Viscoelastic Dynamics

CEE 541. Structural Dynamics
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1 Voigt

a spring in parallel with a dash-pot

$$f_{\mathsf{v}}(t) = k_{\mathsf{v}} x(t) + c_{\mathsf{v}} \dot{x}(t) \tag{1}$$

2 Maxwell

a spring in series with a dash-pot

$$
\begin{array}{rcl}
f_{\mathsf{m}}(t) = k_{\mathsf{m}} x_1(t) & x(t) = x_1(t) + x_2(t) & f_{\mathsf{m}}(t) = c_{\mathsf{m}} \dot{x}_2(t) \\
\dot{f}_{\mathsf{m}}(t) = k_{\mathsf{m}} \dot{x}_1(t) & \dot{x}(t) = \dot{x}_1(t) + \dot{x}_2(t) & f_{\mathsf{m}}(t) = c_{\mathsf{m}}((\dot{x}(t) - \dot{x}_1) \\
\dot{x}_1(t) = \dot{f}_{\mathsf{m}}(t)/k_{\mathsf{m}} & & f_{\mathsf{m}}(t) = c_{\mathsf{m}}((\dot{x}(t) - \dot{f}_{\mathsf{m}}(t)/k_{\mathsf{m}}
\end{array}
$$

$$\dot{f}_{\mathsf{m}}(t) = -(k_{\mathsf{m}}/c_{\mathsf{m}}) f_{\mathsf{m}}(t) + k_{\mathsf{m}} \dot{x}(t) \tag{2}$$

3 Zener (standard viscoelastic solid)

a Maxwell element in parallel with a spring

$$
\begin{aligned}
f_{\mathsf{z}}(t) &= f_{\mathsf{m}}(t) + k_{\mathsf{z}} x(t) \\
\dot{f}_{\mathsf{z}}(t) &= \dot{f}_{\mathsf{m}}(t) + k_{\mathsf{z}} \dot{x}(t) \\
\dot{f}_{\mathsf{z}}(t) &= -(k_{\mathsf{m}}/c_{\mathsf{m}}) f_{\mathsf{m}}(t) + k_{\mathsf{m}} \dot{x}(t) + k_{\mathsf{z}} \dot{x}(t) \\
\dot{f}_{\mathsf{z}}(t) &= -(k_{\mathsf{m}}/c_{\mathsf{m}})(f_{\mathsf{z}}(t) - k_{\mathsf{z}} x(t)) + k_{\mathsf{m}} \dot{x}(t) + k_{\mathsf{z}} \dot{x}(t) \\
\dot{f}_{\mathsf{z}}(t) &= -(k_{\mathsf{m}}/c_{\mathsf{m}}) f_{\mathsf{z}}(t) + (k_{\mathsf{m}} k_{\mathsf{z}}/c_{\mathsf{m}}) x(t) + (k_{\mathsf{m}} + k_{\mathsf{z}}) \dot{x}(t) \qquad (3)
\end{aligned}
$$

a generalization of a Voigt element ($k_{\mathsf{m}} \to \infty$) and a Maxwell element ($k_{\mathsf{z}} \to 0$)

3.1 Zener viscoelasticity in the Laplace (frequency $s = i\omega$) domain

$$\frac{F(s)}{k_{\sf z} X(s)} = \frac{1 + (\tau_{\sf z}/\tau_{\sf m} + 1)\tau_{\sf m} s}{1 + \tau_{\sf m} s} \tag{4}$$

where $\tau_{\sf m} = c_{\sf m}/k_{\sf m}$, the Maxwell element characteristic time constant, $\tau_{\sf z} = c_{\sf m}/k_{\sf z}$, and $\tau_{\sf z}/\tau_{\sf m} = k_{\sf m}/k_{\sf z}$. The real part of $F(\omega)/X(\omega)$ is the *storage modulus.* The imaginary part of $F(\omega)/X(\omega)$ is the *loss modulus.*

3.2 Creep Function

$x(t)$ for $f(t) = F_o H(t)$ ('Heaviside step function')
$x(0) = 0,\ f(t) = 0\ \forall\ t\ \le\ 0,\ f(t) = F_o\ \forall\ t\ >\ 0,\ (\dot{f}(t) = 0\ \forall\ t)$

$$\dot{x}(t) = -\frac{1}{c_{\sf m}} \frac{k_{\sf m} k_{\sf z}}{k_{\sf m} + k_{\sf z}} x(t) + \frac{1}{k_{\sf m} + k_{\sf z}} \frac{k_{\sf m}}{c_{\sf m}} f(t) \tag{5}$$

$$x_{\sf creep}(t)/F_o = \frac{1}{k_{\sf z}} + \left(\frac{1}{k_{\sf m} + k_{\sf z}} - \frac{1}{k_{\sf z}} \right) \exp\left[-\frac{k_{\sf m}}{c_{\sf m}} \frac{k_{\sf z}}{k_{\sf m} + k_{\sf z}} \right] t \tag{6}$$

By the principle of superposition, the viscoelastic displacement response to a set of incremental step loads $f(\tau_j) = \sum_{k=1}^{j} df(\tau_k)$, is

$$x(t_k) = \sum_{j=1}^{k} x_{\sf creep}(t_k - \tau_j) df(\tau_j) \Rightarrow \int_0^t x_{\sf creep}(t - \tau) \frac{df(\tau)}{d\tau} d\tau \tag{7}$$

3.3 Relaxation Function

$f(t)$ for $x(t) = H(t)$ ('Heaviside step function')
$f(0) = 0,\ x(t) = 0\ \forall\ t\ \le\ 0,\ x(t) = X_o\ \forall\ t\ >\ 0,\ (\dot{x}(t) = 0\ \forall\ t)$

$$\dot{f}(t) = -\frac{k_{\sf m}}{c_{\sf m}} f(t) + \frac{k_{\sf m} k_{\sf z}}{c_{\sf m}} x(t) \tag{8}$$

$$f_{\sf relax}(t)/X_o = k_{\sf z} + k_{\sf m} \exp\left[-\frac{k_{\sf m}}{c_{\sf m}} \right] t \tag{9}$$

By the principle of superposition, the viscoelastic force response to a set of incremental step displacements $x(\tau_j) = \sum_{k=1}^{j} dx(\tau_k)$, is

$$f(t_k) = \sum_{j=1}^{k} x_{\sf relax}(t_k - \tau_j) dx(\tau_j) \Rightarrow \int_0^t x_{\sf relax}(t - \tau) \frac{dx(\tau)}{d\tau} d\tau \tag{10}$$

Note that the time constants for creep and relaxation are different ...

$$\tau_{\mathsf{creep}} = (c_{\mathsf{m}}/k_{\mathsf{m}})((k_{\mathsf{m}} + k_{\mathsf{z}})/k_{\mathsf{z}}) \quad \text{and} \quad \tau_{\mathsf{relax}} = c_{\mathsf{m}}/k_{\mathsf{m}}.$$

For the standard viscoelastic solid $\tau_{\mathsf{creep}} > \tau_{\mathsf{relax}}$.

3.4 Zener viscoelasticity in state-space

3.4.1 motion-driven forces

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ f_{\mathsf{z}}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k_{\mathsf{z}}k_{\mathsf{m}}/c_{\mathsf{m}} & -k_{\mathsf{m}}/c_{\mathsf{m}} \end{bmatrix}\begin{bmatrix} x(t) \\ f_{\mathsf{z}}(t) \end{bmatrix} + \begin{bmatrix} 1 \\ k_{\mathsf{m}} + k_{\mathsf{z}} \end{bmatrix}\dot{x}(t) \tag{11}$$

The input is the velocity of the motion, as in the convolution above.

3.4.2 force-driven motions

$$\frac{d}{dt}\begin{bmatrix} f_{\mathsf{z}}(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (k_{\mathsf{m}}/c_{\mathsf{m}})(1/(k_{\mathsf{m}} + k_{\mathsf{z}})) & -(k_{\mathsf{m}}/c_{\mathsf{m}})(k_{\mathsf{z}}/(k_{\mathsf{m}} + k_{\mathsf{z}})) \end{bmatrix}\begin{bmatrix} f_{\mathsf{z}}(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1/(k_{\mathsf{m}} + k_{\mathsf{z}}) \end{bmatrix}\dot{f}_{\mathsf{z}}(t) \tag{12}$$

The input is the rate of the force, as in the convolution above.

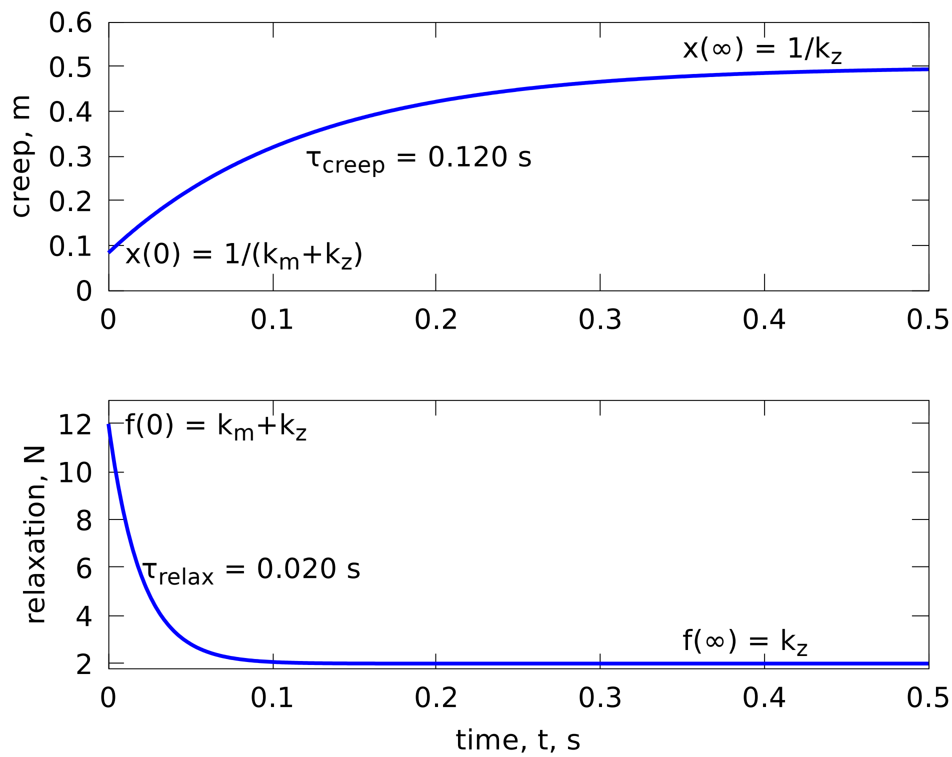

Figure 1. Creep and Relaxation functions for a standard viscoelastic solid $F_o = 1$ N, $X_o = 1$ m, $k_{\mathrm{m}} = 10$ N/m, $c_{\mathrm{m}} = 0.2$ N/m/s, and $k_{\mathrm{z}} = 2$ N/m.

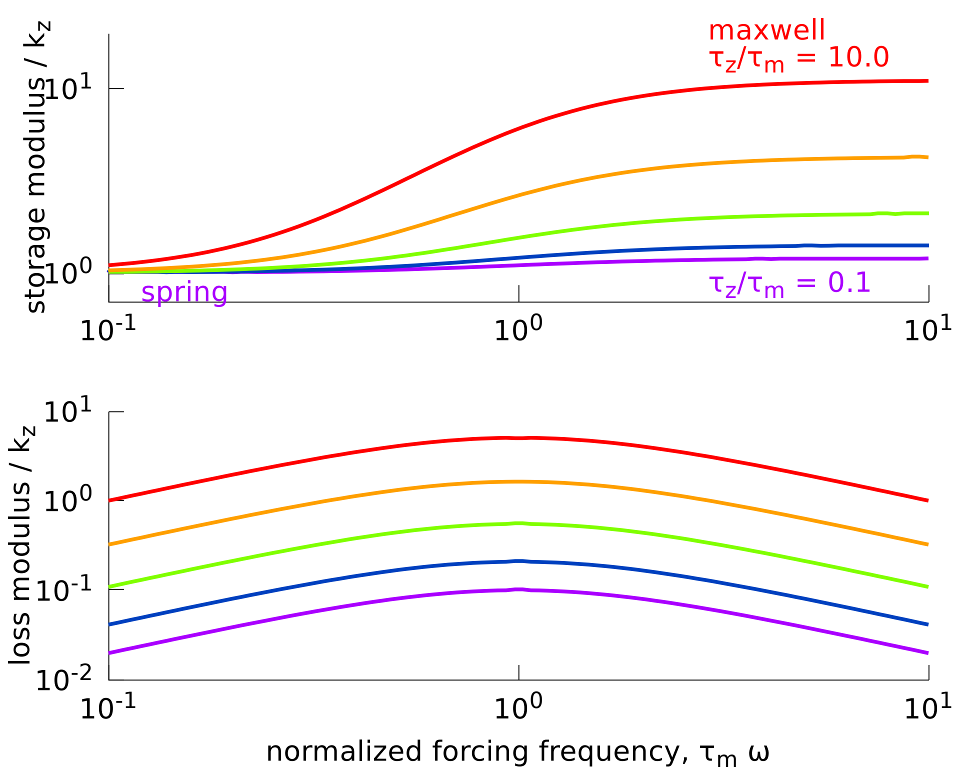

Figure 2. Storage modulus and loss modulus for Zener viscoelasticity .