Model Condensation

CEE 541. Structural Dynamics
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Problem Statement

Given a symmetric stiffness matrix $\boldsymbol{K}$ and a symmetric mass matrix $\boldsymbol{M}$ of a "full" model with N coordinates and given a subset of n primary coordinates, p ($1 \leq p_i \leq N$, $i = 1, \cdots, n$); find condensed stiffness and mass matrices $\boldsymbol{K}_{\rm c}$ and $\boldsymbol{M}_{\rm c}$ such that the dynamic characteristics of $\boldsymbol{K}$ and $\boldsymbol{M}$ are retained at the n condensed coordinates, p.

1 Static Condensation

Consider the partitioning of the full stiffness matrix equation

$$\left[\begin{array}{c} \boldsymbol{f}_p \\ \boldsymbol{f}_s \end{array} \right] = \left[\begin{array}{cc} \boldsymbol{K}_{pp} & \boldsymbol{K}_{ps} \\ \boldsymbol{K}_{sp} & \boldsymbol{K}_{ss} \end{array} \right]_{N \times N} \left[\begin{array}{c} \boldsymbol{r}_p \\ \boldsymbol{r}_s \end{array} \right] \tag{1}$$

where $\boldsymbol{K}_{pp}$ is $n \times n$. Assume the external forces on the secondary coordinates are all zero and that the displacements at the secondary coordinates are of no interest.

$$\begin{aligned} \boldsymbol{f}_s &= \boldsymbol{K}_{sp}\boldsymbol{r}_p + \boldsymbol{K}_{ss}\boldsymbol{r}_s = 0 \qquad \rightarrow \qquad \boldsymbol{r}_s = -\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sp}\boldsymbol{r}_p & (2) \\ \boldsymbol{f}_p &= \boldsymbol{K}_{pp}\boldsymbol{r}_p + \boldsymbol{K}_{ps}\boldsymbol{r}_s \qquad = \qquad \boldsymbol{K}_{pp}\boldsymbol{r}_p - \boldsymbol{K}_{sp}\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sp}\boldsymbol{r}_p & (3) \\ \boldsymbol{K}_{\rm c} &= [\boldsymbol{K}_{pp} - \boldsymbol{K}_{ps}\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sp}] & (4) \end{aligned}$$

Since $\boldsymbol{f}_s$ is assumed to be zero, $\boldsymbol{r}_s$ may be computed from $\boldsymbol{r}_p$ directly.

$$\begin{aligned} \left[\begin{array}{c} \boldsymbol{r}_p \\ \boldsymbol{r}_s \end{array} \right] &= \left[\begin{array}{c} \boldsymbol{I}_{n \times n} \\ -\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sp} \end{array} \right]_{N \times n} \left[\, \boldsymbol{r}_p \, \right] & (5) \\ \boldsymbol{r} &= \qquad \boldsymbol{T}_{\rm c} \qquad\qquad \boldsymbol{r}_p & (6) \end{aligned}$$

Using the coordinate transformation matrix $\boldsymbol{T}_{\rm c}$,

$$\begin{aligned} \boldsymbol{K}_{\rm c} &= \boldsymbol{T}_{\rm c}^{\mathsf{T}} \; \boldsymbol{K} \; \boldsymbol{T}_{\rm c} & (7) \\ &= \left[\begin{array}{cc} \boldsymbol{I}_{n \times n} & -\boldsymbol{K}_{ps}\boldsymbol{K}_{ss}^{-1} \end{array} \right] \left[\begin{array}{cc} \boldsymbol{K}_{pp} & \boldsymbol{K}_{ps} \\ \boldsymbol{K}_{sp} & \boldsymbol{K}_{pp} \end{array} \right] \left[\begin{array}{c} \boldsymbol{I}_{n \times n} \\ -\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sp} \end{array} \right] & (8) \end{aligned}$$

You can confirm that equation (8) simplifies to equation (4).

2 Guyan Condensation

In the Guyan Condensation method, the static condensation coordinate transformation is simply applied to the mass matrix,

$$\boldsymbol{M}_{\rm c} = \boldsymbol{T}_{\rm c}^{\mathsf{T}} \; \boldsymbol{M} \; \boldsymbol{T}_{\rm c} \tag{9}$$

Stiffness and mass matrices condensed in this way preserve potential and kinetic energy.

$$V = \frac{1}{2} \, \boldsymbol{r}^{\mathsf{T}} \; \boldsymbol{K} \; \boldsymbol{r} \tag{10}$$

$$= \frac{1}{2} \, \boldsymbol{r}_p^{\mathsf{T}} \boldsymbol{T}_{\rm c}^{\mathsf{T}} \; \boldsymbol{K} \; \boldsymbol{T}_{\rm c} \boldsymbol{r}_p = \frac{1}{2} \boldsymbol{r}_p^{\mathsf{T}} \; \boldsymbol{K}_{\rm c} \; \boldsymbol{r}_p \tag{11}$$

$$T = \frac{1}{2} \, \dot{\boldsymbol{r}}^{\mathsf{T}} \; \boldsymbol{M} \; \dot{\boldsymbol{r}} \tag{12}$$

$$= \frac{1}{2} \, \dot{\boldsymbol{r}}_p^{\mathsf{T}} \boldsymbol{T}_{\rm c}^{\mathsf{T}} \; \boldsymbol{M} \; \boldsymbol{T}_{\rm c} \dot{\boldsymbol{r}}_p = \frac{1}{2} \dot{\boldsymbol{r}}_p^{\mathsf{T}} \; \boldsymbol{M}_{\rm c} \; \dot{\boldsymbol{r}}_p \tag{13}$$

The P.E. and the K.E. of the condensed model is the same as that of the full model, but the natural frequencies and mode shapes of the condensed model do not match those of the full model. The dynamics characteristics of $\boldsymbol{K}_{\rm c}$ and $\boldsymbol{M}_{\rm c}$ are closer to those of $\boldsymbol{K}$ and $\boldsymbol{M}$ at the lower frequencies.

3 Dynamic Condensation

Consider the partitioning of the undamped and unforced second order equations of motion

$$\begin{bmatrix} \boldsymbol{M}_{pp} & \boldsymbol{M}_{ps} \\ \boldsymbol{M}_{sp} & \boldsymbol{M}_{ss} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{r}}_p \\ \ddot{\boldsymbol{r}}_s \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{pp} & \boldsymbol{K}_{ps} \\ \boldsymbol{K}_{sp} & \boldsymbol{K}_{ss} \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_p \\ \boldsymbol{r}_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{14}$$

Assuming a solution

$$\boldsymbol{r}_p = \bar{\boldsymbol{r}}_p \cos \omega t \tag{15}$$

$$\boldsymbol{r}_s = \bar{\boldsymbol{r}}_s \cos \omega t \tag{16}$$

$$\begin{bmatrix} \boldsymbol{K}_{pp} - \omega^2 \boldsymbol{M}_{pp} & \boldsymbol{K}_{ps} - \omega^2 \boldsymbol{M}_{ps} \\ \boldsymbol{K}_{sp} - \omega^2 \boldsymbol{M}_{sp} & \boldsymbol{K}_{ss} - \omega^2 \boldsymbol{M}_{ss} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{r}}_p \\ \bar{\boldsymbol{r}}_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

Applying the static condensation equation directly to this partitioned eigenvalue problem,

$$\left[\boldsymbol{K}_c - \omega^2 \boldsymbol{M}_c \right] = \left[\boldsymbol{K}_{pp} - \omega^2 \boldsymbol{M}_{pp} \right] - \left[\boldsymbol{K}_{ps} - \omega^2 \boldsymbol{M}_{ps} \right] \left[\boldsymbol{K}_{ss} - \omega^2 \boldsymbol{M}_{ss} \right]^{-1} \left[\boldsymbol{K}_{sp} - \omega^2 \boldsymbol{M}_{sp} \right] \tag{18}$$

By defining a new coordinate transformation matrix

$$\boldsymbol{T}_{\rm c} = \begin{bmatrix} \boldsymbol{I}_{n \times n} \\ \left[\boldsymbol{K}_{ss} - \omega^2 \boldsymbol{M}_{ss} \right]^{-1} \left[\boldsymbol{K}_{sp} - \omega^2 \boldsymbol{M}_{sp} \right] \end{bmatrix}_{N \times n} , \tag{19}$$

$$\boldsymbol{T}_{\rm c}^{\mathsf{T}} \left[\boldsymbol{K} - \omega^2 \boldsymbol{M} \right] \boldsymbol{T}_{\rm c} = \boldsymbol{T}_{\rm c}^{\mathsf{T}} \boldsymbol{K} \boldsymbol{T}_{\rm c} - \omega^2 \boldsymbol{T}_{\rm c}^{\mathsf{T}} \boldsymbol{M} \boldsymbol{T}_{\rm c} \tag{20}$$

So, the condensed stiffness and mass matrices are,

$$\boldsymbol{K}_{\rm c} = \boldsymbol{T}_{\rm c}^{\mathsf{T}} \boldsymbol{K} \boldsymbol{T}_{\rm c} \tag{21}$$

$$\boldsymbol{M}_{\rm c} = \boldsymbol{T}_{\rm c}^{\mathsf{T}} \boldsymbol{M} \boldsymbol{T}_{\rm c} \tag{22}$$

These condensed stiffness and mass matrices match the dynamics of the full model matrices $\boldsymbol{K}$ and $\boldsymbol{M}$ *exactly* at a frequency ω.

Note:

- The frequency ω can be any frequency, e.g., the first mode frequency, a higher mode frequency, or any intermediate frequency.
- If ω is set to zero, the dynamic condensation method reduces to Guyan condensation.
- A condensed stiffness matrix computed using dynamic condensation preserves the bandedness of the full system stiffness matrix.
- This method may be applied recursively in order to compute modal frequencies.
  - at iteration i, select a frequency $\omega^{(i)}$;
  - compute $\boldsymbol{T}_{\rm c}$ using $\omega^{(i)}$;
  - compute $\boldsymbol{K}_{\rm c}$ and $\boldsymbol{M}_{\rm c}$ using $\boldsymbol{T}_{\rm c}$, $\boldsymbol{K}$ and $\boldsymbol{M}$;
  - solve the eigenvalue problem for the smaller-dimensioned $\boldsymbol{K}_{\rm c}$ and $\boldsymbol{M}_{\rm c}$ in order to calculate the modal frequency for the next iteration $\omega^{(i+1)}$;
  - increment the iteration counter, $i = i + 1$, and go to step 2.

4 Modal Condensation

Given a matrix of n of the modal vectors from the full model ($n < N$)

$$\bar{\boldsymbol{R}} = \left[\begin{array}{cccc} | & | & & | \\ \bar{\boldsymbol{r}}_1 & \bar{\boldsymbol{r}}_2 & \cdots & \bar{\boldsymbol{r}}_n \\ | & | & & | \end{array} \right]_{N\times n} \tag{23}$$

and the n corresponding natural frequencies

$$\boldsymbol{\Omega}^2 = \left[\begin{array}{cccc} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{array} \right] \tag{24}$$

find symmetric $n \times n$ matrices $\boldsymbol{K}_{\rm c}$ and $\boldsymbol{M}_{\rm c}$ that have the same modal vectors and frequencies. Note that modal vectors in $\bar{\boldsymbol{R}}$ and the modal frequencies in ^{2} may be any subset of the modal vectors and frequencies, not only the first n. Assume that the modal vectors are mass normalized;

$$\bar{\boldsymbol{R}}_{N\times N}^{\mathsf{T}} \; \boldsymbol{M}_{N\times N} \; \bar{\boldsymbol{R}}_{N\times N} \;\; = \;\; \boldsymbol{I}_{N\times N} \tag{25}$$

$$\bar{\boldsymbol{R}}_{N\times N}^{\mathsf{T}} \; \boldsymbol{K}_{N\times N} \; \bar{\boldsymbol{R}}_{N\times N} \;\; = \;\; \boldsymbol{\Omega}_{N\times N}^2 \tag{26}$$

Define a square modal matrix $\bar{\boldsymbol{R}}_p$ as the rows of the full modal matrix $\bar{\boldsymbol{R}}$ at n the primary coordinates and n selected modes (not necessarily the first n modes). Columns of the matrix $\bar{\boldsymbol{R}}_s$ are the modal vectors at the n selected modes at the remaining (secondary) coordinates. (The matrix $\bar{\boldsymbol{R}}_p$ may be scaled by a factor $\sqrt{N/n}$ in order to match the condensed modal vectors.) Now the problem is to find $\boldsymbol{K}_{\rm c}$ and $\boldsymbol{M}_{\rm c}$ such that

$$\bar{\boldsymbol{R}}_p^{\mathsf{T}} \; \boldsymbol{M}_{\rm c} \; \bar{\boldsymbol{R}}_p \;\; = \;\; \boldsymbol{I}_{n\times n} \tag{27}$$

$$\bar{\boldsymbol{R}}_p^{\mathsf{T}} \; \boldsymbol{K}_{\rm c} \; \bar{\boldsymbol{R}}_p \;\; = \;\; \boldsymbol{\Omega}_{n\times n}^2 \tag{28}$$

Because the matrix $\bar{\boldsymbol{R}}_p$ may be ill-conditioned, using the regularized pseudo-inverse helps with numerical conditioning of $\boldsymbol{K}_{\rm c}$ and $\boldsymbol{M}_{\rm c}$;

$$\bar{\boldsymbol{R}}_p^{+} = \left[\bar{\boldsymbol{R}}_p \bar{\boldsymbol{R}}_p^{\mathsf{T}} + \beta \; \mathrm{diag}(\bar{\boldsymbol{R}}_p \bar{\boldsymbol{R}}_p^{\mathsf{T}}) \right]^{-1} \bar{\boldsymbol{R}}_p \tag{29}$$

where the regularization factor β can be small ($\approx 10^{-4}$). Using this pseudo-inverse, the condensed mass and stiffness matrices can be computed to meet the desired goal of matching the dynamics at the primary coordinates and the selected modes.

$$\boldsymbol{M}_{\rm c} \;\; = \;\; \bar{\boldsymbol{R}}_p^{+} \; \boldsymbol{I}_{n\times n} \; \bar{\boldsymbol{R}}_p^{+\mathsf{T}} \; (N/n) \tag{30}$$

$$\boldsymbol{K}_{\rm c} \;\; = \;\; \bar{\boldsymbol{R}}_p^{+} \; \boldsymbol{\Omega}_{n\times n}^2 \; \bar{\boldsymbol{R}}_p^{+\mathsf{T}} \; (N/n) \tag{31}$$

Responses in the n selected modes of the $N - n$ secondary coordinates may be assembled from the primary coordinate responses,

$$\boldsymbol{r}_s(t) = \bar{\boldsymbol{R}}_s \; \bar{\boldsymbol{R}}_p^{+} \; \boldsymbol{r}_p(t) \tag{32}$$

5 Modal Assurance Criteria

In order to assess the performance of model condensation methods, the modal vectors of the full model and condensed model can be compared. Orthogonality of the full-model modal vectors with the condensed-model modal vectors indicates that the condensed model captures the dynamics of the full model.

Defining $\bar{\boldsymbol{R}}_p$ as the rows of the full modal matrix $\bar{\boldsymbol{R}}$ at n the primary coordinates and the first n modes. and $\tilde{\boldsymbol{R}}_p$ as the eigenvectors of $\boldsymbol{M}_{\mathrm{c}}$ and $\boldsymbol{K}_{\mathrm{c}}$, the modal assurance criteria is

$$\mathsf{MAC} = |\bar{\boldsymbol{R}}_p^{\mathsf{T}} \tilde{\boldsymbol{R}}| \tag{33}$$

Ideally, the MAC is close to identity.

References

[1] Guyan RJ. Reduction of stiffness and mass matrices. AIAA J 1965;3:380

[2] Paz, M. Dynamic condensation. AIAA J 1984;22(5):724-727.
http://dx.doi.org/10.2514/3.48498

[3] Friswell, M.I., Garvey, S.D., and Penny, J.E.T., Model Reduction using Dynamic and Iterated IRS Techniques, Journal of Sound and Vibration (1995) 186(2),311-323.

6 ModelCondensation.m

```
% ModelCondensation.m
% test matrix condensation methods for dynamic model reduction
% RJ Guyan, Reduction of Stiffness and Mass Matrices, AIAA J, v3, n2, 1965, p380
% M Paz, Dynamic condensation. AIAA J, v22, n5, 1984, pp724-727

% March 4 2010, July 2 2015

newSystem = 1
if newSystem  % - mass and stiffness matrices correspond to a series of springs

  N = 100;      % total number of coordinates

  % diagonal mass matrix
    M = eye(N);  diag(1+0.2*rand(1,N));
  % tri-diagonal stiffness matrix
    ki = N^2 * ones(1,N); %  (1+0.3*randn(1,N));
    K = diag([ ki(1:N-1)+ki(2:N), ki(N)]) - diag(ki(2:N),1) - diag(ki(2:N),-1);
  p = [ 5:10:100 ];                               % primary coordinates
  s = zeros(1,N);  s(p) = 1;  s = find(~s);       % secondary coordinates
  n = length(p);                                  % number of primary coordinates

  [X,Omega2] = eigs( K, M, n, 0 );                % solve the full eigenvalue prob

  [Omega2, idx] = sort(diag(Omega2)); X = X(:,idx); % sort to increasing nat'l frq

end   % newSystem

% Guyan Condensation and Dynamic Condensation - Guyan 1965 and Paz 1984 -----

% pick the smallest eigen-value (natural freuency squared)
 w2 = min(Omega2)                % dynamic condensation (Paz)
%w2 = 0.0;                       % Guyan   condensation

 D = K - w2*M;

 T = zeros(N,n);

 T(p,1:n) = eye(n);                      % coordinate transformation
 T(s,1:n) = -D(s,s) \ D(s,p);

 Mc_d = T' * M * T;

 Kc_d = T' * K * T;

 [Xd, w2_d] = eigs( Kc_d, Mc_d, n, 0 );

 [w2_d, idx] = sort(diag(w2_d)); Xd = Xd(:,idx); % sort to increasing nat'l frq

 ms = diag(diag (Xd'*Mc_d*Xd));          % modal mass
 Xd  =  Xd / sqrt(ms);                   % mass normalized

% Modal Condensation ---------------------------------------------------------

% "n" selected modes to match for modal condensation

%m = [ 1 2 3 6 7 9 12 15 16 17 ];
 m = [1:n];

 Xp = X(p,m) * sqrt(N/n);        % "m" mode shapes at the primary coordinates

 beta = 0e-4;                    % regularization factor

 Xp_pi = ( Xp*Xp' + beta*diag(diag(Xp*Xp')) ) \ Xp;      % pseudo-inverse

 Mc_m = Xp_pi * eye(length(m))  * Xp_pi' * N/n;

 Kc_m = Xp_pi * diag(Omega2(m)) * Xp_pi' * N/n;
```

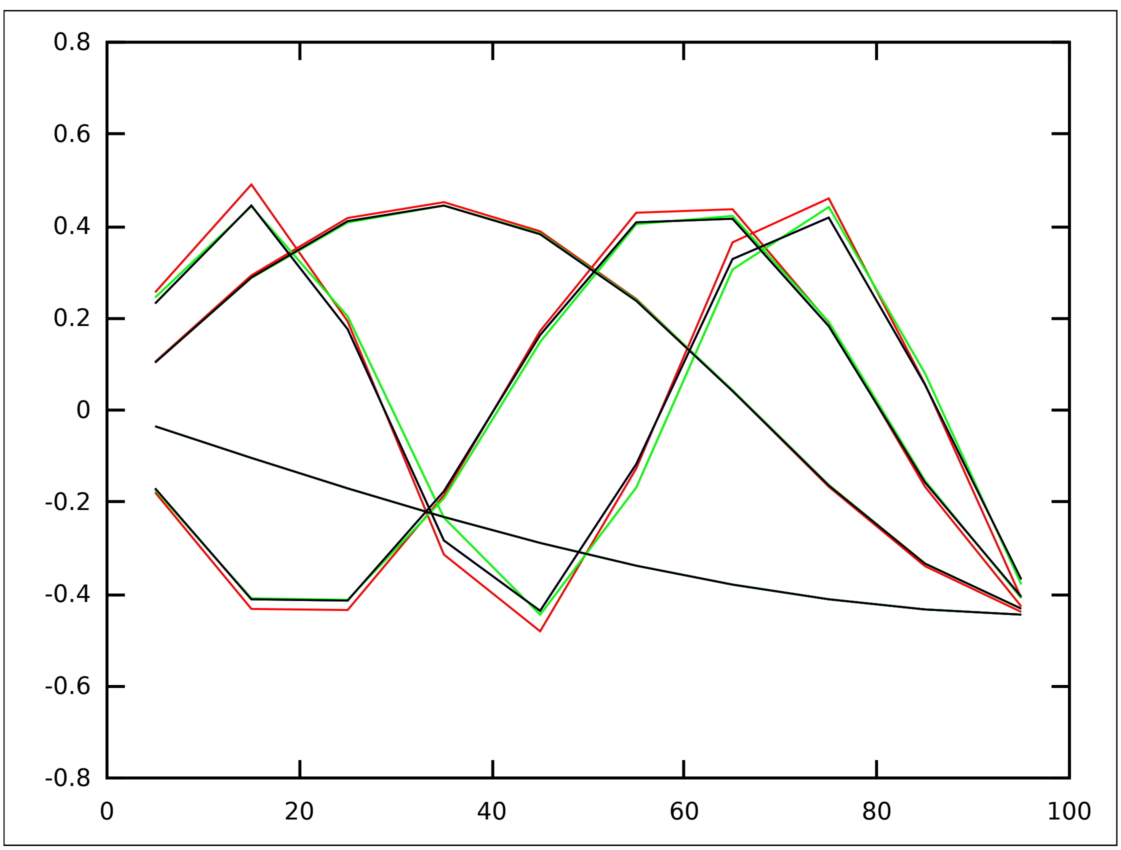
```
 [Xm, w2_m] = eigs( Kc_m, Mc_m, n, 0 );

 [w2_m, idx] = sort(diag(w2_m)); Xd = Xd(:,idx); % sort to increasing nat'l frq

 ms = diag(diag (Xm'*Mc_m*Xm));          % modal mass
 Xm  =  Xm / sqrt(ms);                   % mass normalized

% Iterated Improved Reduced System (IIRS) - Frisewell 1998 ----------------

 T = zeros(N,n);
 To = -K(s,s) \ K(s,p);      % static condensation coordinate transformation
 Ti = To;
 T(p,1:n) = eye(n);          % coordinate transformation
 for iter = 1:7
   T(s,1:n) = Ti;
   Mc_r = T'*M*T;
   Kc_r = T'*K*T;
   Ti   = To + K(s,s) \ [ M(s,p) , M(s,s) ] * T * (Mc_r \ Kc_r);
 end

 [Xr, w2_r] = eigs( Kc_r, Mc_r, n, 0 );

 [w2_r, idx] = sort(diag(w2_r)); Xr = Xr(:,idx); % sort to increasing nat'l frq

 ms = diag(diag (Xr'*Mc_r*Xr));          % modal mass
 Xr  =  Xr / sqrt(ms);                   % mass normalized

%
```

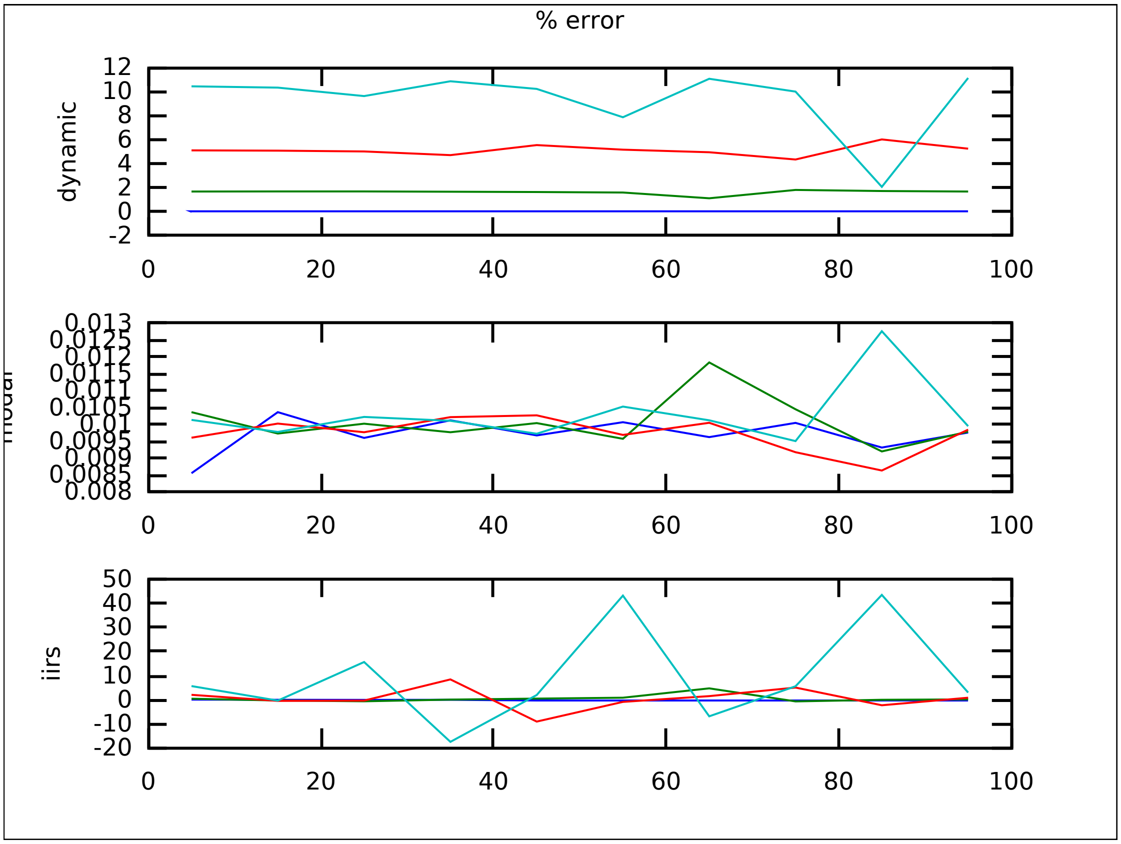
```

% Display Results ----------------------------------------------------------

epsPlots = 0; if epsPlots, formatPlot(1); else formatPlot(0); end

fprintf(' Natural Frequencies\n')
fprintf('     \t\tDynamic\t\tModal\t\tIIRS \n')
fprintf(' Full\t\tCondensation\tCondensation\tCondensation\n')
format
for i=1:10
 fprintf('%8.6f\t%8.6f\t%8.6f\t%8.6f\n', ...
           sqrt(Omega2(i))/2/pi , ...
           sqrt(w2_d(i))/2/pi , ...
           sqrt(w2_m(i))/2/pi , ...
           sqrt(w2_r(i))/2/pi )
end

% match signs of mode shapes and scale mode vectors to full model values
 Xd = Xd * diag ( sign( X(N,1:n) .* Xd(n,1:n) ) ) * sqrt(N/n);
 Xm = Xm * diag ( sign( X(N,1:n) .* Xm(n,1:n) ) ) * sqrt(N/n);
 Xr = Xr * diag ( sign( X(N,1:n) .* Xr(n,1:n) ) ) * sqrt(N/n);

% first "n" mode shapes at the primary coordinates
 format bank
 actual_mode_shapes = Xp
 dynamic_mode_shapes = Xd
 modal_condensation_mode_shapes = Xm
 iirs_condensation_mode_shapes = Xr

% error of condensed mode shapes
 err_d =  (Xd-Xp)./Xp;
 err_m =  (Xm-Xp)./Xp;
 err_r =  (Xr-Xp)./Xp;

% modal assurance criteria "MAC"
 MACpd = abs( Xp' * Xd ); % - eye(n);
 MACpm = abs( Xp' * Xm ); % - eye(n);
 MACpr = abs( Xp' * Xr ); % - eye(n);
```

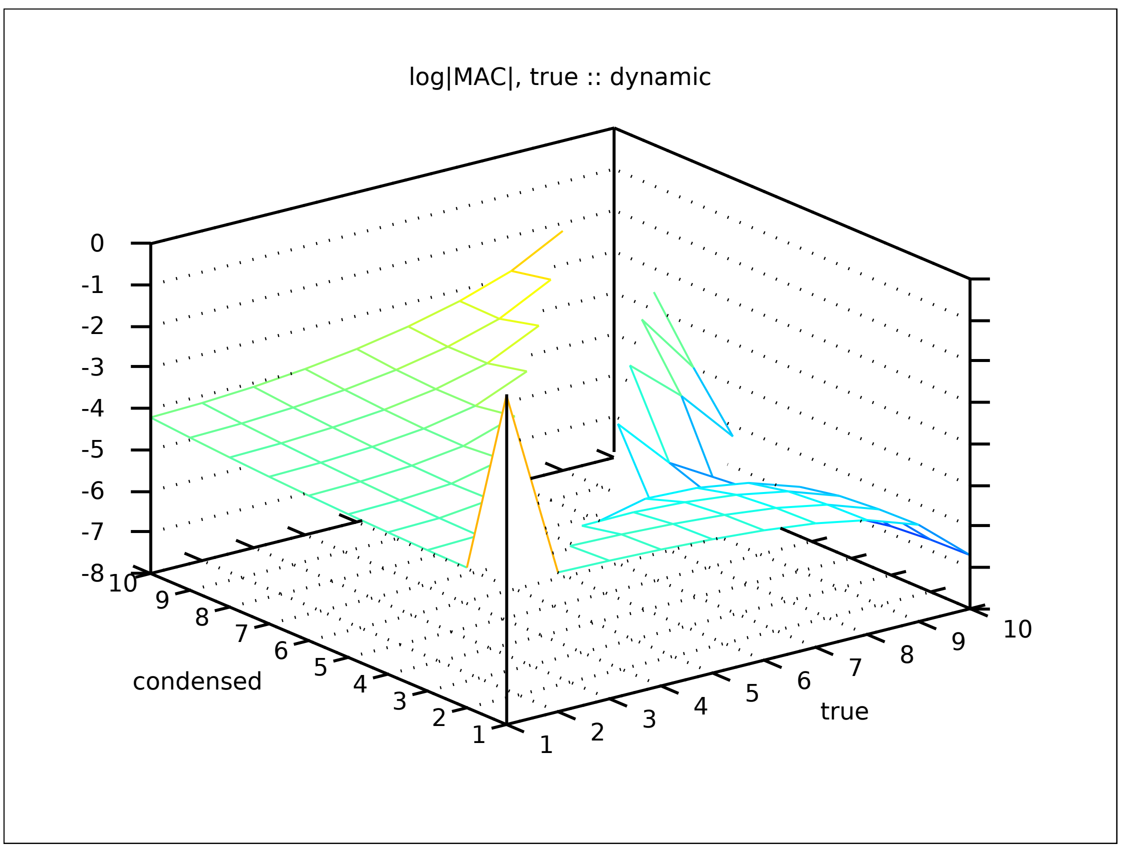
7 Results

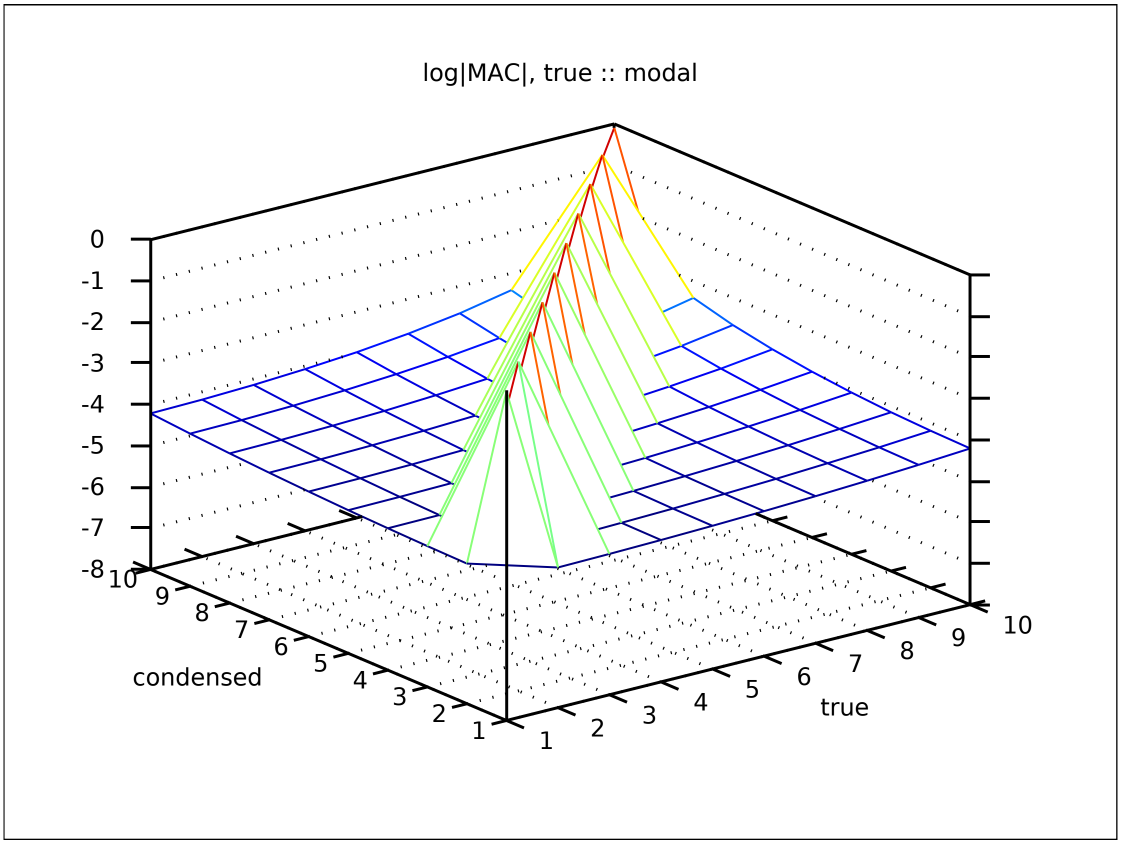
```
                Dynamic         Modal           IIRS
 Full           Condensation    Condensation    Condensation
0.248754        0.248754        0.248754        0.248756
0.746200        0.751695        0.746200        0.746553
1.243465        1.273221        1.243465        1.247959
1.740425        1.825566        1.740425        1.766281
2.236961        2.420037        2.236961        2.334217

Mc_d =

   5.06   1.66   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00
   1.66   6.72   1.66   0.00   0.00   0.00   0.00   0.00   0.00   0.00
   0.00   1.66   6.72   1.66   0.00   0.00   0.00   0.00   0.00   0.00
   0.00   0.00   1.66   6.72   1.66   0.00   0.00   0.00   0.00   0.00
   0.00   0.00   0.00   1.66   6.72   1.66   0.00   0.00   0.00   0.00
   0.00   0.00   0.00   0.00   1.66   6.72   1.66   0.00   0.00   0.00
   0.00   0.00   0.00   0.00   0.00   1.66   6.72   1.66   0.00   0.00
   0.00   0.00   0.00   0.00   0.00   0.00   1.66   6.72   1.66   0.00
   0.00   0.00   0.00   0.00   0.00   0.00   0.00   1.66   6.72   1.66
   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   1.66   8.89

Mc_m =

   10.11   -0.12    0.12   -0.12    0.13   -0.13    0.14   -0.15    0.17   -0.20
   -0.12   10.12   -0.12    0.12   -0.13    0.14   -0.14    0.16   -0.17    0.20
    0.12   -0.12   10.12   -0.13    0.13   -0.14    0.15   -0.16    0.18   -0.21
   -0.12    0.12   -0.13   10.13   -0.14    0.14   -0.15    0.17   -0.19    0.22
    0.13   -0.13    0.13   -0.14   10.14   -0.15    0.16   -0.18    0.20   -0.24
   -0.13    0.14   -0.14    0.14   -0.15   10.16   -0.18    0.20   -0.23    0.28
    0.14   -0.14    0.15   -0.15    0.16   -0.18   10.19   -0.22    0.26   -0.33
   -0.15    0.16   -0.16    0.17   -0.18    0.20   -0.22   10.25   -0.31    0.42
    0.17   -0.17    0.18   -0.19    0.20   -0.23    0.26   -0.31   10.40   -0.59
   -0.20    0.20   -0.21    0.22   -0.24    0.28   -0.33    0.42   -0.59   11.07

Mc_r =

    5.04    1.31    2.22   -2.49    1.26   -0.32   -0.05    0.09   -0.05    0.02
    1.31    4.63    2.58    0.57   -1.01    0.31    0.09   -0.14    0.07    0.00
    2.22    2.58    7.38    0.47    0.81   -0.68    0.22   -0.04    0.04   -0.09
   -2.49    0.57    0.47    9.31    0.04    0.58   -0.60    0.38   -0.25    0.24
    1.26   -1.01    0.81    0.04    9.61   -0.06    0.58   -0.66    0.47   -0.38
   -0.32    0.31   -0.68    0.58   -0.06   10.02   -0.41    0.65   -0.61    0.45
   -0.05    0.09    0.22   -0.60    0.58   -0.41   10.64   -0.75    0.63   -0.48
    0.09   -0.14   -0.04    0.38   -0.66    0.65   -0.75   11.03   -0.86    0.50
   -0.05    0.07    0.04   -0.25    0.47   -0.61    0.63   -0.86   11.18   -0.91
    0.02    0.00   -0.09    0.24   -0.38    0.45   -0.48    0.50   -0.91   11.33


round( Kc_d)

   3000  -1000      0      0      0      0      0      0      0      0
  -1000   2000  -1000      0      0      0      0      0      0      0
      0  -1000   2000  -1000      0      0      0      0      0      0
      0      0  -1000   2000  -1000      0      0      0      0      0
      0      0      0  -1000   2000  -1000      0      0      0      0
      0      0      0      0  -1000   2000  -1000      0      0      0
      0      0      0      0      0  -1000   2000  -1000      0      0
      0      0      0      0      0      0  -1000   2000  -1000      0
      0      0      0      0      0      0      0  -1000   2000  -1000
      0      0      0      0      0      0      0      0  -1000   1000

round(Kc_m)

   5343  -2563    796   -427    289   -222    184   -159    134    -60
  -2563   3576  -2195    658   -361    252   -199    168   -139     62
    796  -2195   3437  -2128    622   -340    238   -188    151    -66
   -427    658  -2128   3401  -2107    609   -331    231   -173     74
```

```
     289    -361     622   -2107    3389   -2100     604    -325     215     -87
    -222     252    -340     609   -2100    3384   -2096     598    -307     113
     184    -199     238    -331     604   -2096    3381   -2089     573    -174
    -159     168    -188     231    -325     598   -2089    3365   -2042     364
     134    -139     151    -173     215    -307     573   -2042    3252   -1597
     -60      62     -66      74     -87     113    -174     364   -1597    1341

round({Kc_r)

    3484   -1789     834    -424     -22     185    -144      63     -15       1
   -1789    3548   -3089    1465    -383    -198     284    -177      70     -15
     834   -3089    5795   -4709    1984    -519    -105     210    -126      33
    -424    1465   -4709    7017   -4995    2023    -634      46      71     -26
     -22    -383    1984   -4995    6603   -4413    1702    -583     144     -28
     185    -198    -519    2023   -4413    5669   -3643    1292    -473     121
    -144     284    -105    -634    1702   -3643    4802   -2969     912    -237
      63    -177     210      46    -583    1292   -2969    4080   -2397     446
     -15      70    -126      71     144    -473     912   -2397    3473   -1660
       1     -15      33     -26     -28     121    -237     446   -1660    1364
```




% error
dynamic
12
10
8
6
4
2
0
-2
0
20
40
60
80
100
0.013
0.0125
0.012
0.0115
0.011
0.0105
0.01
0.0095
0.009
0.0085
0.008
iirs
50
40
30
20
10
0
-10
-20


log|MAC|, true :: dynamic
0
-1
-2
-3
-4
-5
-6
-7
-8
condensed
true
1
2
3
4
5
6
7
8
9
10



log|MAC|, true :: modal
0
-1
-2
-3
-4
-5
-6
-7
-8
condensed
1
2
3
4
5
6
7
8
9
10
true
1
2
3
4
5
6
7
8
9
10