indices, q , contain N consecutive points, the indices may be shifted arbitrarily.

$$\begin{aligned} q &\in \{ -N/2 + 1, \dots, -1, 0, 1, 2, \dots, N/2 - 1, N/2 \} \\ q &\in \{ 0, \dots, N/2 - 2, N/2 - 1, N/2, -N/2 + 1, \dots, -2, -1 \} \\ q &\in \{ 0, \dots, N/2 - 2, N/2 - 1, N/2, N/2 + 1, \dots, N - 2, N - 1 \} \end{aligned}$$

The N -point discrete Fourier approximation is computed by the inverse discrete Fourier transform,

$$f(t_n) = \sum_{q=-N/2+1}^{N/2} F_q \cdot \exp \left[i \frac{2\pi qn}{N} \right] = \sum_{q=0}^{N-1} F_q \cdot \exp \left[i \frac{2\pi qn}{N} \right] \quad (28)$$

With the normalization given in equations (27) and (28), the mean square in the time domain is the sum-square of the Fourier amplitudes.

$$\frac{1}{N} \sum_{n=0}^{N-1} f_n^2 = \sum_{q=0}^{N-1} |F_q|^2,$$

The number of operations required to compute a DFT (equation (27)) increases with N^2 . The *fast* Fourier transform¹ (FFT) algorithm computes the DFT by simply scaling and re-ordering the sequence f_n . The number of operations required to compute a DFT using the FFT algorithm increases with $N \log_2(N)$: if it takes 1 minute to compute an FFT with $N = 10^6$, it would take almost 35 days to carry out the same DFT using equation (27)!

By convention, the fast Fourier transform (FFT) computes the DFT without the $(1/N)$ factor. Also, by convention, the Fourier frequencies, $\omega_q = 2\pi q/T$ are sorted as follows: $q \in \{0, 1, \dots, N/2, -N/2 + 1, \dots, -2, -1\}$. Here is an animation for the steps in the Danielson-Lanczos FFT algorithm: http://www.duke.edu/~hpgavin/cee541/fft_anim.m.

¹Heideman, Michael T.; Johnson, Don H.; Burrus, C. Sidney (1985-09-01). “Gauss and the history of the fast Fourier transform,” *Archive for History of Exact Sciences*, 34(3): 265-277. doi:10.1007/BF00348431

6 Fourier Series and Periodic Responses of Dynamic Systems

The response of a system described by a frequency response function $H(\omega)$ to arbitrary periodic forces $f(t)$, described by a Fourier series $f(t) = \sum F_q \exp[i\omega_q t]$, may be represented in the frequency domain as Fourier coefficients $X_q = H(\omega_q)F_q$, or in the time domain as a Fourier series $x(t) = \sum X_q \exp[i\omega_q t]$.

Complex exponential notation allows us to directly determine the steady-state periodic response to general periodic forcing, in terms of both the magnitude of the response and the phase of the response. Recall the relationship between the complex magnitudes X and F for a sinusoidally-driven spring-mass-damper oscillator is

$$X_q = H(\omega_q) F_q = \frac{1}{(k - m\omega_q^2) + i(c\omega_q)} F_q \quad (29)$$

The function $H(\omega)$ is called the *frequency response function* of a simple oscillator relating the input $f(t)$ to the output $x(t)$. The frequency response function $H(\omega)$ for any *linear* dynamic system (such as a simple oscillator) may be derived from the system's differential equations.

By definition for any *linear* dynamic system, if the response to $f_1(t)$ is $x_1(t)$, and if the response to $f_2(t)$ is $x_2(t)$, then the response to $c_1 f_1(t) + c_2 f_2(t)$ is $c_1 x_1(t) + c_2 x_2(t)$. More generally, then,

$$x(t) = \sum_{q=-\infty}^{q=\infty} H(\omega_q) F_q e^{i\omega_q t} = \sum_{q=-\infty}^{q=\infty} \frac{1}{(k - m\omega_q^2) + i(c\omega_q)} F_q e^{i\omega_q t} \quad (30)$$

Equivalently, the periodic response can be expressed in terms of the Fourier series coefficients a_q and b_q (equation (1)),

$$\begin{aligned} x(t) &= \frac{1}{2}a_0/k + \sum_{q=1}^{\infty} \frac{1/k}{(1 - \Omega_q^2)^2 + (2\zeta\Omega_q)^2} \cdot \\ &\quad \left[\left(a_q(1 - \Omega_q^2) - b_q(2\zeta\Omega_q) \right) \cos \omega_q t + \left(a_q(2\zeta\Omega_q) + b_q(1 - \Omega_q^2) \right) \sin \omega_q t \right] \\ &= \frac{1}{2}a_0/k + \sum_{q=1}^{\infty} \frac{1/k}{\sqrt{(1 - \Omega_q^2)^2 + (2\zeta\Omega_q)^2}} \cdot \left[a_q \cos(\omega_q t + \theta_q) + b_q \sin(\omega_q t + \theta_q) \right] \end{aligned} \quad (31)$$

where $\omega_q = 2\pi q/T$, $\Omega_q = \omega_q/\omega_n$, and $\tan \theta_q = -2\zeta\Omega_q/(1 - \Omega_q^2)$.

For this frequency response function $H(\omega)$, the series expansion for the response $x(t)$ converges with fewer terms than the Fourier series for the external forcing, because $|H(\omega_q)|$ decreases with ω . (In this example $|H(\omega)|$ decreases as $1/\omega^2$).

Note that ω_q is *not* the same symbol as ω_n . The symbol ω_n denotes the *natural* frequency. (The subscript n is *not italicized* because it does denote an index number.) The symbol $\omega_q = 2\pi q/T$ is the frequency of a Fourier component of a periodic forcing function of period T , and q denotes the q^{th} term in the series.

7 Examples

In the following examples, the external forcing is periodic with a period, T , of 2π (seconds). For functions periodic in 2π seconds, the frequency increment in the Fourier series ($\Delta\omega$ of equation (14)) is 1 radian/second. So in this case, the frequency index number, q , is also the frequency, ω_q , in radians per second. (Though in general, $\omega_q = 2\pi q/T$.)

The system is a forced spring-mass-damper oscillator,

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = \frac{1}{m}f^{\text{ext}}(t) . \quad (32)$$

The complex-valued frequency response function from the external force, $f^{\text{ext}}(t)$, to the response displacement, $x(t)$, is given by

$$H(\omega) = \frac{1/\omega_n^2}{1 - \Omega^2 + i 2\zeta\Omega} , \quad (33)$$

where Ω is the frequency ratio, ω/ω_n ,

In the following three numerical examples, $\omega_n = 10$ rad/s, $\zeta = 0.1$, $m = 1$ kg, and $Q = 16$ terms. The three examples consider external forcing in the form of a square-wave, a sawtooth-wave, and a triangle-wave. In each example six plots are provided.

In the (a) plots, the solid line represents the exact form of $f(t)$, the dashed lines represent the real-valued form of the Fourier approximation and the complex-valued form of the Fourier approximation, and the circles represent $2Q$ sample points of the function $f(t)$ for use in fast Fourier transform (FFT) computations. The two dashed lines are exactly equal.

In the (b) plots, the triangles show the values of the Fourier coefficients, F_q , found by evaluating the Fourier integral, equation (24), and the circles represent the Fourier coefficients computed by the FFT. They are nearly (not exactly) identical,

In the (c) plots, the red solid lines show the cosine terms of the Fourier series and the blue dashed lines show the sine terms of the Fourier series.

The (d) and (f) plots show the magnitude and phase of the transfer function, $H(\omega)$ as a solid line, the Fourier coefficients, F_q , as the green circles, and $H(\omega_q)F_q$ as the blue circles.

In the (e) plots, the solid line represents the true analytical solution for $x(t)$ represented by equation (30), the circles represent the real part of the result of the inverse FFT calculation, and the dots represent the imaginary part of the result of the inverse FFT calculation (which are nearly exactly zero, as they should be). The FFT calculation with 16 coefficients is close to the analytical solution.

The numerical details involved in the correct representation of periodic functions and frequency indexing for FFT computations are provided in the attached Matlab code.

7.1 Example 1: square wave

The external forcing is given by

$$f^{\text{ext}}(t) = \text{sgn}(t - \pi) ; \quad 0 \leq t < 2\pi \tag{34}$$

The Fourier coefficients for the real-valued Fourier series are:

$$a_q = 0 \text{ and } b_q = (2/(\pi q)) (-1^q - 1).$$

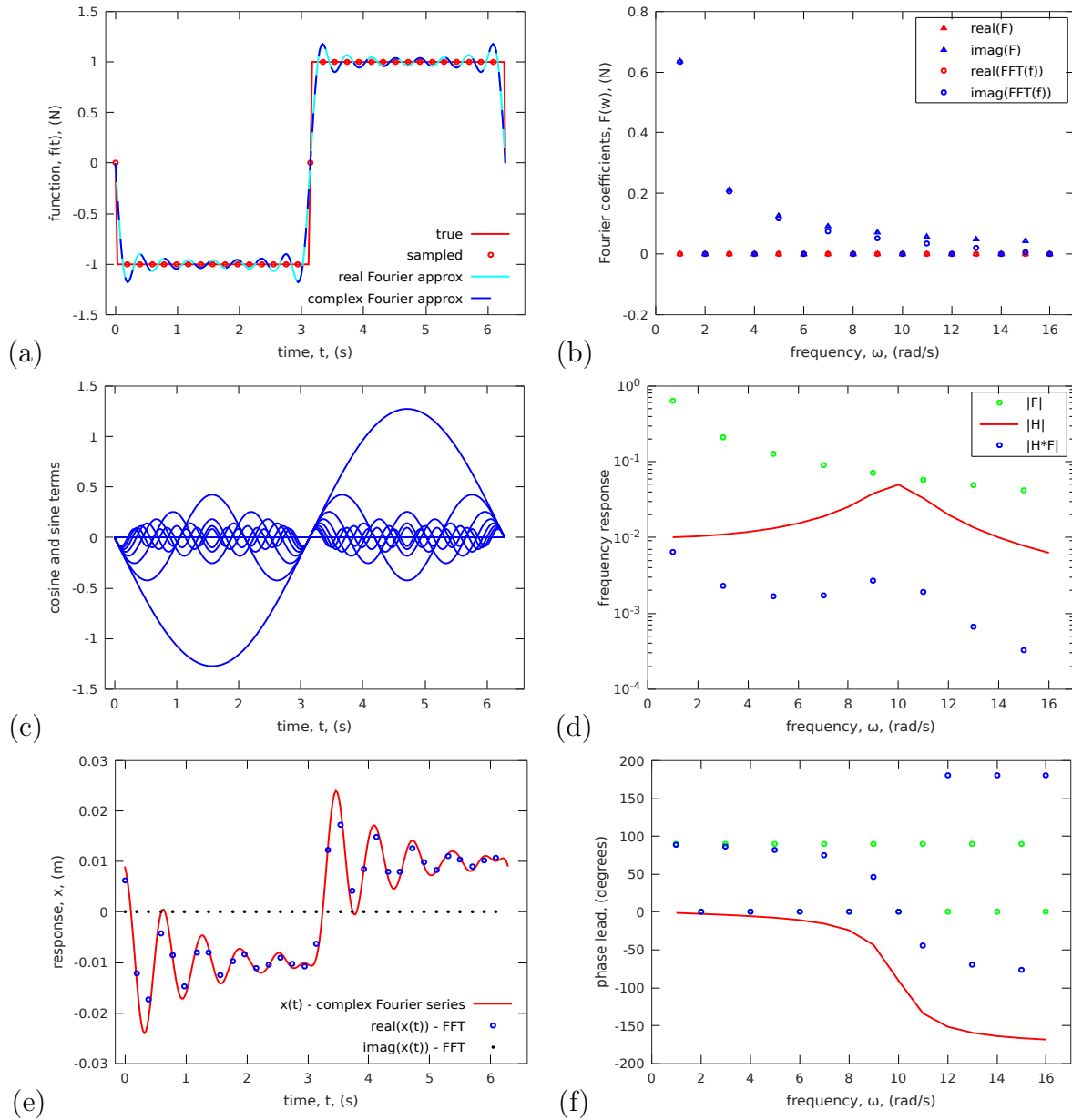


Figure 1. (a): $\text{—} = f^{\text{ext}}(t)$; $\text{--} = \tilde{f}^{\text{ext}}(t)$. (b): $\triangle = \text{Re}(F_q)$; $\triangle = \text{Im}(F_q)$; $\circ = F_q$ from FFT. (c): $\text{—} = \text{cosine components}$, $\text{—} = \text{sine components}$. (d): $\text{—} = |H(\omega)|$; $\circ = |F_q|$; $\circ = |H(\omega_q)F_q|$. (e): $\text{—} = x(t)$; $\circ = \text{Re}(x(t))$ from IFFT; $\cdots = \text{Im}(x(t))$ from IFFT. (f): $\text{—} = \text{phase of } H(\omega)$; $\circ = \text{phase of } F_q$; $\circ = \text{phase of } H(\omega_q)F_q$.

7.2 Example 2: sawtooth wave

The external forcing is given by

$$f^{\text{ext}}(t) = t - \pi ; \quad 0 \leq t < 2\pi \quad (35)$$

The Fourier coefficients for the real-valued Fourier series are:

$$a_q = 0 \text{ and } b_q = -(2/q).$$

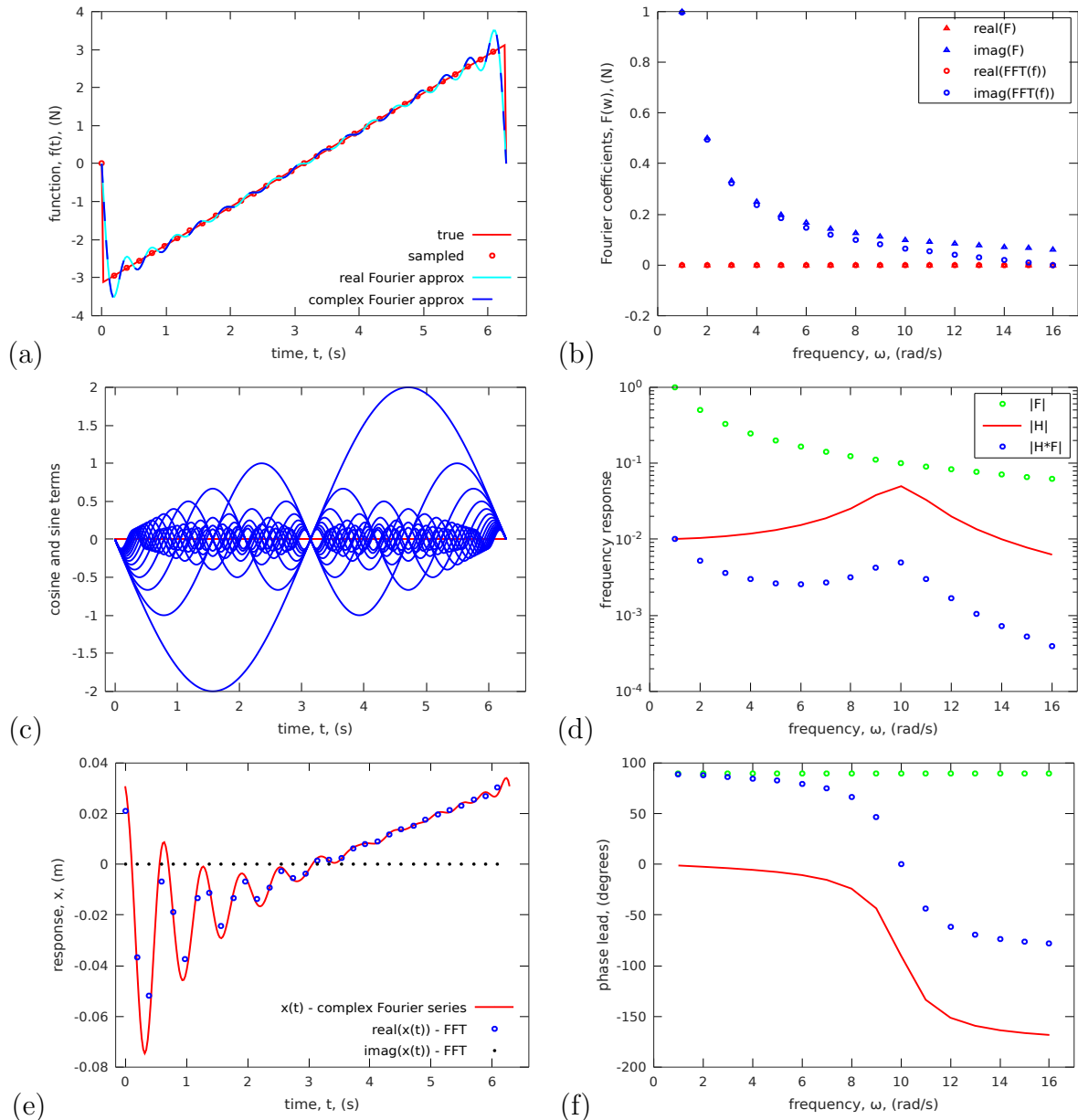


Figure 2. (a): $\text{—} = f^{\text{ext}}(t)$; $\text{--} = \tilde{f}^{\text{ext}}(t)$. (b): $\triangle = \text{Re}(F_q)$; $\triangle = \text{Im}(F_q)$; $\circ = F_q$ from FFT. (c): $\text{—} = \text{cosine components}$, $\text{—} = \text{sine components}$. (d): $\text{—} = |H(\omega)|$; $\circ = |F_q|$; $\circ = |H(\omega_q)F_q|$. (e): $\text{—} = x(t)$; $\circ = \text{Re}(x(t))$ from IFFT; $\cdots = \text{Im}(x(t))$ from IFFT. (f): $\text{—} = \text{phase of } H(\omega)$; $\circ = \text{phase of } F_q$; $\circ = \text{phase of } H(\omega_q)F_q$.

7.3 Example 3: triangle wave

The external forcing is given by

$$f^{\text{ext}}(t) = \pi/2 - (t - \pi) \operatorname{sgn}(t - \pi) ; \quad 0 \leq t < 2\pi \tag{36}$$

The Fourier coefficients for the real-valued Fourier series are:

$$a_q = (2/(\pi q^2))(-1^q - 1) \text{ and } b_q = 0.$$

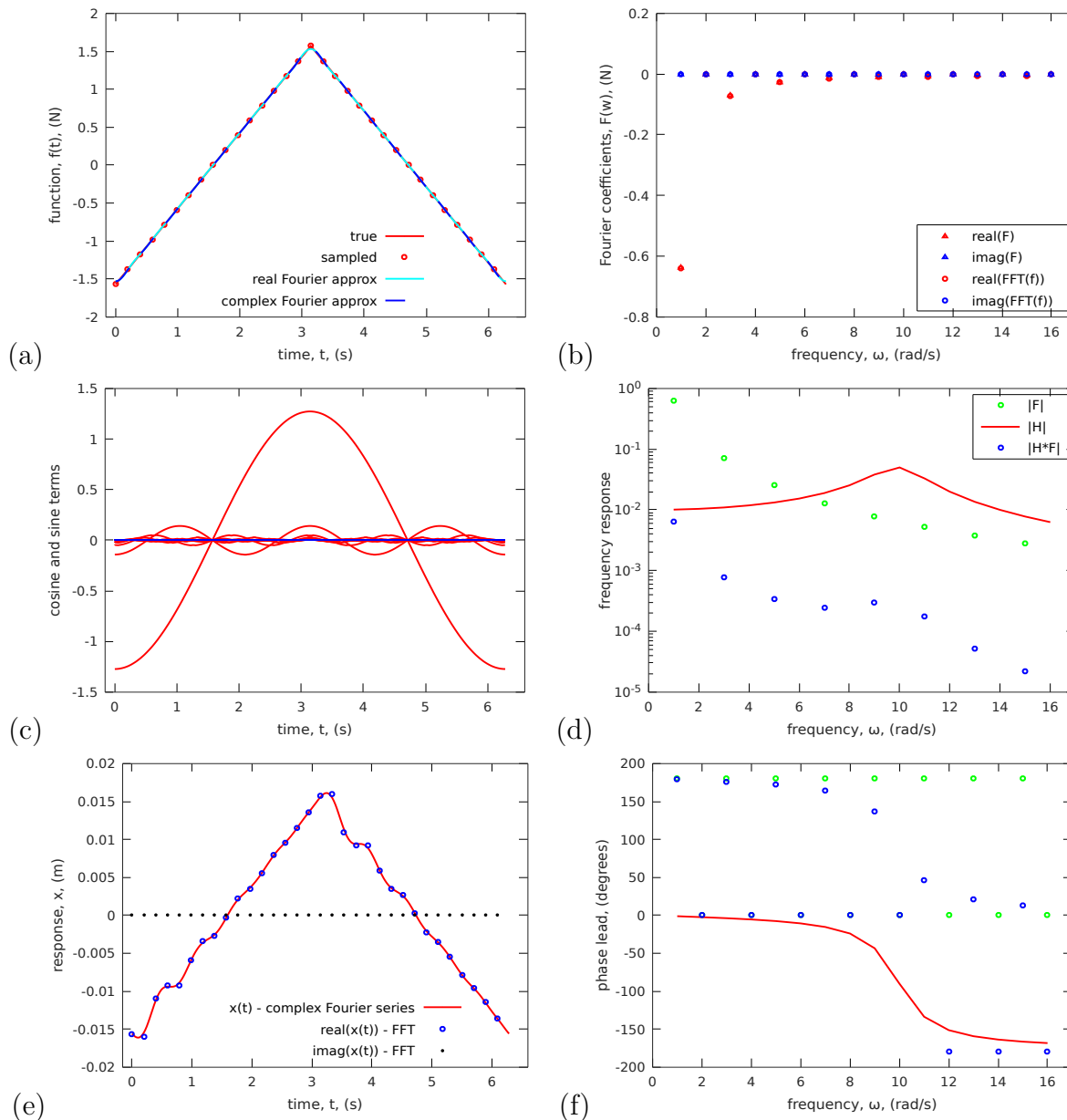


Figure 3. (a): $\text{—} = f^{\text{ext}}(t)$; $\text{--} = \tilde{f}^{\text{ext}}(t)$. (b): $\triangle = \operatorname{Re}(F_q)$; $\triangle = \operatorname{Im}(F_q)$; $\circ = F_q$ from FFT. (c): $\text{—} =$ cosine components, $\text{—} =$ sine components. (d): $\text{—} = |H(\omega)|$; $\circ = |F_q|$; $\circ = |H(\omega)F_q|$. (e): $\text{—} = x(t)$; $\circ = \operatorname{Re}(x(t))$ from IFFT; $\cdots = \operatorname{Im}(x(t))$ from IFFT. (f): $\text{—} =$ phase of $H(\omega)$; $\circ =$ phase of F_q ; $\circ =$ phase of $H(\omega)F_q$.

7.4 Matlab code

```

1  function [Fq,wq,x,t] = Fourier(type,Q,wn,z)
2  % [F,w,x] = Fourier(type,Q,wn,z)
3  % Compute the Fourier series coefficients of a periodic signal, f(t),
4  % in two different ways:
5  % * complex exponential expansion (i.e., sine and cosine expansion)
6  % * fast Fourier transform
7  %
8  % The 'type' of periodic signal may be
9  %   'square' a square wave
10 %   'sawtooth' a saw-tooth wave
11 %   'triangle' a triangle wave
12 %
13 % The period of the signal, f(t), is fixed at 2*pi (second).
14 %
15 % The number of Fourier series coefficients is input as Q.
16 % This results in a 2*Q Fourier transform coefficients.
17 %
18 % The Fourier approximation is then used to compute the steady-state
19 % response of a single degree of freedom (SDOF) oscillator, described by
20 %
21 %
22 %  $x''(t) + 2 z wn x'(t) + wn x(t) = f(t)$  ,
23 %
24 % where the mass of the system is fixed at 1 (kg).
25 %
26 % see: http://en.wikipedia.org/wiki/Fourier\_series
27 %      http://www.jhu.edu/~signals/fourier2/index.html
28 %
29
30 if nargin < 4, help Fourier; return; end
31
32 T = 2*pi;           % period of external forcing, s
33
34 a = zeros(1,Q);    % coefficients of cosine part of Fourier series
35 b = zeros(1,Q);    % coefficients of sine part of Fourier series
36
37 q = [1:Q];         % positive frequency index value
38 wq = 2*pi*q/T;     % positive frequency values, rad/sec, eq'n (14)
39
40 dt = T/(2*Q);      % 2Q discrete points in time for FFT sampling
41 ts = [0:2*Q-1]*dt; % 2Q discrete points in time for FFT sampling
42 P = 128;           % length of the time-record (for plotting purposes only)
43 t = [0:2*P]*T/2/P; % time axis
44
45 % =====
46 % Fourier series approximation of general periodic functions
47
48 % sin(q*pi) = 0 , cos(q*pi) = (-1).^q
49 % f_true is used only for plotting
50 % f_samp is used in FFT computations
51
52 if strcmp(type, 'square' )           % square wave
53     b = 2./(q*pi) .* ((-1).^q - 1);
54     f_true = sign(t-pi); f_true(1) = 0; f_true(2*P+1) = 0;
55     f_samp = sign(ts-pi);
56     f_samp(1) = 0;                   % get periodicity right, f(0) = 0
57 end
58
59 if strcmp(type, 'sawtooth' )         % sawtooth wave
60     b = -(2./q);
61     f_true = t-pi; f_true(1) = 0; f_true(2*P+1) = 0;
62     f_samp = ts-pi;
63     f_samp(1) = 0;                   % get periodicity right, f(-pi) = 0
64 end
65
66 if strcmp(type, 'triangle' )         % triangle wave
67     a = (2./(pi*q.^2)) .* ( (-1).^q - 1 );
68     f_true = pi/2 - (t-pi) .*sign(t-pi);
69     f_samp = pi/2 - (ts-pi) .*sign(ts-pi);
70 end

```

```

71 f_approxR = a * cos(wq'*t) + b * sin(wq'*t); % real Fourier series
72
73
74 Fq = (a - i*b)/2; % complex Fourier coefficients
75
76 % Complex Fourier series (positive and negative exponents)
77 f_approxC = Fq * exp(i*wq'*t) + conj(Fq) * exp(-i*wq'*t);
78
79 % The imaginary part of the complex Fourier series is exactly zero!
80 imag_over_real_1 = max(abs(imag(f_approxC))) / max(abs(real(f_approxC)))
81
82
83 % The fast Fourier transform (FFT) method
84 % In Matlab, the forward Fourier transform has a negative exponent.
85 % According to a convention of the FFT method, index number 1 is for time = 0
86
87 F_FFT = fft( f_samp ) / (2*Q); % Fourier coeff's
88
89 % =====
90 % Steady-state response of a simple oscillator to general periodic forcing.
91
92 % Complex-valued frequency response function, H(w), has units of [m/N]
93 H = (1/wn^2) ./ ( 1 - (wq/wn).^2 + 2*i*z*(wq/wn) );
94
95 % Complex Fourier series (positive and negative exponents)
96 x_approxC = (H .* Fq) * exp(i*wq'*t) + (conj(H) .* conj(Fq)) * exp(-i*wq'*t);
97
98 % In Matlab, the inverse Fourier transform has a positive exponent.
99 % ... see the table at the end of this file for the sorting convention.
100 x_FFT = ifft( [ (1/wn^2) , H , conj(H(Q-1:-1:1)) ] .* F_FFT ) * (2*Q);
101
102 % The imaginary part is exactly zero for the complex exponent series method!
103 imag_over_real_2 = max(abs(imag(x_approxC))) / max(abs(real(x_approxC)))
104
105 % The imaginary part is practically zero for the FFT method!
106 imag_over_real_3 = max(abs(imag(x_FFT))) / max(abs(real(x_FFT)))
107
108 % =====
109 % SORTING of the FFT coefficients number of data points = N = 2*Q
110 % note: f_max = 1/(2*dt); df = f_max/(N/2) = 1/(N*dt) = 1/T;
111 %
112 %
113 % index time frequency
114 %
115 % 1 0 0 imag(F)=0
116 % 2 dt df
117 % 3 2*dt 2*df
118 % 4 3*dt 3*df
119 % : : :
120 % N/2-1 (N/2-2)*dt f_max-2*df = (N/2-2)*df
121 % N/2 (N/2-1)*dt f_max-df = (N/2-1)*df
122 % N/2+1 (N/2)*dt +/- f_max = (N/2)*df imag(F)=0
123 % N/2+2 (N/2+1)*dt -f_max+df = (-N/2+1)*df
124 % N/2+3 (N/2+2)*dt -f_max+2*df = (-N/2+2)*df
125 % : : :
126 % N-2 (N-3)*dt -3*df
127 % N-1 (N-2)*dt -2*df
128 % N (N-1)*dt -df
129 % =====

```