Modal Superposition

CEE 541. Structural Dynamics
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1 Solve the eigen-value problem and diagonalize the matrix second order o.d.e's

Given a matrix second order ordinary differential equation for a structural system with N degrees of freedom,

$$\boldsymbol{M}\ddot{\boldsymbol{r}}(t) + \boldsymbol{C}\dot{\boldsymbol{r}}(t) + \boldsymbol{K}\boldsymbol{r}(t) = \boldsymbol{f}(t) \tag{1}$$

with positive definite mass and stiffness matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ and a diagonalizeable damping matrix $\boldsymbol{C}$ (or a set of modal damping ratios, ζ_j), solve the generalized eigen-value problem,

$$[\boldsymbol{K} - {\omega_{\mathrm{n}}}^2_j \boldsymbol{M}]\bar{\boldsymbol{r}}_j = \boldsymbol{0} \tag{2}$$

for a number of eigen-values (squared natural frequencies), ${\omega_{\mathrm{n}}}^2_j$, and eigen-vectors (modal vectors), $\bar{\boldsymbol{r}}_j$. The modal vectors will be assumed to be mass-normalized, $\bar{\boldsymbol{r}}_j^{\mathsf{T}}\boldsymbol{M}\bar{\boldsymbol{r}}_j = m_j^* = 1$ and $\bar{\boldsymbol{r}}_j^{\mathsf{T}}\boldsymbol{K}\bar{\boldsymbol{r}}_j = k_j^* = {\omega_{\mathrm{n}}}^2_j$. This can be done by dividing each modal vector $\bar{\boldsymbol{r}}_j$ by $\sqrt{m_j^*}$. For mass-normalized modal vectors the diagonalized (un-coupled) second order equations of motion are

$$\ddot{q}_j(t) + 2\zeta_j\omega_{\mathrm{n}j}\dot{q}_j(t) + {\omega_{\mathrm{n}}}^2_j q_j(t) = \bar{\boldsymbol{r}}_j^{\mathsf{T}}\boldsymbol{f}(t) = p_j(t) \ . \tag{3}$$

A relatively large value of the modal forcing $p_j(t)$ indicates that the forcing vector $\boldsymbol{f}(t)$ couples strongly to the j^{th} mode, $\bar{\boldsymbol{r}}_j$ at time t.

2 Assemble Modal Responses into Structural Responses

If the modal forcing, $p_j(t)$, is an impulse, simple harmonic, or periodic, obtain $q_j(t)$ analytically. Otherwise solve for $q_j(t)$ numerically. Once $q_j(t)$ has been found for a selected number of modes, assemble the modes to get the response of all the structural coordinates.

$$\boldsymbol{r}(t) = \sum_j \bar{\boldsymbol{r}}_j q_j(t) \tag{4}$$

2.1 Transient response to Arbitrary Initial Conditions

$$\boldsymbol{r}(0) = \boldsymbol{d}_o \qquad \dot{\boldsymbol{r}}(0) = \boldsymbol{v}_o \tag{5}$$

$$q_j(0) = q_{oj} = \bar{\boldsymbol{r}}_j^{\mathsf{T}}\boldsymbol{M}\boldsymbol{d}_o \qquad \dot{q}_j(0) = \dot{q}_{oj} = \bar{\boldsymbol{r}}_j^{\mathsf{T}}\boldsymbol{M}\boldsymbol{v}_o \tag{6}$$

$$q_j(t) = e^{-\zeta_j\omega_{\mathrm{n}j}t}\left(q_{oj}\cos\omega_{\mathrm{d}j}t + \frac{\zeta_j\omega_{\mathrm{n}j}q_{oj} + \dot{q}_{oj}}{\omega_{\mathrm{d}j}}\sin\omega_{\mathrm{d}j}t\right) \tag{7}$$

$$\boldsymbol{r}(t) = \sum_j \bar{\boldsymbol{r}}_j q_j(t) \tag{8}$$

2.2 Transient response to Impulse Forcing

$$\boldsymbol{f}(t) = \bar{\boldsymbol{f}}\ \delta(t) \tag{9}$$

$$p_j(t) = \bar{\boldsymbol{r}}_j^{\mathsf{T}} \bar{\boldsymbol{f}}\ \delta(t) = \bar{p}_j\ \delta(t) \tag{10}$$

$$q_j(t) = \bar{p}_j h_j(t) = \bar{p}_j \frac{1}{\omega_{\mathrm{d}j}} e^{-\zeta_j \omega_{\mathrm{n}j} t} \sin \omega_{\mathrm{d}j} t \tag{11}$$

$$\boldsymbol{r}(t) = \sum_j \bar{\boldsymbol{r}}_j \bar{p}_j \frac{1}{\omega_{\mathrm{d}j}} e^{-\zeta_j \omega_{\mathrm{n}j} t} \sin \omega_{\mathrm{d}j} t \tag{12}$$

2.3 Steady-state response to Simple Harmonic Forcing

$$\boldsymbol{f}(t) = \bar{\boldsymbol{f}} \cos \omega t \tag{13}$$

$$p_j(t) = \bar{\boldsymbol{r}}_j^{\mathsf{T}} \bar{\boldsymbol{f}} \cos \omega t = \bar{p}_j \cos \omega t \tag{14}$$

$$q_j(t) = \bar{p}_j\ \bar{q}_j\ [(1 - \Omega_j^2) \cos \omega t + (2 \zeta_j \Omega_j) \sin \omega t] \tag{15}$$

$$\bar{q}_j = \frac{1/{\omega_{\mathrm{n}}}^2{}_j}{(1 - \Omega_j^2)^2 + (2 \zeta_j \Omega_j)^2} \qquad \Omega_j = \omega / \omega_{\mathrm{n}j} \tag{16}$$

Defining $\bar{q}_{\mathrm{c}j} = \bar{q}_j (1 - \Omega_j^2)$ and $\bar{q}_{\mathrm{s}j} = \bar{q}_j (2 \zeta_j \Omega_j)$,

$$\boldsymbol{r}(t) = \sum_j \bar{\boldsymbol{r}}_j \bar{p}_j \bar{q}_{\mathrm{c}j} \cos \omega t + \sum_j \bar{\boldsymbol{r}}_j \bar{p}_j \bar{q}_{\mathrm{s}j} \sin \omega t \tag{17}$$

A relatively large value of $\bar{p}_j$ indicates that the distribution $\bar{\boldsymbol{f}}$ of harmonic loads within a structure couples strongly to the j^{th} mode, $\bar{\boldsymbol{r}}_j$. A relatively large value of $\bar{q}_j$ indicates that the forcing frequency, ω, is close to the resonant frequency of the j^{th} mode, $\omega_{\mathrm{n}j}$. The modal response is scaled by the product $(\bar{p}_j \bar{q}_j)$. What are the units of $\bar{\boldsymbol{r}}_j$, $\bar{p}_j$, $\bar{q}_j$, and $(\bar{\boldsymbol{r}}_j \bar{p}_j \bar{q}_j)$?

2.4 Steady-state response to Periodic Forcing

$$\boldsymbol{f}(t) = \sum_{k=-K}^{K} \bar{\boldsymbol{f}}_k e^{i \omega_k t} \tag{18}$$

$$p_j(t) = \sum_k \bar{\boldsymbol{r}}_j^{\mathsf{T}} \bar{\boldsymbol{f}}_k e^{i \omega_k t} = \sum_k \bar{p}_{jk} e^{i \omega_k t} \tag{19}$$

$$q_j(t) = \sum_k \bar{p}_{jk} H_j(\omega_k) e^{i \omega_k t} \tag{20}$$

$$= \sum_k \bar{p}_{jk} \frac{1/{\omega_{\mathrm{n}}}^2{}_j}{1 - \Omega_{jk}^2 + i 2 \zeta_j \Omega_{jk}} e^{i \omega_k t} \qquad \Omega_{jk} = \omega_k / \omega_{\mathrm{n}j} \tag{21}$$

$$\boldsymbol{r}(t) = \sum_j \bar{\boldsymbol{r}}_j \sum_k \bar{p}_{jk} \frac{1/{\omega_{\mathrm{n}}}^2{}_j}{1 - \Omega_{jk}^2 + i 2 \zeta_j \Omega_{jk}} e^{i \omega_k t} \tag{22}$$