Inelastic Response Spectra

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Department of Civil and Environmental Engineering
Duke University
Henri P. Gavin

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Modeling

The following nonlinear ordinary differential equation describes the behavior of an inelastic single degree of freedom oscillator responding to accelerations at its base,

$$m\ \ddot{r}(t) + c\ \dot{r}(t) + R(r(t), \dot{r}(t)) = -m\ \ddot{w}(t). \tag{1}$$

The variable $r(t)$ represents the relative displacement of the mass, m, with respect to the base, the variable c is a constant viscous damping coefficient and $R(r(t), \dot{r}(t))$ represents an inelastic restoring force. Now, if R_y is a yield force level and r_y is a yield displacement level, then the stiffness, k, is R_y/r_y, the *ductility response*, $\mu(t)$, is defined by $r(t)/r_\text{y}$, and equation (1) may be divided by $(m\ r_\text{y})$ to yield,

$$\ddot{\mu}(t) + 2\zeta\omega_\text{n}\ \dot{\mu}(t) + \frac{R(r(t), \dot{r}(t))}{m\ r_\text{y}} = -\frac{\ddot{w}(t)}{r_\text{y}}\ . \tag{2}$$

Defining a normalized restoring force, $z(t)$, as the ratio $R(t)/R_\text{y}$, note that

$$\frac{R(t)}{m\ r_\text{y}} = \frac{R_\text{y}\ z(t)}{m\ r_\text{y}} = \frac{k\ r_\text{y}\ z(t)}{m\ r_\text{y}} = \omega_\text{n}^2\ z(t)\ . \tag{3}$$

Defining a yield strength coefficient, C_y, as the ratio of the yield force level to the weight of the structure, note that

$$C_\text{y} = \frac{R_\text{y}}{mg} = \frac{kr_\text{y}}{mg} = \frac{\omega_\text{n}^2 r_\text{y}}{g}\ . \tag{4}$$

With these new variables equation (2) may be written,

$$\ddot{\mu}(t) + 2\zeta\omega_\text{n}\ \dot{\mu}(t) + \omega_\text{n}^2\ z(t) = -\frac{\omega_\text{n}^2}{C_\text{y} g}\ \ddot{w}(t)\ , \tag{5}$$

where, for example,

$$\dot{z}(t) = [\ 1 - |z(t)|^p\ \text{sgn}(\dot{\mu}(t) z(t))\ ]\ \dot{\mu}(t)\ . \tag{6}$$

If the inelastic restoring force has a post-yield stiffness, κk, then equation (5) may be generalized to

$$\ddot{\mu}(t) + 2\zeta\omega_\text{n}\ \dot{\mu}(t) + \omega_\text{n}^2((1-\kappa)z(t) + \kappa\mu(t)) = -\frac{\omega_\text{n}^2}{C_\text{y} g}\ \ddot{w}(t)\ . \tag{7}$$

For structures with a specified natural frequency, $\sqrt{R_\text{y}/(mr_\text{y})}$, and damping ratio, $c/(2\sqrt{mR_\text{y}/r_\text{y}})$, doubling the maximum earthquake acceleration and also doubling the yield strength coefficient, results in identical ductility responses.

Ductility demand response spectra

A ductility demand response spectrum is a plot of the maximum ductility response for a particular earthquake

$$\mu_{\max}(T_{\rm n}; \zeta, C_{\rm y}, \kappa, \ddot{w}(t)) = \max |\mu(t; T_{\rm n}, \zeta, C_{\rm y}, \kappa, \ddot{w}(t))| \ , \tag{8}$$

as a function of the structure's natural period for a particular damping ratio, a particular yield force coefficient, and a particular post-yield stiffness coefficient, κ.

Here is a procedure for computing ductility demand response spectra:

1. Select a particular earthquake accelogram record, $\ddot{w}(t)$.
2. Select a damping ratio, ζ, a yield force coefficient, $C_{\rm y}$, a post-yield stiffness factor, κ, and a set of natural periods.
3. For each natural period, $T_{\rm n} = 2\pi/\omega_{\rm n}$,
   (a) Solve the coupled ordinary differential equations (6) and (7).
   (b) Record:
   $T_{\rm n}$
   $\mu_{\max}(T_{\rm n}; \zeta, C_{\rm y}, \kappa, \ddot{w}(t))$
   $A_{\max}(T_{\rm n}; \zeta, C_{\rm y}, \kappa, \ddot{w}(t)) = \max[\ddot{\mu}(t) r_{\rm y} + \ddot{w}(t)]$
   $\dot{r}_{\max}(T_{\rm n}; \zeta, C_{\rm y}, \kappa, \ddot{w}(t)) = \dot{\mu}_{\max} r_{\rm y}$
   $r_{\max}(T_{\rm n}; \zeta, C_{\rm y}, \kappa, \ddot{w}(t)) = \mu_{\max} r_{\rm y}$
4. Plot the ductility demand spectrum $\mu_{\max}$ vs. $T_{\rm n}$
   Plot the response spectrum, $\log(\dot{r}_{\max})$ vs. $\log(T_{\rm n})$
   Plot the acceleration-displacement response spectrum, $A_{\max}$ vs. $r_{\max}$
   etc.

Equal ductility response spectra

An equal ductility response spectrum is a response spectrum for which every structure responds with an equal maximum ductility, $\max|\mu(t)|$. In order for the maximum ductility to be equal for every natural period, the required yield force coefficient, $C_{\rm y}$, of the structural system must depend on the natural period, $T_{\rm n}$, for each point in an equal ductility response spectrum. A trial-and-error approach using bisection or some other nonlinear root-finding method is required to find the proper value of the yield force coefficient for each natural period. In many cases, this nonlinear optimization problem does not have a unique solution, as documented in the references below, and enforcing a very small convergence tolerance may have questionable value.

The following procedure for computing equal ductility response spectra endeavors to start the algorithm with a pair of good initial guesses of values of $C_{\rm y}$ corresponding to a target ductility demand μ_t. The algorithm makes use of a combination of hyperbolic fits, the secant method, and Newton-Raphson iterations.

1. Select a particular earthquake accelogram record, $\ddot{w}(t)$.
2. Select a damping ratio, ζ, a target maximum ductility, μ_t, a post-yield stiffness factor, κ, and a set of natural periods sorted from short to long.
3. Select values of the yield strength coefficient $C_{\rm y}$ representative of the smallest and largest expected values, $C_{\rm y(min)} = 0.01$ and $C_{\rm y(max)} = 2.0$. The values of $C_{\rm y(min)}$ and $C_{\rm y(max)}$ need not bound the solution.
4. For the shortest natural period in the set:
   (a) Solve the coupled ordinary differential equations (6) and (7) for $C_{\rm y} = C_{\rm y(min)}$ and record the resulting ductility demand as $\mu_{\rm (max)}$.
   (b) Solve the coupled ordinary differential equations (6) and (7) for $C_{\rm y} = C_{\rm y(max)}$ and record the resulting ductility demand as $\mu_{\rm (min)}$.
   (c) A hyperbola of the form $\mu = a/(C_{\rm y}+b)$ passing through coordinates $(C_{\rm y(min)}, \mu_{\rm (max)})$ and $(C_{\rm y(max)}, \mu_{\rm (min)})$ has coefficients

$$a = \mu_{\rm (min)}\mu_{\rm (max)} \frac{C_{\rm y(max)} - C_{\rm y(min)}}{\mu_{\rm (max)} - \mu_{\rm (min)}}$$

   and

$$b = \frac{C_{\rm y(max)}\mu_{\rm (min)} - C_{\rm y(min)}\mu_{\rm (max)}}{\mu_{\rm (max)} - \mu_{\rm (min)}}$$

   (d) Using this hyperbola as an approximation for the relationship between the yield strength coefficient and the ductility demand, determine values $C_{\rm y1}$ and $C_{\rm y2}$ that correspond to $1.2\mu_t$ and $0.8\mu_t$.

$$C_{\rm y1} = \max[(a/(1.2\mu_t) - b), C_{\rm y(min)}]$$

$$C_{\rm y2} = \min[(a/(0.8\mu_t) - b), C_{\rm y(max)}]$$

5. For each natural period, $T_{\rm n} = 2\pi/\omega_{\rm n}$:
   (a) For the shortest natural period in the set, use $C_{\rm y1}$ and $C_{\rm y2}$ from step 4.(d), otherwise assign $C_{\rm y1} = 1.5C_{\rm y(old)}$ and $C_{\rm y2} = 1.4C_{\rm y(old)}$, where $C_{\rm y(old)}$ is the value of $C_{\rm y}$ determined previously for the next shortest natural period. Choosing $C_{\rm y1}$ and $C_{\rm y2}$ to be larger than what was found for the previous natural period helps the following steps to converge to the largest yield strength coefficient providing the target ductility demand.
   (b) Solve the coupled ordinary differential equations (6) and (7) for $C_{\rm y} = C_{\rm y1}$ and record the resulting ductility demand as μ_1.

(c) Solve the coupled ordinary differential equations (6) and (7) for $C_\text{y} = C_\text{y2}$ and record the resulting ductility demand as μ_2.

(d) Compute the slope of the straight line connecting coordinates (C_y1, μ_1) and (C_y2, μ_2).

$$S = (\mu_2 - \mu_1)/(C_\text{y2} - C_\text{y1})$$

(e) Evaluate $\Delta C_\text{y} = \min\left[|\mu_t - \mu_2|/S, 0.1(C_\text{y1} + C_\text{y2})\right]$.

(f) If $(\mu_t - \mu_2)/S < 0$, then change the sign of ΔC_y.

(g) If $S > 0$ and $\mu_2 < \mu_t$, then assign $\Delta C_\text{y} = -0.5 C_\text{y2}$.

(h) If $S > 0$ and $\mu_2 > \mu_t$, then assign $\Delta C_\text{y} = 0.1 C_\text{y2}$.

(i) If $C_\text{y2} + \Delta C_\text{y} < 0$, then use a secant interpolation,

$$\Delta C_\text{y} = \frac{(C_{\text{y(min)}} - C_\text{y2})(\mu_t - \mu_2)}{\mu_{(\text{max})} - \mu_2}$$

(j) Assign C_y2 to C_y1 and μ_2 to μ_1.

(k) Update C_y2. $C_\text{y2} := C_\text{y2} + \Delta C_\text{y}$.

(l) Solve the coupled ordinary differential equations (6) and (7) for $C_\text{y} = C_\text{y2}$ and record the resulting peak ductility as μ_2.

(m) Check for convergence. If $|C_\text{y1} - C_\text{y2}| < 10^{-6}$ or $|\mu_1 - \mu_2| < 10^{-6}$ or $|\mu_2 - \mu_t| < 10^{-2}$, then assign $C_\text{y} = C_\text{y2}$, and continue to step 5.(n)
otherwise, repeat steps 5.(a) through 5.(m).

(n) Record:
T_n
$C_\text{y}(T_\text{n}; \zeta, \mu_t, \kappa, \ddot{w}(t))$
$r_\text{max}(T_\text{n}; \zeta, \mu_t, \kappa, \ddot{w}(t)) = \mu_\text{max} r_\text{y}$
$\dot{r}_\text{max}(T_\text{n}; \zeta, \mu_t, \kappa, \ddot{w}(t)) = \dot{\mu}_\text{max} r_\text{y}$
$A_\text{max}(T_\text{n}; \zeta, \mu_t, \kappa, \ddot{w}(t)) = \max[\ddot{\mu}(t) r_\text{y} + \ddot{w}(t)]$

6. Plot the equal ductility yield force coefficient spectrum, C_y vs. T_n
Plot the equal ductility response spectrum, $\log(\dot{r}_\text{max})$ vs. $\log(T_\text{n})$
Plot the acceleration-displacement response spectrum, A_max vs. r_max
etc.

This algorithm requires four time history analyses to initialize the iterations for the first (shortest) natural period in the set, and two time history analyses to initialize the iterations for the second and all subsequent natural periods. Convergence is achieved typically within six iterations (ten time history analyses) for the first natural period and within two iterations (four time history analyses) for the second and all subsequent natural periods. Determining responses with a specified target ductility demand greater than five in structures with natural periods greater than five seconds, and subjected to low-level earthquakes, can require up to thirty time history analyses. The algorithm can be made more efficient in such cases by setting $C_{\text{y(min)}}$ to 0.001.

Numerical Solution for Inelastic Behavior

A number of numerical methods may be used to solve equations (6) and (7) for the response of inelastic systems to transient excitation. The method described here involves casting these equations into an incremental form, in which the dynamic variables are the *changes* in displacement, velocity, and accelerations from one time step to the next. The incremental velocity and incremental acceleration are related to the incremental displacement and the values of velocity and acceleration at the current time step using the Newmark-β method. Newton-Raphson iterations are required to solve for the incremental displacement at each time step because of the nonlinearity in equation (6). Once the incremental displacement is found, the incremental velocity is computed from the Newmark-β approximation, the displacement and velocity are updated, and the acceleration at the next time step is computed from the equation of dynamic equilibrium, equation (7).

Consider equations (6) and (7) evaluated at time $t_{i+1} = (i+1)h$,

$$\dot{z}_{i+1} = [\ 1 - |z_{i+1}|^p \ \mathrm{sgn}(\dot{\mu}_{i+1} z_{i+1})\ ]\ \ \dot{\mu}_{i+1}\ , \tag{9}$$

$$\ddot{\mu}_{i+1} + 2\zeta\omega_{\mathrm{n}}\ \dot{\mu}_{i+1} + \omega_{\mathrm{n}}^2((1-\kappa)z_{i+1} + \kappa\mu_{i+1}) = -\frac{\omega_{\mathrm{n}}^2}{C_{\mathrm{y}} g}\ \ddot{w}_{i+1}\ , \tag{10}$$

and at time $t_i = ih$,

$$\dot{z}_i = [\ 1 - |z_i|^p \ \mathrm{sgn}(\dot{\mu}_i z_i)\ ]\ \ \dot{\mu}_i\ , \tag{11}$$

$$\ddot{\mu}_i + 2\zeta\omega_{\mathrm{n}}\ \dot{\mu}_i + \omega_{\mathrm{n}}^2((1-\kappa)z_i + \kappa\mu_i) = -\frac{\omega_{\mathrm{n}}^2}{C_{\mathrm{y}} g}\ \ddot{w}_i\ , \tag{12}$$

where h is the constant time step increment.[1] Subtracting equation (12) from (10),

$$\delta\ddot{\mu}_i + 2\zeta\omega_{\mathrm{n}}\ \delta\dot{\mu}_i + \omega_{\mathrm{n}}^2((1-\kappa)(z_{i+1} - z_i) + \kappa\delta\mu_i) = -\frac{\omega_{\mathrm{n}}^2}{C_{\mathrm{y}} g}\ \delta\ddot{w}_i\ , \tag{13}$$

where $\mu_{i+1} = \mu_i + \delta\mu_i$, and $\ddot{w}_{i+1} = \ddot{w}_i + \delta\ddot{w}_i$.

The Newmark-β approximation for $\delta\ddot{\mu}_i$, $\delta\dot{\mu}_i$, and z_{i+1} are

$$\delta\ddot{\mu}_i = \frac{1}{\beta h^2}\delta\mu_i - \frac{1}{\beta h}\dot{\mu}_i - \frac{1}{2\beta}\ddot{\mu}_i \tag{14}$$

$$\delta\dot{\mu}_i = \frac{\gamma}{\beta h}\delta\mu_i - \frac{\gamma}{\beta}\dot{\mu}_i + h\left(1 - \frac{\gamma}{2\beta}\right)\ddot{\mu}_i \tag{15}$$

$$z_{i+1} = z_i + h[(1-\gamma)\dot{z}_i + \gamma\dot{z}_{i+1}]\ , \tag{16}$$

where β and γ are algorithmic constants. Recommended values are $\beta = 1/6$ and $\gamma = 1/2$. Substituting equations (14) and (15) into equation (13) results in

$$\frac{1}{\beta h^2}\delta\mu_i - \frac{1}{\beta h}\dot{\mu}_i - \frac{1}{2\beta}\ddot{\mu}_i + 2\zeta\omega_{\mathrm{n}}\left(\frac{\gamma}{\beta h}\delta\mu_i - \frac{\gamma}{\beta}\dot{\mu}_i + h\left(1 - \frac{\gamma}{2\beta}\right)\ddot{\mu}_i\right) + \kappa\omega_{\mathrm{n}}^2\delta\mu_i + (1-\kappa)\omega_{\mathrm{n}}^2(z_{i+1} - z_i) = -\frac{\omega_{\mathrm{n}}^2}{C_{\mathrm{y}} g}\delta\ddot{w}_i\ . \tag{17}$$

[1] Note: ω_{n} is the n*atural frequency* and $\mu_i = \mu(t_i) = \mu(ih)$.

At time step i, equation (17) has two unknowns, $\delta\mu_i$ and z_{i+1}. Substituting equations (9), (11), and (15) into (16) results in the second equation required to solve for the two unknowns.

$$
\begin{aligned}
z_{i+1} &= z_i + h\Bigg[(1-\gamma)\dot{z}_i + \\
&\quad \gamma \left[1 - |z_{i+1}|^p \, \mathrm{sgn}\left(\left(\dot{\mu}_i + \frac{\gamma}{\beta h}\delta\mu_i - \frac{\gamma}{\beta}\dot{\mu}_i + h\left(1-\frac{\gamma}{2\beta}\right)\ddot{\mu}_i \right) z_{i+1} \right) \right] \times \\
&\quad \left(\dot{\mu}_i + \frac{\gamma}{\beta h}\delta\mu_i - \frac{\gamma}{\beta}\dot{\mu}_i + h\left(1-\frac{\gamma}{2\beta}\right)\ddot{\mu}_i \right) \Bigg] . \qquad (18)
\end{aligned}
$$

Equations (17) and (18) are the two equations for the unknowns $\delta\mu_i$ and z_{i+1}. At time step t_i, all the remaining variables in these equations are known. Equation (17) is linear in $\delta\mu_i$ and z_{i+1}. Solving equation (17) for $\delta\mu_i$ in terms of z_{i+1},

$$
\begin{aligned}
\left(\frac{1}{\beta h^2} + 2\zeta\omega_{\mathrm{n}}\frac{\gamma}{\beta h} + \kappa\omega_{\mathrm{n}}^2 \right)\delta\mu_i^{(n)} &= \left(\frac{1}{\beta h} + 2\zeta\omega_{\mathrm{n}}\frac{\gamma}{\beta} \right)\dot{\mu}_i \\
&\quad + \left(\frac{1}{2\beta} + 2\zeta\omega_{\mathrm{n}} h\left(1-\frac{\gamma}{2\beta}\right) \right)\ddot{\mu}_i \\
&\quad - (1-\kappa)\omega_{\mathrm{n}}^2\left(z_{i+1}^{(n)} - z_i\right) - \frac{\omega_{\mathrm{n}}^2}{C_{\mathrm{y}} g}\delta\ddot{w}_i \, , \qquad (19)
\end{aligned}
$$

where the superscript (n) refers to Newton-Raphson iteration number n.

Given $\dot{\mu}_i$, $\ddot{\mu}_i$, z_i, $\delta\ddot{w}_i$ and an estimate for $z_{i+1}^{(n)}$, at Newton-Raphson iteration i, equation (19) gives the incremental ductility $\delta\mu_i^{(n)}$. The dimensionless inelastic force $z_{i+1}^{(n)}$ is found from the root of the function $f\left(z_{i+1}^{(n)}\right)$,

$$
\begin{aligned}
f\left(z_{i+1}^{(n)}\right) &= z_{i+1}^{(n)} - z_i - h\Bigg[(1-\gamma)\dot{z}_i + \\
&\quad \gamma \left[1 - \left|z_{i+1}^{(n)}\right|^p \, \mathrm{sgn}\left(\left(\dot{\mu}_i + \frac{\gamma}{\beta h}\delta\mu_i^{(n)} - \frac{\gamma}{\beta}\dot{\mu}_i + h\left(1-\frac{\gamma}{2\beta}\right)\ddot{\mu}_i \right) z_{i+1} \right) \right] \times \\
&\quad \left(\dot{\mu}_i + \frac{\gamma}{\beta h}\delta\mu_i^{(n)} - \frac{\gamma}{\beta}\dot{\mu}_i + h\left(1-\frac{\gamma}{2\beta}\right)\ddot{\mu}_i \right) \Bigg] , \qquad (20)
\end{aligned}
$$

where equation (19) is solved for $\delta\mu_i^{(n)}$ as a function of $z_{i+1}^{(n)}$ at each Newton-Raphson iteration.

At each time step, n, Newton-Raphson iterations are carried out with the knowledge that $-1 < z(t) < 1$. The following algorithm attempts to start the Newton-Raphson iterations with a pair of very good initial guesses for z_{i+1}.

Algorithm for Numerical Solution for Inelastic Behavior

1. Initialization:
   Specify a convergence tolerance, $\epsilon = 10^{-12}$.
   Initialize the Newton-Raphson iteration counter, $n = 0$.
   Provide a first initial guess for the solution at $i+1$ as the solution at i, $z_{i+1}^{(0)} = z_i$.
   Solve equation (19) for $\delta\mu_i^{(0)}$ using $\left(z_{i+1}^{(0)} - z_i\right) = 0$.
   Evaluate $\dot{\mu}_{i+1}^{(0)} = (1-\gamma/\beta)\dot{\mu}_i + (\gamma/(\beta h))\delta\mu_i^{(0)} + h(1-\gamma/(2\beta))\ddot{\mu}_i$.
   Evaluate $\dot{z}_i = \left[1 - |z_i|^p \text{sgn}(\dot{\mu}_i z_i)\right]\dot{\mu}_i$.
   Evaluate $\dot{z}_{i+1}^{(0)} = \left[1 - |z_i|^p \text{sgn}\left(\dot{\mu}_{i+1}^{(0)} z_i\right)\right]\dot{\mu}_{i+1}^{(0)}$.
   Evaluate $f(z_{i+1}^{(0)}) = -h\left[(1-\gamma)\dot{z}_i + \gamma\dot{z}_{i+1}^{(0)}\right]$.
   If $\dot{z}_{i+1}^{(0)} \neq 0$, assign $\Delta z_{i+1}^{(0)} = \dot{z}_{i+1}^{(0)} h$,
   otherwise, assign $\Delta z_{i+1}^{(0)} = 10\,\epsilon\,\text{sgn}\left(\dot{\mu}_{i+1}^{(0)}\right)$.
   Increment the Newton-Raphson iteration counter, $n = 1$.
   Evaluate a second guess for the solution, $z_{i+1}^{(1)} = z_{i+1}^{(0)} + \Delta z_{i+1}^{(0)}$.
   This completes the initialization.

2. Solve equation (19) for $\delta\mu_i^{(n)}$ using $\left(z_{i+1}^{(n)} - z_i\right)$.

3. Evaluate $\dot{\mu}_{i+1}^{(n)} = (1-\gamma/\beta)\dot{\mu}_i + (\gamma/(\beta h))\delta\mu_i^{(n)} + h(1-\gamma/(2\beta))\ddot{\mu}_i$.

4. Evaluate $\dot{z}_{i+1}^{(n)} = \left[1 - \left|z_{i+1}^{(n)}\right|^p \text{sgn}\left(\dot{\mu}_{i+1}^{(n)} z_{i+1}^{(n)}\right)\right]\dot{\mu}_{i+1}^{(n)}$.

5. Evaluate $f(z_{i+1}^{(n)}) = z_{i+1}^{(n)} - z_{i+1}^{(n-1)} - h\left[(1-\gamma)\dot{z}_i + \gamma\dot{z}_{i+1}^{(n)}\right]$.

6. If $\left|z_{i+1}^{(n)} - z_{i+1}^{(n-1)}\right| < 10^{-9}$ or $\left|f(z_{i+1}^{(n)}) - f(z_{i+1}^{(n-1)})\right| < 10^{-6}$, then assign $\Delta z_{i+1}^{(n)} = 0$,
   otherwise, approximate the derivative of the function $f\left(z_{i+1}^{(n)}\right)$ with respect to $z_{i+1}^{(n)}$,
   $$f'^{(n)} = \left(f\left(z_{i+1}^{(n)}\right) - f\left(z_{i+1}^{(n-1)}\right)\right) / \left(z_{i+1}^{(n)} - z_{i+1}^{(n-1)}\right)$$
   and determine the Newton-Raphson increment in z_{i+1}
   $$\Delta z_{i+1}^{(n)} = -f\left(z_{i+1}^{(n)}\right) / f'^{(n)} \; .$$

7. Increment the dimensionless inelastic force $z_{i+1}^{(n+1)} = z_{i+1}^{(n)} + \Delta z_{i+1}^{(n)}$

8. If $z_{i+1}^{(n+1)} > 1$, then assign $z_{i+1}^{(n+1)} = 1$.
   If $z_{i+1}^{(n+1)} < -1$, then assign $z_{i+1}^{(n+1)} = -1$.

9. If $\left|\Delta z_{i+1}^{(n)}\right| < \epsilon$, then convergence is achieved,
   otherwise, increment the iteration counter, $n := n+1$, and return to step 2.

Convergence of the Newton-Raphson iterations provides accurate values for $\delta\mu_i$ and z_{i+1}. The displacement is updated using $\mu_{i+1} = \mu_i + \delta\mu_i$, the new velocity is $\dot{\mu}_{i+1}^{(n)}$ and the new acceleration is computed by solving equation (10) for $\ddot{\mu}_{i+1}$. This completes the computations for time step i. The time step counter is then incremented, $i := i + 1$, and the Newton-Raphson iterations resume for the next time step.

This numerical method is unconditionally stable, is exceptionally accurate, and executes faster than the fourth-order Runge-Kutta method with constant time steps.

References

1. Chopra, Anil, "Elastic response spectrum: a historical note," *Earthquake Engineering and Structural Dynamics,* vol 36, pp 3-12 (2007).

2. Inel, M., Aschhmeit, M.A., Abrams, D.P., "An Algorithm for Computing Isoductile Response Spectra," *Journal of Earthquake Engineering,* vol 6, pp 375-390 (2002).

3. Riddell, Raphael, "Inelastic response spectrum: early history," *Earthquake Engineering and Structural Dynamics,* vol 37, pp 1175-1183 (2008).

4. Tena-Colunga, A., "Displacement ductility demand spectra for the seismic evaluation of structures," *Engineering Structures,* vol 23, pp 1319-1330 (2001).

5. Wen, Y-K. "Method for Random Vibration of Hysteretic Systems," *Journal of Engineering Mechanics,* vol 102, no EM2, pp 249-263 (1976).