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Fourier series and Fourier transforms

This document describes the forced-response of a lightly-damped simple oscillator to general periodic loading. The analysis is carried out using Fourier series approximations to the periodic external forcing and the resulting periodic steady-state response.

1 Fourier Series

Suppose an external forcing, $f(t)$, is persistent and periodic with period T.

$$\cdots = f(t-2T) = f(t-T) = f(t) = f(t+T) = f(f+2T) = \cdots \tag{1}$$

Any periodic function may be represented as a series expansion of sines and cosines, in a Fourier series,

$$f(t) = \frac{1}{2}a_0 + \sum_{q=1}^{\infty} a_q \cos\frac{2\pi q t}{T} + \sum_{q=1}^{\infty} b_q \sin\frac{2\pi q t}{T} \ , \tag{2}$$

where the Fourier coefficients, a_q and b_q are given by the Fourier integrals,

$$a_q \ = \ \frac{2}{T}\int_{t_o}^{t_o+T} f(t) \ \cos\frac{2\pi q t}{T} \ dt \ , \quad q = 0, 1, 2, \ldots \tag{3}$$

$$b_q \ = \ \frac{2}{T}\int_{t_o}^{t_o+T} f(t) \ \sin\frac{2\pi q t}{T} \ dt \ , \quad q = 0, 1, 2, \ldots \tag{4}$$

and the time t_o is arbitrary. This section emphasizes the connection between real and complex-valued notations for an oscillation. The Fourier series (2) may be represented using complex exponential notation.

$$f(t) \ = \ \frac{1}{2}a_0 + \sum_{q=1}^{\infty}\left[a_q \cos\frac{2\pi q t}{T} + b_q \sin\frac{2\pi q t}{T}\right] \tag{5}$$

$$= \sum_{q=-\infty}^{q=\infty} F_q \exp\left[i\frac{2\pi q}{T}t\right] \tag{6}$$

$$= \sum_{q=-\infty}^{q=\infty} F_q \ e^{i\omega_q t} \tag{7}$$

as long as $F_q = F^*_{-q}$.

Proof:

$$
\begin{aligned}
f(t) &= \sum_{q=-\infty}^{q=\infty} F_q \; e^{i\omega_q t} && (8)\\
&= \sum_{q=-\infty}^{q=0} F_q \; e^{i\omega_q t} + \sum_{q=1}^{q=\infty} F_q \; e^{i\omega_q t} && (9)\\
&= \sum_{q=0}^{q=\infty} F_{-\mathrm{q}} e^{-i\omega_q t} + \sum_{q=1}^{n=\infty} F_q e^{i\omega_q t} && (10)\\
&= F_0 + \sum_{q=1}^{\infty} \left[F_q \; e^{i\omega_q t} + F_{-q} \; e^{-i\omega_q t} \right] && (11)\\
&= f_0 + \sum_{q=1}^{\infty} \left[(F'_q + iF''_q)\,(\cos\omega_q t + i\sin\omega_q t) + (F'_q - iF''_q)\,(\cos\omega_q t - i\sin\omega_q t) \right] && (12)\\
&= F_0 + \sum_{q=1}^{\infty} \left[2F'_q \cos\omega_q t - 2F''_q \sin\omega_q t \right] && (13)
\end{aligned}
$$

So, the real part of F_q, F'_q, is half of a_q, the imaginary part of F_q, F''_q, is half of $-b_q$, and $F_q = F^*_{-q}$. The complex coefficients may be found directly from

$$
F_q = \frac{1}{T} \int_{t_o}^{t_o+T} f(t) \;\; \exp\left[-\frac{2\pi i q t}{T} \right] \; dt \; . \qquad (14)
$$

Complex exponential notation allows us to directly determine the steady-state periodic response to general periodic forcing, in terms of both the magnitude of the response and the phase of the response. Recall the relationship between the complex magnitudes X and F for a sinusoidally-driven spring-mass-damper oscillator is

$$
\begin{aligned}
X_q &= \frac{1}{(k - m\omega_q^2) + i(c\omega_q)} \; F_q && (15)\\
X_q &= H(\omega_q) \; F_q && (16)
\end{aligned}
$$

The function $H(\omega)$ is called the frequency response function for the dynamic system relating the input $f(t)$ to the output $x(t)$.

Because the oscillator is linear, if the response to $f_1(t)$ is $x_1(t)$, and the response to $f_2(t)$ is $x_2(t)$, then the response to $c_1 f_1(t) + c_2 f_2(t)$ is $c_1 x_1(t) + c_2 x_2(t)$. More generally, then,

$$
x(t) = \sum_{q=-\infty}^{q=\infty} \frac{1}{(k - m\omega_q^2) + i(c\omega_q)} \; F_q \; e^{i\omega_q t} \qquad (17)
$$

where $\omega_q = 2\pi q/T$.

Note that ω_q is *not* the same symbol as ω_{n}; $\omega_{\mathrm{n}} = \sqrt{k/m}$, $\omega_q = 2\pi q/T$.

The series expansion for the response $x(t)$ converges with fewer terms than the Fourier series for the external forcing, because $|H(\omega_q)|$ decreases as $1/\omega_q^2$.

2 Fourier Transforms

Recall for periodic functions of period, T, the Fourier series expansion is

$$f(t) = \sum_{q=-\infty}^{q=\infty} F_q \; e^{i\omega_q t} \; , \tag{18}$$

where the Fourier coefficients, F_q, have the same units as $f(t)$, and are given by the Fourier integral,

$$F_q = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \; e^{-i\omega_q t} \; dt \; , \tag{19}$$

in which the interval of integration arbitrarily starts at $-T/2$.

Now, consider a change of variables, by defining a frequency increment, $\Delta\omega$, and a scaled amplitude, $F(\omega_q)$.

$$\Delta\omega \;\triangleq\; \omega_1 = \frac{2\pi}{T} \qquad (\omega_q = q\;\Delta\omega) \tag{20}$$

$$F(\omega_q) \;\triangleq\; T F_q = \frac{2\pi}{\Delta\omega} F_q \tag{21}$$

Where the scaled amplitude, $F(\omega_q)$, has units of $f(t) \cdot t$ or $f(t)/\omega$.

Using these new variables,

$$f(t) \;=\; \frac{1}{2\pi} \sum_{q=-\infty}^{q=\infty} F(\omega_q) \; e^{i\omega_q t} \; \Delta\omega \; , \tag{22}$$

$$F(\omega_q) \;=\; \int_{-T/2}^{T/2} f(t) \;\; e^{-i\omega_q t} \; dt \; . \tag{23}$$

Finally, taking the limit as $T \to \infty$, implies $\Delta\omega \to d\omega$ and $\sum \to \int$

$$f(t) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \; e^{i\omega t} \; d\omega \; , \tag{24}$$

$$F(\omega) \;=\; \int_{-\infty}^{\infty} f(t) \;\; e^{-i\omega t} \; dt \; . \tag{25}$$

These expressions are the famous Fourier transform pair. Equation (24) is commonly called the *inverse* Fourier transform and equation (25) is commonly called the *forward* Fourier transform. They differ only by the sign of the exponent and the factor of 2π.

By convention, the forward fast Fourier transform (FFT) of an N-point time series of duration T ($x_k = x((k-1)\Delta t)$, $k = 1, \cdots, N$) scales the N, complex-valued, Fourier amplitudes/coefficients as follows: $\mathrm{FFT}(x) = X(\omega_q)/\Delta t = N X_q$, where the Fourier-transform frequencies, ω_q, are given by $\omega_q = 2\pi q/T$, and are sorted as follows:
$q = 0, \cdots, N/2, -N/2+1, \cdots, -1$.

3 Fourier Approximation

Any periodic function may be *approximated* as a *truncated* series expansion of sines and cosines, as a Fourier series,

$$\tilde{f}(t) = \frac{1}{2}a_0 + \sum_{q=1}^{Q} a_q \cos\frac{2\pi qt}{T} + \sum_{q=1}^{Q} b_q \sin\frac{2\pi qt}{T} \ , \tag{26}$$

where the Fourier coefficients, a_q and b_q may be found by solving the Fourier integrals,

$$a_q \ = \ \frac{2}{T}\int_{-T/2}^{T/2} f(t) \ \cos\frac{2\pi qt}{T} \ dt \ , \quad q = 0, 1, 2, \ldots, Q \tag{27}$$

$$b_q \ = \ \frac{2}{T}\int_{-T/2}^{T/2} f(t) \ \sin\frac{2\pi qt}{T} \ dt \ , \quad q = 0, 1, 2, \ldots, Q \tag{28}$$

The Fourier approximation (26) may also be represented using complex exponential notation.

$$\tilde{f}(t) = \sum_{q=-Q}^{q=Q} F_q \ e^{i\omega_q t} = \frac{1}{T}\sum_{q=-Q}^{q=Q} F(\omega_q) \ e^{i\omega_q t} \tag{29}$$

where $e^{i\omega_q t} = \cos\omega_q t + i\sin\omega_q t$, $\omega_q = 2\pi q/T$, $F_q = (a_q - ib_q)/2$, $F_{-q} = F_q^*$, and

$$F_q = \frac{1}{T}\int_{-T/2}^{T/2} f(t) \ e^{-i\omega_q t} \ dt \ . \tag{30}$$

The response of a system described by a frequency response function $H(\omega)$ to arbitrary periodic forces described by a Fourier series may be found in the frequency domain,

$$X_q \ = \ H(\omega_q) \ F_q \ , \tag{31}$$

or in the time domain,

$$x(t) = \sum_{q=-Q}^{q=Q} H(\omega_q) \ F_q \ e^{i\omega_q t} = \frac{1}{T}\sum_{q=-Q}^{q=Q} H(\omega_q) \ F(\omega_q) \ e^{i\omega_q t} \ . \tag{32}$$

4 Examples

In the following examples, the external forcing is periodic with a period, T, of 2π (seconds). For functions periodic in 2π seconds, the frequency increment in the Fourier series is 1 radian/second. In this case, the frequency index number, q, is also the frequency, ω_q, in radians per second, otherwise, $\omega_q = 2\pi q/T$.

The system is a forced spring-mass-damper oscillator,

$$\ddot{x}(t) + 2\zeta\omega_\mathrm{n}\dot{x}(t) + \omega_\mathrm{n}^2 x(t) = \frac{1}{m} f^\mathrm{ext}(t) \ . \tag{33}$$

The complex-valued frequency response function from the external force, $f^\mathrm{ext}(t)$, to the response displacement, $x(t)$, is given by

$$H(\omega) = \frac{1/\omega_\mathrm{n}^2}{1 - \Omega^2 + i\ 2\zeta\Omega} \ , \tag{34}$$

where Ω is the frequency ratio, ω/ω_n,

In the following three numerical examples, $\omega_\mathrm{n} = 10$ rad/s, $\zeta = 0.1$, $m = 1$ kg, and $Q = 16$ terms. The three examples consider external forcing in the form of a square-wave, a sawtooth-wave, and a triangle-wave. In each example six plots are provided.

In the (a) plots, the solid line represents the exact form of $f(t)$, the dashed lines represent the real-valued form of the Fourier approximation and the complex-valued form of the Fourier approximation, and the circles represent $2Q$ sample points of the function $f(t)$ for use in fast Fourier transform (FFT) computations. The two dashed lines are exactly equal.

In the (b) plots, the impulse-lines show the values of the Fourier coefficients, F_q, found by evaluating the Fourier integral, equation (30), and the circles represent the Fourier coefficients returned by the FFT.

In the (c) plots, the red solid lines show the cosine terms of the Fourier series and the blue dashed lines show the sine terms of the Fourier series.

The (d) and (e) plots show the magnitude and phase of the transfer function, $H(\omega)$ as a solid line, the Fourier coefficients, F_q, as the green circles, and $H(\omega_q)F_q$ as the blue circles.

In the (f) plots, the solid line represents $x(t)$ as computed using equation (32), the circles represent the real part of the result of the inverse FFT calculation and the dots represent the imaginary part of the result of the inverse FFT calculation.

The numerical details involved in the correct representation of periodic functions and frequency indexing for FFT computations are provided in the attached Matlab code.

4.1 Example 1: square wave

The external forcing is given by

$$f^{\text{ext}}(t) = \text{sgn}(t) \; ; \quad -\pi < t < \pi \tag{35}$$

The Fourier coefficients for the real-valued Fourier series are:
$a_q = 0$ and $b_q = -(2/(q\pi))\,(-1^q - 1)$.

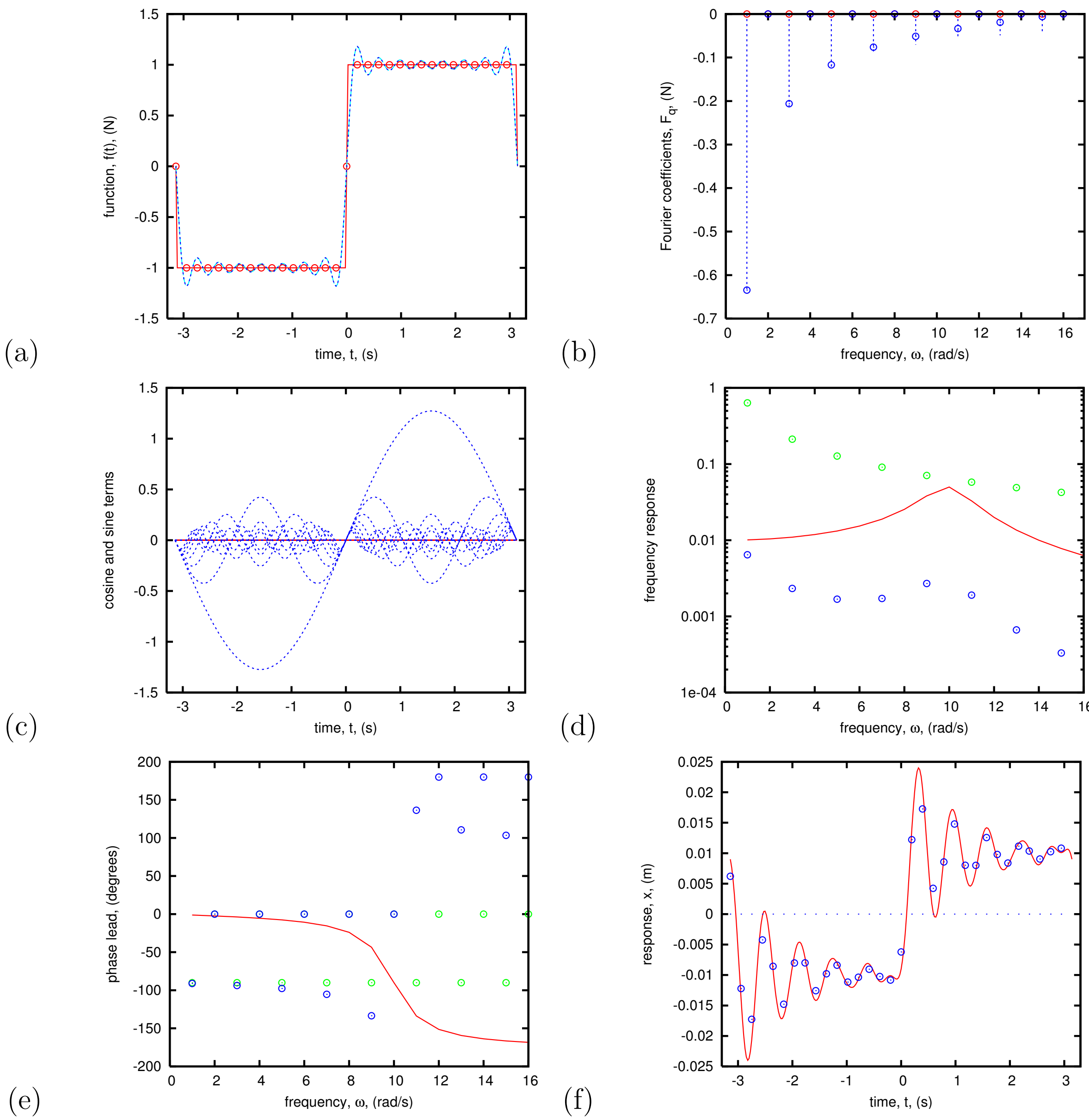

Figure 1. (a): — = $f^{ext}(t)$; - - = $\tilde{f}^{ext}(t)$. (b): red solid = $\Re(F_q)$; blue dashed = $\Im(F_q)$; *: F_q from FFT. (c): red solid = component cosine terms, blue dashed = component sine terms. (d): — = $|H(\omega)|$; *: $|F_q|$ and $|H(\omega_q)F_q|$. (e): — = phase lead of $H(\omega)$; *: phase leads of F_q and $H(\omega_q)F_q$. (f): — = $x(t)$; *: $x(t)$ from IFFT; $\cdots$: $\Im(x(t))$ from IFFT.

4.2 Example 2: sawtooth wave

The external forcing is given by

$$f^{\mathrm{ext}}(t) = x \; ; \quad -\pi < t < \pi \tag{36}$$

The Fourier coefficients for the real-valued Fourier series are:
$a_q = 0$ and $b_q = -(2/q)\ (-1^q)$.

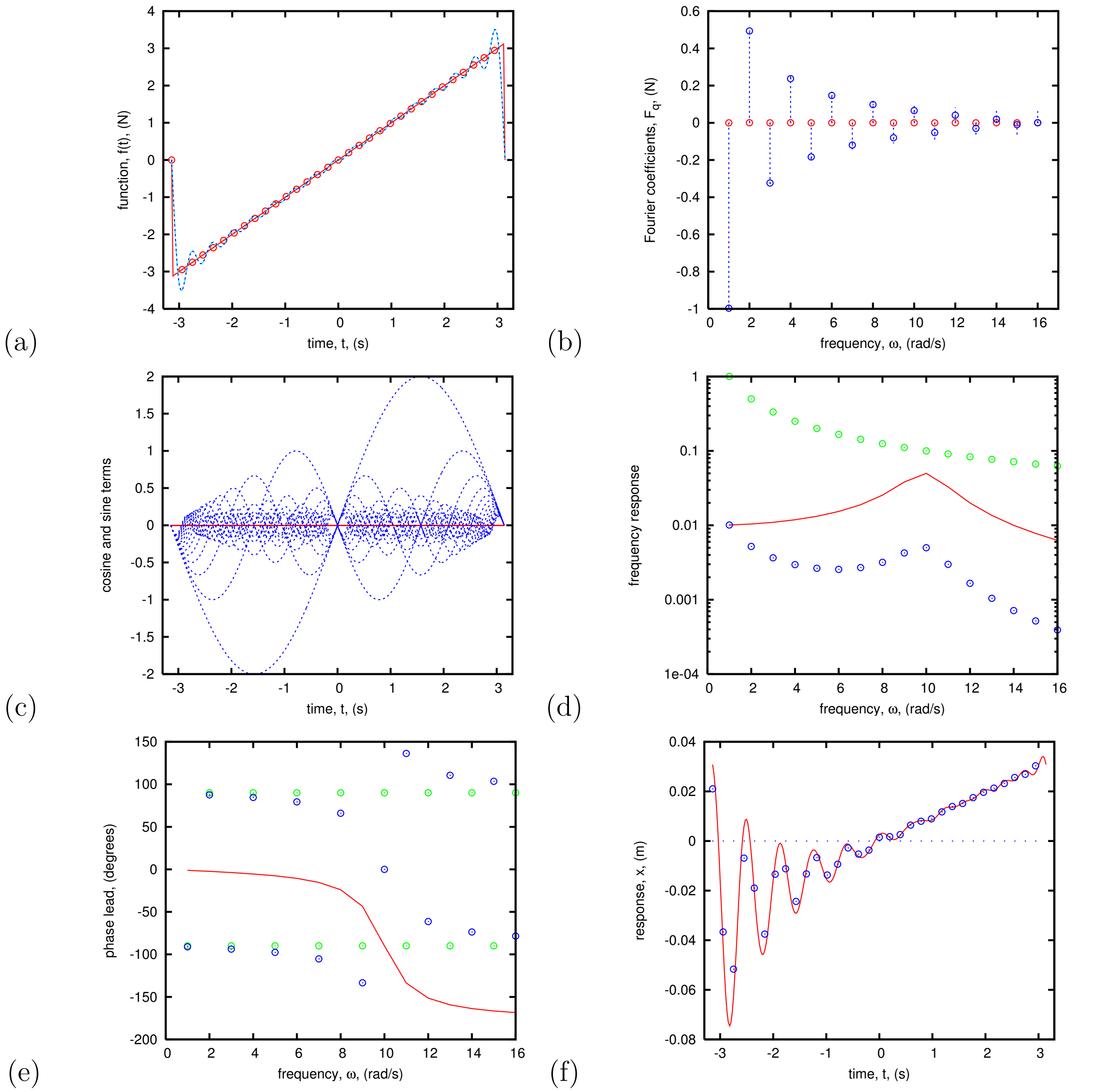

Figure 2. (a): — = $f^{ext}(t)$; - - = $\tilde{f}^{\mathrm{ext}}(t)$. (b): red solid = $\Re(F_q)$; blue dashed = $\Im(F_q)$; *: F_q from FFT. (c): red solid = component cosine terms, blue dashed = component sine terms. (d): — = $|H(\omega)|$; *: $|F_q|$ and $|H(\omega_q)F_q|$. (e): — = phase lead of $H(\omega)$; *: phase leads of F_q and $H(\omega_q)F_q$. (f): — = $x(t)$; *: $x(t)$ from IFFT; $\cdots$: $\Im(x(t))$ from IFFT.

4.3 Example 3: triangle wave

The external forcing is given by

$$f^{\text{ext}}(t) = \pi/2 - t\ \text{sgn}(t)\ ; \quad -\pi < t < \pi \tag{37}$$

The Fourier coefficients for the real-valued Fourier series are:
$a_q = -(2/q^2\pi)(-1^q - 1)$ and $b_q = 0$.

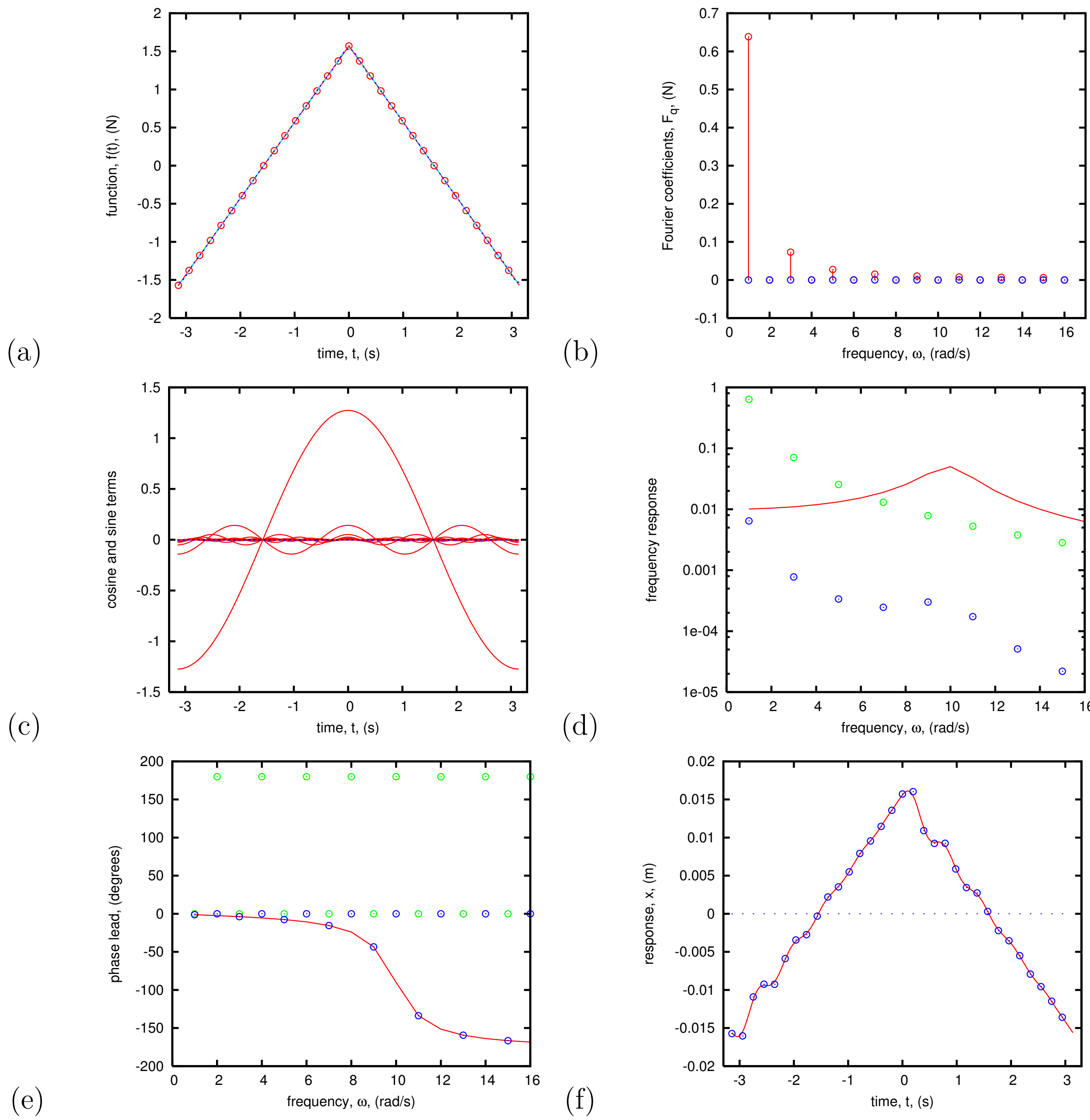

Figure 3. (a): — $= f^{ext}(t)$; - - $= \tilde{f}^{\text{ext}}(t)$. (b): red solid $= \Re(F_q)$; blue dashed $= \Im(F_q)$; *: F_q from FFT. (c): red solid = component cosine terms, blue dashed = component sine terms. (d): — $= |H(\omega)|$; *: $|F_q|$ and $|H(\omega_q)F_q|$. (e): — = phase lead of $H(\omega)$; *: phase leads of F_q and $H(\omega_q)F_q$. (f): — $= x(t)$; *: $x(t)$ from IFFT; $\cdots$: $\Im(x(t))$ from IFFT.

4.4 Matlab code

```
function [Fq,wq,x,t] = Fourier(type,Q,wn,z)
% [F,w,x] = Fourier(type,Q,wn,z)
% Compute the Fourier series coefficients of a periodic signal, f(t),
% in two different ways:
%  * complex exponential expansion   (i.e., sine and cosine expansion)
%  * fast Fourier transform
%
% The 'type' of periodic signal may be
%      'square' a square wave
%      'sawtooth' a saw-tooth wave
%      'triangle' a triangle wave
%
% The period of the signal, f(t), is fixed at 2*pi (second).
%
% The number of Fourier series coefficients is input as Q.
% This results in a 2*Q Fourier transform coefficients.
%
% The Fourier approximation is then used to compute the steady-state
% response of a single degree of freedom (SDOF) oscillator, described by
%
%                          2
%  x"(t) + 2 z wn x'(t) + wn x(t) = f(t) ,
%
% where the mass of the system is fixed at 1 (kg).
%
% see: http://en.wikipedia.org/wiki/Fourier_series
%      http://www.jhu.edu/~signals/fourier2/index.html
%

if nargin < 4
   help Fourier
   return
end


T = 2*pi;                          % period of external forcing, s

a  = zeros(1,Q);                   % coefficients of cosine part of Fourier series
b  = zeros(1,Q);                   % coefficients of   sine part of Fourier series

q  = [1:Q];                        % positive frequency index value
wq = 2*pi*q/T;                     % positive frequency values, rad/sec,  eq'n (20)

dt = T/(2*Q);                      % 2Q discrete points in time for FFT sampling
ts = [-Q:Q-1]*dt;                  % 2Q discrete points in time for FFT sampling

% "shifted" time ... -T/2 < t < T/2 ...
%  ts(1)   = -Q*dt    = -T/2      = -pi    ... f(-T/2) = f(T/2) by extension
%  ts(Q)   = -dt      = -T/(2*Q)  = -pi/Q
%  ts(Q+1) =   0                           ... f(0)    = f(T)   by extension
%  ts(Q+2) =  dt      =  T/(2*Q)  =  pi/Q
%  ts(2*Q) = (Q-1)*dt = T/2 - dt  =  pi - pi/Q

P = 128;  % number of points in the time-record (for plotting purposes only)
t  = [-P:P]*T/2/P;                 % time axis

% ======================================================================
% Fourier series approximation of general periodic functions

% sin(q*pi) = 0 , cos(q*pi) = (-1).^q
% f_true is used only for plotting
% f_samp is used in FFT computations



if strcmp( type,'square' )                      % square wave
   b = - 2./(q*pi) .* ((-1).^q - 1);
   f_true = sign(t);   f_true(1) = 0; f_true(2*P+1) = 0;
   f_samp = sign(ts);
   f_samp(1) = 0;                       % get periodicity right, f(-pi) = 0
```

```
end

if strcmp( type,'sawtooth' )                        % sawtooth wave
   b = -(2./q) .* (-1).^q;
   f_true = t;   f_true(1) = 0; f_true(2*P+1) = 0;
   f_samp = ts;
   f_samp(1) = 0;                           % get periodicity right, f(-pi) = 0
end

if strcmp( type,'triangle' )                        % triangle wave
   a = -(2./(pi*q.^2)) .* ( (-1).^q - 1 );
   f_true = pi/2 - t .*sign(t);
   f_samp = pi/2 - ts.*sign(ts);
end


f_approxR = a * cos(q'*t) + b * sin(q'*t);      % real Fourier series

Fq = (a - i*b)/2;                                   % complex Fourier coefficients

% Complex Fourier series (positive and negative exponents)
f_approxC = Fq * exp(i*wq'*t) + conj(Fq) * exp(-i*wq'*t);


% The imagniary part of the complex Fourier series is exactly zero!
imag_over_real_1 = max(abs(imag(f_approxC))) / max(abs(real(f_approxC)))


% The fast Fourier transform (FFT) method
% In Matlab, the forward Fourier transform has a negative exponent.
% According to a convention of the FFT method, index number 1 is for time = dt
% ts(Q+1)=0 ; ts(2*Q) = (Q-1)*dt = T/2-dt ; ts(1) = -Q*dt = -T/2 ; ts(Q) = -dt
%   ... see the table at the end of this file.

F_FFT = fft( [ f_samp(Q+1:2*Q) f_samp(1:Q) ] ) / (2*Q);  % Fourier coeff's


% coef = [ Fq' F_FFT(2:Q+1)' ]  % display the Fourier coefficients

% ------- Plotting --------

figure(1);
  plot(t,f_true,'-r' , ts,f_samp,'*r' , t,f_approxR,'-c' , t,f_approxC,'-b' )
   xlabel('time, t, (s)')
   ylabel('function, f(t), (N)')
   axis (1.05*[-pi,pi])
   legend('true','sampled','real Fourier approx','complex Fourier approx')
  plot(wq,real(Fq),'^r', wq,imag(Fq),'^b' , ...
       wq,real(F_FFT(2:Q+1)),'*r', wq,imag(F_FFT(2:Q+1)),'*b' )
   ylabel('Fourier coefficients, F(w), (N)')
   xlabel('frequency, {/Symbol w}, (rad/s)')
   legend('real(F)', 'imag(F)', 'real(FFT(f))', 'imag(FFT(f))')


figure(2);
  plot(t,a'*ones(1,2*P+1).*cos(q'*t),'-r' , t,b'*ones(1,2*P+1).*sin(q'*t),'-b');
   xlabel('time, t, (s)')
   ylabel('cosine and sine terms')
   axis (1.05*[-pi,pi])


% ============================================================
% Steady-state response of a SDOF oscillator to general periodic forcing.

% Complex-valued frequency response function, H(w), has units of [m/N]
H = (1/wn^2) ./ ( 1 - (wq/wn).^2 + 2*i*z*(wq/wn) );

% Complex Fourier series (positive and negative exponents)
x_approxC = (H .* Fq) * exp(i*wq'*t) + (conj(H) .* conj(Fq)) * exp(-i*wq'*t);

% In Matlab, the inverse Fourier transform has a positive exponent.
```

```
%   ... see the table at the end of this file for the sorting convention.
x_FFT = ifft( [ (1/wn^2) , H , conj(H(Q-1:-1:1)) ] .* F_FFT ) * (2*Q);

figure(3);                                  % plot frequency response functions
  semilogy(wq,abs(Fq),'*g', wq,abs(H),'-r', wq,abs(H.*Fq),'*b' )
   xlabel('frequency, {/Symbol w}, (rad/s)')
   ylabel('frequency response')
   legend('|F|', '|H|', '|H*F|')

  plot (wq,angle(Fq)*180/pi,'*g', wq,angle(H)*180/pi,'-r', wq,angle(H.*Fq)*180/pi,'*b')
   xlabel('frequency, {/Symbol w}, (rad/s)')
   ylabel('phase lead, (degrees)')

figure(4);                                % plot time response
  ts = [ ts(Q+1:2*Q) ts(1:Q) ];          % re-sort the time axis
  plot ( t,x_approxC,'-r' , ts, real(x_FFT),'*b' , ts, imag(x_FFT),'.b' )
   axis (1.05*[-pi,pi])
   xlabel('time, t, (s)')
   ylabel('response, x, (m)')
   legend('x(t) - complex Fourier series', 'real(x(t)) - FFT', 'imag(x(t)) - FFT')


% The imaginary part is exactly zero for the complex exponent series method!
imag_over_real_2 = max(abs(imag(x_approxC))) / max(abs(real(x_approxC)))

% The imaginary part is practically zero for the FFT method!
imag_over_real_3 = max(abs(imag(x_FFT))) / max(abs(real(x_FFT)))

% ---------------------------------------------------------------------------
% SORTING of the FFT coefficients    number of data points = N = 2*Q
% note:  f_max = 1/(2*dt);    df = f_max/(N/2) = 1/(N*dt) = 1/T;
%
%
%       index          time              frequency
%       =====          ====              =========
%         1               0                  0
%         2              dt                 df
%         3            2*dt               2*df
%         4            3*dt               3*df
%         :               :                  :
%       N/2-1        (N/2-2)*dt         f_max-2*df  = ( N/2-2)*df
%       N/2          (N/2-1)*dt         f_max-df    = ( N/2-1)*df
%       N/2+1        (N/2)*dt       +/- f_max       = ( N/2  )*df
%       N/2+2        (N/2+1)*dt        -f_max+df    = (-N/2+1)*df
%       N/2+3        (N/2+2)*dt        -f_max+2*df  = (-N/2+2)*df
%         :               :                  :               :
%       N-2          (N-3)*dt           -3*df
%       N-1          (N-2)*dt           -2*df
%       N            (N-1)*dt             -df
% ---------------------------------------------------------------------------
```

5 Random Loading

5.1 Response to Random Loading with Finite Period or Duration

The frequency content of a random process is characterized by its power spectral density. The power spectral density of a random process of finite duration, T, is the expectation of the process's squared Fourier amplitudes divided by its duration

$$P_{xx}(\omega_q) = \frac{1}{T}\,\mathsf{E}[|X(\omega_q)|^2] = \frac{1}{T}\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}|X_m(\omega_q)|^2 = \frac{1}{T}\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\left|\int_0^T x_m(t)e^{-i\omega_q t}\,dt\right|. \tag{38}$$

where $x_m(t)$ is a record in a sample of M records representative of the process. *Unit-intensity white noise*, $u(t)$, is a random process with a power spectral density of unity over all frequencies, so $\mathsf{E}[|U(\omega_q)|^2] = T$. If $u(t)$ is periodic in T then $\mathsf{E}[|U(\omega_q)|^2] = |U_m(\omega_q)|^2 \;\forall\; n$, and there is no uncertainty in the Fourier magnitudes, $|U(\omega_q)|$. In such a situation, the uncertainty lies entirely in the ratio of the real to the imaginary parts of $U(\omega_q)$, i.e., the phase, $\phi_q = \tan^{-1}(U''(\omega_q)/U'(\omega_q))$. The complex Fourier amplitudes of unit-intensity noise with period T (and no spectral magnitude variability) may therefore be computed from

$$U(\omega_q) = U'(\omega_q) + iU''(\omega_q) = \sqrt{T}\cos\phi_q + i\sqrt{T}\sin\phi_q\ ; \tag{39}$$

where ϕ_q is uniformly distributed between $-\pi$ and π.

If $u(t)$ is periodic (or if $u(t)$ has finite duration and is assumed to be periodic) the response $y(t)$ of a system with frequency response $H(\omega)$ to $u(t)$ is also periodic and may be approximated from the finite series

$$y(t) = \frac{1}{T}\sum_{q=-Q}^{q=Q} H(\omega_q)\;U(\omega_q)\;e^{i\omega_q t}\ . \tag{40}$$

where $\omega_q = 2\pi q/T$. If $U(\omega)$ is the Fourier transform of a unit-intensity random process for which $\mathsf{E}[|U(\omega_q)|^2] = |U(\omega_q)|^2$, the periodic response may be computed from

$$y(t) = 2\sqrt{\Delta f}\sum_{q=1}^{q=Q}|H(\omega_q)|\cos(\omega_q t + \theta_q)\ , \tag{41}$$

where θ_q is uniformly distributed between $-\pi$ and π. This response has a period T and $\Delta f = 1/T$.

Since the frequency response, $H(\omega)$ represents the input-output relationship of real-valued functions, $H(-\omega) = H^*(\omega)$, $U(-\omega) = U^*(\omega)$, and $Y(-\omega) = Y^*(\omega)$. The system frequency response function may be represented in terms of real and imaginary parts, $H(\omega) = H'(\omega) + iH''(\omega)$ where $H'(\omega) = |H(\omega)|\cos\psi(\omega)$ and $H''(\omega) = |H(\omega)|\sin\psi(\omega)$, and where $|H(\omega)|$ is the magnitude of the frequency response function and $\psi(\omega)$ is the phase of the frequency response function. The following derivation makes use of the notation $U(\omega_q)/T = U_q$ and $H(\omega_q)/T = H_q$.

Breaking the sum of equation (40) into negative and non-negative values of the index q,

$$
\begin{aligned}
y(t) &= \frac{1}{T}\sum_{q=-Q}^{q=-1} H(\omega_q)\ U(\omega_q)\ e^{i\omega_q t} + \frac{1}{T}\sum_{q=0}^{q=Q} H(\omega_q)\ U(\omega_q)\ e^{i\omega_q t} \\
&= T\sum_{q=-Q}^{-1} H_q U_q(\cos\omega_q t + i\sin\omega_q t) + T\sum_{q=0}^{Q} H_q U_q(\cos\omega_q t + i\sin\omega_q t) \\
&= T\sum_{q=1}^{Q} H_q^* U_q^*(\cos\omega_q t - i\sin\omega_q t) + T\sum_{q=0}^{Q} H_q U_q(\cos\omega_q t + i\sin\omega_q t) \\
&= TH_0U_0(\cos\omega_0 t + i\sin\omega_0 t) + \\
&\quad T\sum_{q=1}^{Q}(H_q'U_q' - H_q''U_q'' - i(H_q'U_q'' + H_q''U_q'))(\cos\omega_q t - i\sin\omega_q t) + \\
&\quad T\sum_{q=1}^{Q}(H_q'U_q' - H_q''U_q'' + i(H_q'U_q'' + H_q''U_q'))(\cos\omega_q t + i\sin\omega_q t) \\
&= TH_0U_0 + T\sum_{q=1}^{Q} H_q'U_q'\cos\omega_q t - H_q''U_q''\cos\omega_q t - i(H_q'U_q'' + H_q''U_q')\cos\omega_q t + \\
&\qquad - iH_q'U_q'\sin\omega_q t + iH_q''U_q''\sin\omega_q t - (H_q'U_q'' + H_q''U_q')\sin\omega_q t + \\
&\qquad T\sum_{q=1}^{Q} H_q'U_q'\cos\omega_q t - H_q''U_q''\cos\omega_q t + i(H_q'U_q'' + H_q''U_q')\cos\omega_q t + \\
&\qquad\quad iH_q'U_q'\sin\omega_q t - iH_q''U_q''\sin\omega_q t - (H_q'U_q'' + H_q''U_q')\sin\omega_q t \\
&= TH_0U_0 + 2T\sum_{q=1}^{Q}(H_q'U_q' - H_q''U_q'')\cos\omega_q t - (H_q'U_q'' + H_q''U_q')\sin\omega_q t \\
&= TH_0U_0 + 2\sqrt{T}\sum_{q=1}^{Q}(H_q'\cos\phi_q - H_q''\sin\phi_q)\cos\omega_q t - (H_q'\sin\phi_q + H_q''\cos\phi_q)\sin\omega_q t \\
&= TH_0U_0 + 2\sqrt{T}\sum_{q=1}^{Q} H_q'(\cos\phi_q\cos\omega_q t - \sin\phi_q\sin\omega_q t) - H_q''(\cos\phi_q\sin\omega_q t + \sin\phi_q\cos\omega_q t) \\
&= TH_0U_0 + 2\sqrt{T}\sum_{q=1}^{Q} H_q'\cos(\omega_q t + \phi_q) - H_q''\sin(\omega_q t + \phi_q) \\
&= TH_0U_0 + 2\sqrt{T}\sum_{q=1}^{Q} |H_q|\cos\psi_q\cos(\omega_q t + \phi_q) - |H_q|\sin\psi_q\sin(\omega_q t + \phi_q) \\
&= TH_0U_0 + 2\sqrt{T}\sum_{q=1}^{Q} |H_q|\cos(\omega_q t + \phi_q + \psi_q) \qquad (42)
\end{aligned}
$$

Since cosine is periodic in $[-\pi : \pi]$ and since ϕ is uniformly distributed within $[-\pi : \pi]$, and assuming either $H_0 = 0$ or $U_0 = 0$, equation (42) simplifies to equation (41)

$$y(t) = 2\sqrt{\Delta f}\sum_{q=1}^{q=Q} |H(\omega_q)|\cos(\omega_q t + \theta_q)$$

where θ_q is uniformly distributed in $[-\pi : \pi]$. Note, again, that over a period $T = 2\pi/\omega_1 = 1/(\Delta f)$, this response is periodic. With frequencies $\omega_q = 2\pi q/T$, and $0 \le t < T$, the magnitude of the Fourier transform of the response to unit-intensity random input, $y(t)$, as computed from equation (41) recovers $|H(\omega)|$ exactly: $\mathsf{E}[|Y(\omega_q)|^2] = |Y(\omega_q)|^2 = |H(\omega_q)|^2$.

5.2 Response to Non-periodic or Persistent Random Loading

The response of a linear system to non-periodic or persistent white noise is not periodic. A strictly-proper linear system,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_o & (43) \\ y(t) &= Cx(t) & (44) \end{aligned}$$

has the frequency response $H(\omega) = C(i\omega I - A)^{-1}B$, which, in turn, has a realization

$$H(\omega) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right] . \qquad (45)$$

Its response may be simulated in discrete-time via numerical integration methods. If the time series $u(t_k)$ and $y(t_k)$ have N points, simulating white noise response of the system (45) requires the generation of N uncorrelated standard Gaussian random values, $u(t_k)$. Sampled unit Gaussian white noise with time step Δt is normally distributed with a mean of zero and a standard deviation of $1/\sqrt{\Delta t}$, has a mean square value of $1/(\Delta t)$, and a power spectral density of 1 over a frequency range $-1/(2\Delta t) < f < 1/(2\Delta t)$.