Tuned Mass Dampers

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1 Damped Primary System, Undamped Tuned Mass Damper

Consider a damped single-degree-of-freedom (SDOF) oscillator (with mass m, stiffness k and damping c driven by a sinusoidal force $f(t) = \bar{f} \cos \omega t$) with an attached undamped and unforced SDOF oscillator, (with mass $m_{\rm a}$ and stiffness $k_{\rm a}$). The coupled equations of motion are

$$
\begin{aligned}
m\ddot{r}(t) + c\dot{r}(t) + kr(t) - k_{\rm a}(r_{\rm a}(t) - r(t)) &= \bar{f} \cos \omega t && (1) \\
m_{\rm a}\ddot{r}_{\rm a}(t) + k_{\rm a}(r_{\rm a}(t) - r(t)) &= 0 && (2)
\end{aligned}
$$

Using complex-exponential notation and retaining the positive-exponent parts, the steady state responses $r(t) = \bar{r}e^{i\omega t}$ and $r_{\rm a}(t) = \bar{r}_{\rm a}e^{i\omega t}$ to forcing $f(t) = \bar{f}e^{i\omega t}$ can be inserted into the equations of motion to obtain

$$
\begin{aligned}
(-\omega^2 m + i\omega c + k + k_{\rm a})\bar{r} - k_{\rm a}\bar{r}_{\rm a} &= \bar{f} && (3) \\
-k_{\rm a}\bar{r} + (-\omega^2 m_{\rm a} + k_{\rm a})\bar{r}_{\rm a} &= 0 && (4)
\end{aligned}
$$

and the frequency response function from forcing to the displacement response of the primary system mass, m

$$
H(\omega) = \frac{\bar{r}}{\bar{f}/k} = \frac{-\omega^2 m_{\rm a} + k_{\rm a}}{(1/k)(-\omega^2 m + i\omega c + k + k_{\rm a})(-\omega^2 m_{\rm a} + k_{\rm a}) - k_{\rm a}^2} \qquad (5)
$$

The frequency response function has a value of zero where $-\omega^2 m_{\rm a} + k_{\rm a} = 0$ which gives the tuning equation for the un-damped tuned mass damper. The TMD should be tuned so that it's natural frequency equals the forcing frequency.

$$
\omega^2 = \frac{k_{\rm a}}{m_{\rm a}} = {\omega_{\rm na}}^2 \qquad (6)
$$

For systems tuned in this way, the motion amplitude of the TMD is out of phase with the forcing, $\bar{r}_{\rm a} = -\bar{f}/k_{\rm a}$. Undamped TMD's are applicable to cases in which the forcing frequency is known precisely and in which the forcing frequency does not vary. Systems with undamped TMD's are very sensitive to mis-tuning errors; they are not robust with respect to variations in forcing frequency.

2 Undamped Primary System, Damped Tuned Mass Damper

Now, neglecting damping in the primary system, but adding damping to the TMD, we consider the Den-Hartog absorber [1].

If $c_{\rm a} = \infty$, the absorber is essentially linked with the primary system and the "linked" natural frequency is $\omega_{{\rm n}l} = \sqrt{k/(m + m_{\rm a})}$. If $c_{\rm a} = 0$, the system is un-damped and $\bar{r}/\bar{f}$ is zero at $\omega = \sqrt{k_{\rm a}/m_{\rm a}}$.

The equations of motion in this case are

$$m\ddot{r}(t) + kr(t) - c_{\rm a}(\dot{r}_{\rm a}(t) - \dot{r}(t)) - k_{\rm a}(r_{\rm a}(t) - r(t)) = \bar{f}\cos\omega t \quad (7)$$

$$m_{\rm a}\ddot{r}_{\rm a}(t) + c_{\rm a}(\dot{r}_{\rm a}(t) - \dot{r}(t)) + k_{\rm a}(r_{\rm a}(t) - r(t)) = 0 \quad (8)$$

Using complex-exponential notation and considering the positive-exponent part of the solution, the frequency-response from forcing $\bar{f}$ to response $\bar{r}$ is

$$H(\omega) = \frac{\bar{r}}{\bar{f}/k} = \frac{-\omega^2 m_{\rm a} + i\omega c_{\rm a} + k_{\rm a}}{(1/k)[(-\omega^2 m + i\omega c_{\rm a} + k + k_{\rm a})(-\omega^2 m_{\rm a} + i\omega c_{\rm a} + k_{\rm a}) - (k_{\rm a} + i\omega c_{\rm a})^2]} \quad (9)$$

For fixed values of m, k, $m_{\rm a}$, and $k_{\rm a}$, and for any value of $c_{\rm a}$, all of the frequency-response curves from $\bar{f}$ to $\bar{r}$ pass through the same coordinates, ω_A, H_A and ω_B, H_B.

Den-Hartog showed that when $H_A \neq H_B$ the peaks of $H(\omega)$ are not at ω_A or ω_B.

Define: the mass ratio $\alpha = m_{\rm a}/m$, the primary system natural frequency $\omega_{\rm np} = \sqrt{k/m}$, the absorber natural frequency $\omega_{\rm na} = \sqrt{k_{\rm a}/m_{\rm a}}$, and the absorber damping ratio $\zeta = c_{\rm a}/(2\sqrt{mk})$. The Den-Hartog design objectives are:

- to select $\omega_{\rm na}$ so that $H_A = H_B$[1].
- to select c_a as the average of the damping values that maximize $H(\omega)$ at ω_A and at ω_B.

The optimal tuning is found from the criterion $H_A = H_B$.

$$\left(\frac{\omega_{\rm na}}{\omega_{\rm np}}\right)_{\rm opt} = \frac{1}{1+\alpha} \; . \quad (10)$$

The values of H_A and H_B are both $\sqrt{(2+\alpha)/\alpha}$ and are located at frequencies

$$\Omega^2_{A,B} = \left(\frac{\omega}{\omega_{\rm np}}\right)^2_{A,B} = \frac{2 + \alpha \pm \sqrt{\alpha^2 + 2\alpha}}{(1+\alpha)(2+\alpha)} \quad (11)$$

The optimal absorber damping is found from averaging the damping values that maximize $H(\omega)$ at ω_A and at ω_B;

$$\zeta_{\rm opt} = \sqrt{\frac{3\alpha^3}{8(1+\alpha)^3}} \; . \quad (12)$$

[1] In this sense Den-Hartog invented H_∞ theory

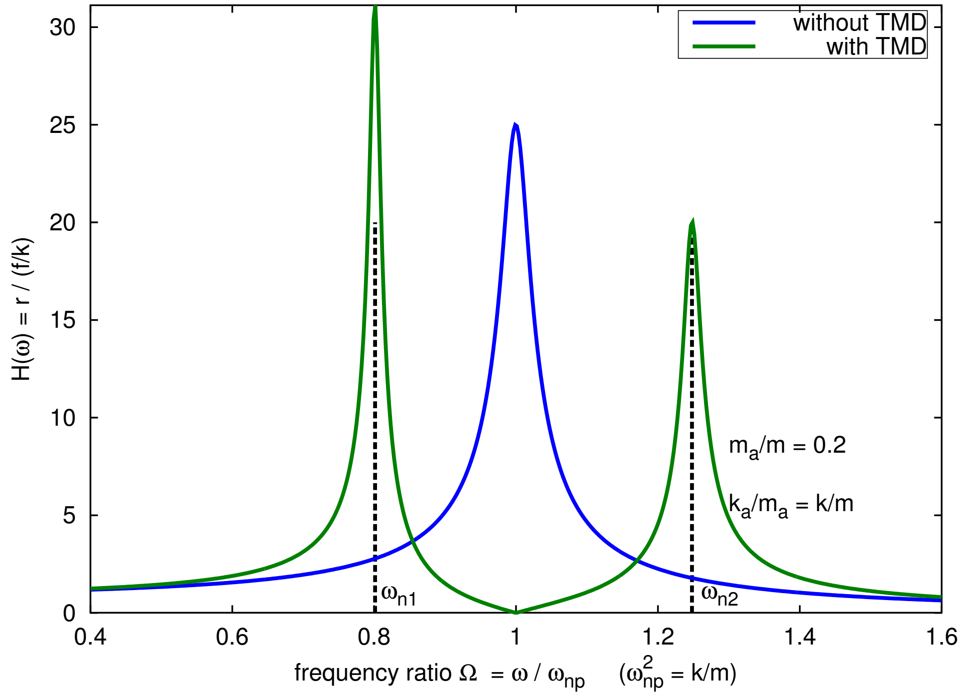

Figure 1. un-damped tuned-mass damper

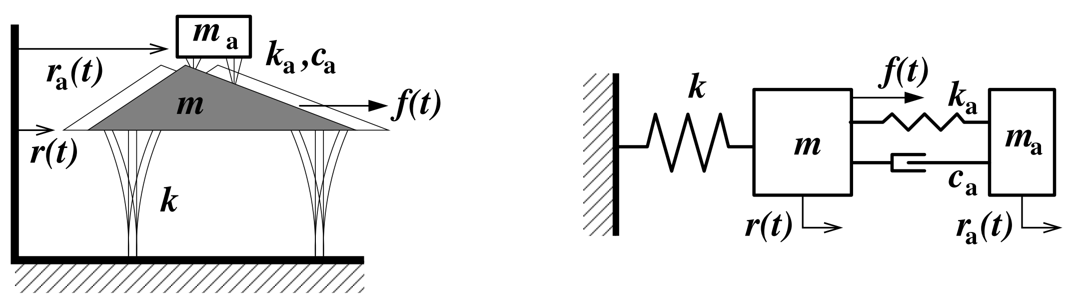

Figure 2. Force and displacement coordinates for the Den Hartog absorber

References

[1] Den Hartog, J.P., *Mechanical Vibration*, Dover 1984, (pp. 93-105).

[2] Hunt, J.B., *Dynamic Vibration Absorbers*, Mechanical Engineering Publications, 1979.

[3] Smith, R., "Dynamic Vibration Absorbers," *Sound & Vibration Magazine*, Nov. 1998, pp. 22-27.

[4] Snowdon, J.C., *Vibration and Shock in Damped Vibrational Systems,* Wiley 1968, Ch. 4.

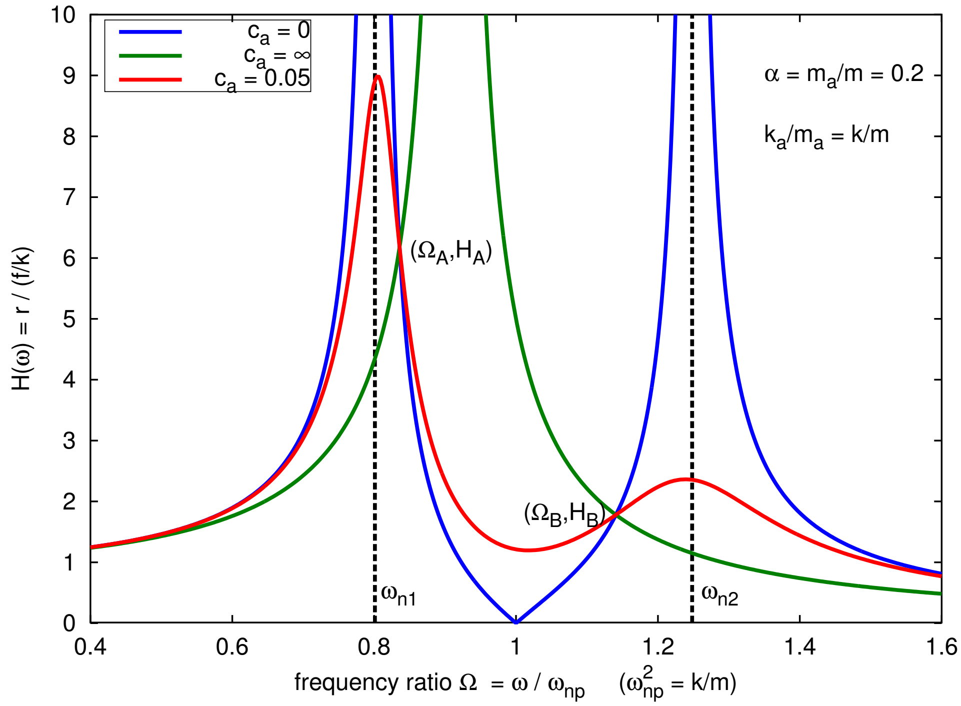

Figure 3. un-tuned tuned-mass damper

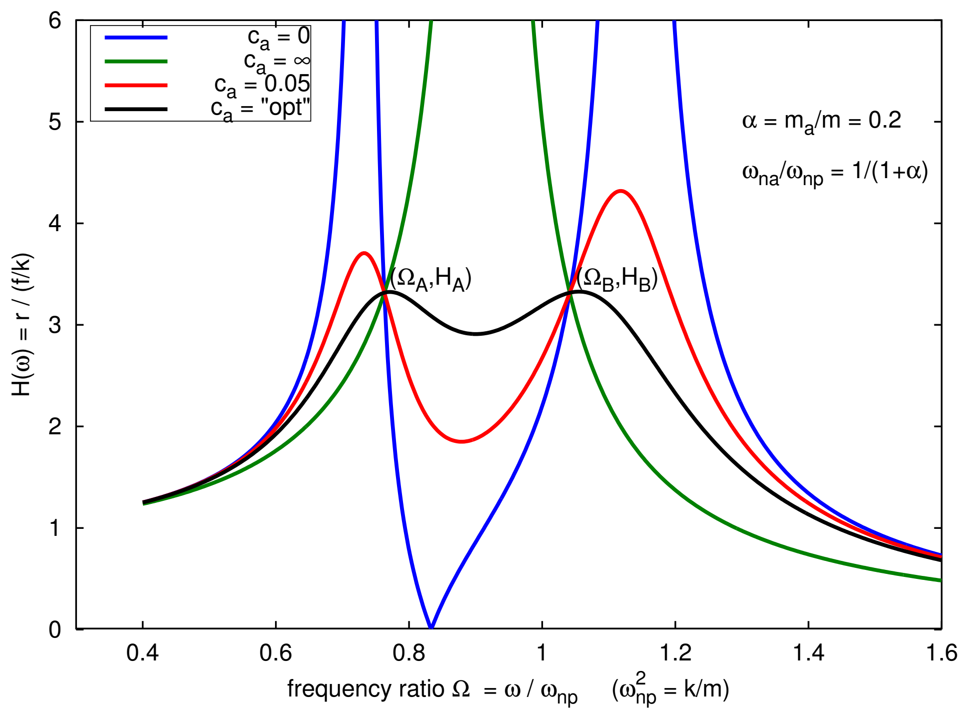

Figure 4. Den-Hartog tuned-mass damper