un-observable and un-controllable subspaces

CEE 629. System Identification

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1 Un-controllable states

In discrete-time, the controllability matrix of block-length R relates a sequence of R inputs to the state $x(R)$ (assuming the state $x(0) = 0$).

$$x(R) = \left[\begin{array}{cccccc} B & AB & A^2B & A^3B & \cdots & A^{R-1}B \end{array} \right] \left[\begin{array}{c} u(R-1) \\ u(R-2) \\ u(R-3) \\ u(R-4) \\ \vdots \\ u(0) \end{array} \right] \qquad (1)$$

or $x(R) = \mathcal{Q}_R u_R$. If $\mathcal{Q}_R$ is full (row) rank then there is an input sequence that can bring the state from $x(0) = 0$ to $x(R) \neq 0$ in R steps. The two statements "$\mathcal{Q}_R$ is full rank" and "the pair (A, B) is *controllable*" are equivalent.

The pair (A, B) is not controllable if two or more rows of $\mathcal{Q}_R$ are linearly dependent. Because the rank of a matrix is invariant under a coordinate transformation, the rank of $\mathcal{Q}_R$ is equal to the rank of $T^{-1}\mathcal{Q}_R$, where T is a coordinate transformation matrix. If two (or more) rows of $\mathcal{Q}_R$ are linearly dependent, then there is a transformation matrix T for which one (or more) row of $T^{-1}\mathcal{Q}_R$ is zero. To examine a case in which (A, B) is not (fully) controllable, it is easier to consider a diagonalized realization. For the diagonalizing coordinate transformation,

$$x(k) = Tq(k) \ , \qquad q(k) = T^{-1}x(k)$$

$$\begin{aligned} x(k+1) &= ATq(k) + Bu(k) \\ q(k+1) &= T^{-1}ATq(k) + T^{-1}Bu(k) \\ T^{-1}AT &= \Lambda = \mathsf{diag}([\lambda_1, ..., \lambda_n]) \end{aligned}$$

the coordinates $q(k)$ are called "modal coordinates." Defining notation for the input matrix in modal coordinates $\bar{B} = T^{-1}B$, the controllability matrix in modal coordinates is

$$\bar{\mathcal{Q}}_R = \left[\begin{array}{cccccc} \bar{B} & \Lambda\bar{B} & \Lambda^2\bar{B} & \Lambda^3\bar{B} & \cdots & \lambda^{R-1}\bar{B} \end{array} \right] \qquad (2)$$

Now consider a case with a single input, so that the input matrix has one column, $B = b$, and $\bar{B} = \bar{b}$. If b is normal to the i-th row of T^{-1},

- the i-th row of $\bar{b}$ is zero,
- the i-th row of $\Lambda^k\bar{b}$ is zero,
- the rank of $\mathcal{Q}_R$ is $n - 1$,
- the input sequence $u(k)$ does not couple to the sequence $q_i(k)$, and
- q_i is in an uncontrollable subspace of $\mathbb{R}^n$.

2 Un-observable states

In discrete-time, the observability matrix of block-length M relates a sequence of M free responses to the initial state $x(0)$ (assuming the inputs $u(k) = 0$).

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ \vdots \\ y(M-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{M-1} \end{bmatrix} x(0) \tag{3}$$

or $y_M = \mathcal{P}_M x(0)$. If $\mathcal{P}_M$ is full (column) rank then there is a unique initial state that is linearly related to the output sequence y_M. The two statements "$\mathcal{P}_M$ is full rank" and the pair (A, C) is *observable*" are equivalent.

The pair (A, C) is not observable if two or more columns of $\mathcal{P}_M$ are linearly dependent. Because the rank of a matrix is invariant under a coordinate transformation, the rank of $\mathcal{P}_M$ is equal to the rank of $\mathcal{P}_M T$, where T is a coordinate transformation matrix. If two (or more) columns of $\mathcal{P}_M$ are linearly dependent, then there is a transformation matrix T for which one (or more) row of $\mathcal{P}_M T$ is zero. To examine a case in which (A, C) is not (fully) observable, it is easier to consider a diagonalized realization. For the diagonalizing coordinate transformation,

$$x(k) = Tq(k) \;, \qquad q(k) = T^{-1}x(k)$$

$$\begin{aligned} q(k+1) &= \Lambda q(k) \\ T^{-1}AT &= \Lambda = \mathsf{diag}([\lambda_1, ..., \lambda_n]) \end{aligned}$$

the coordinates $q(k)$ are called "modal coordinates." Defining notation for the output matrix in modal coordinates $\bar{C} = CT$, the observability matrix in modal coordinates is

$$\bar{\mathcal{P}}_M = \begin{bmatrix} \bar{C} \\ \bar{C}\Lambda \\ \bar{C}\Lambda^2 \\ \bar{C}\Lambda^3 \\ \vdots \\ \bar{C}\lambda^{M-1} \end{bmatrix} \tag{4}$$

Now consider a case with a single output, so that the output matrix has one row, $C = c$, and $\bar{C} = \bar{c}$. If c is normal to the j-th column of T,

- the j-th column of $\bar{c}$ is zero,
- the j-th column of $\Lambda^k \bar{c}$ is zero,
- the rank of $\mathcal{P}_M$ is $n-1$,
- the output sequence $y(k)$ does not couple to the sequence $q_j(k)$, and
- q_j is in an unobservable subspace of $\mathbb{R}^n$.

3 Example: string vibration controllability

Consider the forced transverse vibration of a string of length L, and mass-per-unit-length μ_s, and under uniform constant tension T_s, in continuous time and continuous space. The (mass-normalized) modes of vibration of a string under tension are

$$\phi_j(x) = \frac{1}{\sqrt{\mu_s}} \sin \frac{j\pi x}{L}$$

the position of the transverse string position at a particular location x ($0 \le x \le L$) can be expressed as a (truncated) expansion of the first n modes

$$r(x,t) = \sum_{j=1}^{n} \phi_j(x) p_j(t)$$

Substituting this modal expansion into the partial differential equation for linear string vibration, and taking the inner product of $\phi_k(x)$ with the PDE, we obtain a set of ordinary differential equations. With mass-normalized modes and a single transverse force acting at $x = \xi_u$, the system of decoupled ODE's is

$$\begin{bmatrix} \ddot{p}_1(t) \\ \vdots \\ \ddot{p}_n(t) \end{bmatrix} + \begin{bmatrix} {\omega_{\mathrm{n}}}_1^2 & & \\ & \ddots & \\ & & {\omega_{\mathrm{n}}}_n^2 \end{bmatrix} \begin{bmatrix} p_1(t) \\ \vdots \\ p_n(t) \end{bmatrix} = \begin{bmatrix} \phi_1(\xi_u) \\ \vdots \\ \phi_n(\xi_u) \end{bmatrix} u(t) \tag{5}$$

Now consider the location along the string $\xi_u = 3L/4$, which is a node of $\phi_{(4,8,12,16,\ldots)}(x)$, that is, $\phi_{(4,8,12,16,\ldots)}(3L/4) = 0$. In matrix form ($p \in \mathbb{R}^n$, $h \in \mathbb{R}^n$), the system of decoupled ODE's is

$$\ddot{p}(t) + \Omega_{\mathrm{n}}^2 p(t) = hu(t)$$

where, in this example, $h_{(4,8,12,16,\ldots)} = 0$. In continuous time state-space form, with states of p and $\dot{p}$,

$$\frac{d}{dt}\begin{bmatrix} p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -\Omega_{\mathrm{n}}^2 & 0_n \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ h \end{bmatrix} u(t) \tag{6}$$

The eigenvalues of this continuous-time dynamics matrix are $\pm \mathrm{i}\omega_{\mathrm{n}}$. Now, following through with a transformation from continuous-time to discrete-time,

$$x(k+1) = Ax(k) + bu(k)$$

where

$$A = \begin{bmatrix} \cos(\Omega_{\mathrm{n}}(\Delta t)) & \Omega_{\mathrm{n}}^{-1}\sin(\Omega_{\mathrm{n}}(\Delta t)) \\ -\Omega_{\mathrm{n}}\sin(\Omega_{\mathrm{n}}(\Delta t)) & \cos(\Omega_{\mathrm{n}}(\Delta t)) \end{bmatrix}_{2n\times 2n} \tag{7}$$

The input matrix of the discrete-time system has one column and (with a zero-order hold on inputs sampled at intervals (Δt)) can be derived as

$$\begin{aligned} b &= \begin{bmatrix} 0 & I_n \\ -\Omega_{\mathrm{n}}^2 & 0_n \end{bmatrix}^{-1} (A - I) \begin{bmatrix} 0 \\ h \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\Omega_{\mathrm{n}}^{-2} \\ I_n & 0_n \end{bmatrix} \begin{bmatrix} \cos(\Omega_{\mathrm{n}}(\Delta t)) - I_n & \Omega_{\mathrm{n}}^{-1}\sin(\Omega_{\mathrm{n}}(\Delta t)) \\ -\Omega_{\mathrm{n}}\sin(\Omega_{\mathrm{n}}(\Delta t)) & \cos(\Omega_{\mathrm{n}}(\Delta t)) - I_n \end{bmatrix} \begin{bmatrix} 0 \\ h \end{bmatrix} \\ &= \begin{bmatrix} -\Omega_n^{-2}(\cos(\Omega_{\mathrm{n}}(\Delta t)) - I_n)h \\ -\Omega_{\mathrm{n}}^{-1}\sin(\Omega_{\mathrm{n}}(\Delta t))h \end{bmatrix} \end{aligned} \tag{8}$$

Since the matrices multiplying h in the top-half and bottom-half of b are diagonal, $b_{(4,8,12,16,...)} = 0$ and $b_{n+(4,8,12,16,...)} = 0$. Diagonalizing this state-space system,

$$q(k+1) = \Lambda q(k) + T^{-1} b u(k) = \Lambda q(k) + \bar{b} u(k)$$

where the columns of T are the eigenvectors of A, and Λ is diagonal with each element lying on the unit circle. Since $T^{-1} e^{A} T = e^{T^{-1}AT}$, the continuous-time dynamics matrix (6) and the discrete-time dynamics matrix (7) have the same eigenvectors, T. Since the continuous-time dynamics matrix is easier to work with, the eigenvectors are more easily derived from it.

$$\left[\begin{array}{cc} \mathrm{i}\omega_{\mathrm{n}k} I_n & I_n \\ -\Omega_{\mathrm{n}}^2 & \mathrm{i}\omega_{\mathrm{n}k} I_n \end{array}\right] \left[\begin{array}{c} t_k^{(1)} \\ t_k^{(2)} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \Leftrightarrow \begin{array}{rcl} -\mathrm{i}\omega_{\mathrm{n}k} t_k^{(1)} & = & t_k^{(2)} \\ \Omega_{\mathrm{n}}^2 t_k^{(1)} & = & \mathrm{i}\omega_{\mathrm{n}k} t_k^{(2)} = \omega_{\mathrm{n}k}^{\;2} t_k^{(1)} \end{array} \tag{9}$$

So

$$T = \left[\begin{array}{cccc} +v_1 & -v_1 & & \\ & \ddots & \ddots & \\ & & +v_n & -v_n \\ \mathrm{i}\omega_{\mathrm{n}1} v_1 & \mathrm{i}\omega_{\mathrm{n}1} v_1 & & \\ & \ddots & \ddots & \\ & & \mathrm{i}\omega_{\mathrm{n}n} v_n & \mathrm{i}\omega_{\mathrm{n}n} v_n \end{array}\right]$$

and

$$T^{-1} = \left[\begin{array}{cccccc} +1/(2v_1) & & & 1/(2\mathrm{i}\omega_{\mathrm{n}1} v_1) & & \\ -1/(2v_1) & & & 1/(2\mathrm{i}\omega_{\mathrm{n}1} v_1) & & \\ & +1/(2v_2) & & & 1/(2\mathrm{i}\omega_{\mathrm{n}2} v_2) & \\ & -1/(2v_2) & \ddots & & 1/(2\mathrm{i}\omega_{\mathrm{n}2} v_2) & \ddots \\ & \ddots & +1/(2v_n) & & \ddots & 1/(2\mathrm{i}\omega_{\mathrm{n}n} v_n) \\ & & -1/(2v_n) & & & 1/(2\mathrm{i}\omega_{\mathrm{n}n} v_n) \end{array}\right]$$

So, if $b_{(4,8,12,16,...)} = 0$ and $b_{n+(4,8,12,16,...)} = 0$, then rows (7,8), (15,16), (23,24), (31,32), ... of $T^{-1}b$ are also zero; and so are these rows of $\Lambda^k \bar{b}$; and so are these rows of $\mathcal{Q}_R$, and so these states of this system (in complex-conjugate pairs) are in an un-controllable subspace of the state-space; and the system is not (fully) controllable.

4 Example: string vibration observability

Continuing with the same example, suppose the response of the string is measured at location $x = \xi_y$, and that $\xi_y = L/3$, which is a node of $\phi_{(3,6,9,12,...)}(x)$, that is, $\phi_{(3,6,9,12,...)}(L/3) = 0$. The output equation in modal coordinates is

$$y(t) = \sum_{j=1}^{n} \phi_j(\xi_y) p_j(t)$$

And in the state-space $p, \dot{p}$,

$$y(t) = \left[\begin{array}{cccccc} \phi_1(\xi_y) & \cdots & \phi_n(\xi_y) & 0 & \cdots & 0 \end{array} \right] \left[\begin{array}{c} p_1(t) \\ \vdots \\ p_n(t) \\ \dot{p}_1(t) \\ \vdots \\ \dot{p}_n(t) \end{array} \right]$$

or, in discrete-time matrix form, $y(k) = cx(k)$. Note that columns (3, 6, 9, 12, ...) of c are zero. In diagonalized discrete-time state-space,

$$y(k) = \left[\begin{array}{cccccc} \phi_1(\xi_y) & \cdots & \phi_n(\xi_y) & 0 & \cdots & 0 \end{array} \right] T q(k) = \bar{c} q(k)$$

where the structure of the modal matrix T is provided in the previous section. So, if columns (3, 6, 9, 12, ...) of c are zero, then columns (5,6, 11,12, 17,18, 23,24, ...) of $\bar{c}$ are zero; and so are these columns of $\bar{c}\Lambda^k$; and so are these columns of $\mathcal{P}_M$, and so these states of this system (in complex-conjugate pairs) are in an un-observable subspace of the state-space, and the system is not (fully) observable.

Note that if modes (4,8,12,...) are in the uncontrollable subspace and if modes (3,6,9,...) are in the unobservable subspace, then modes (12, 24, 36, ...) are in a subspace that is both uncontrollable and unobservable.

5 Bases for the Kalman decomposition

The previous example illustrates how the state-space can be categorized into subspaces that are (controllable and observable , co), (controllable and unobservable , $c\bar{o}$), (uncontrollable and observable , $\bar{c}o$), (uncontrollable and unobservable , $\bar{c}\bar{o}$).

The range-space of $\mathcal{Q}$ is a basis for the controllable subspace, $\hat{B}_c = \mathcal{R}(\mathcal{Q})$. Orthonormal bases for the controllable (c) and uncontrollable ($\bar{c}$) subspaces can be found from the left singular vectors of the controllability matrix $\mathcal{Q}$.

$$x = \mathcal{Q}u = [B \ , \ AB \ , \ A^2B \ , ...]u = [\hat{B}_c \ , \ \hat{B}_{\bar{c}}] \begin{bmatrix} \sigma_{c,1} & & & \\ & \ddots & & \\ & & \sigma_{c,n_c} & \\ & & & 0 \end{bmatrix} V^{\mathsf{T}} u \tag{10}$$

The null-space of $\mathcal{P}$ is a basis for the unobservable subspace, $\hat{B}_{\bar{o}} = \mathcal{N}(\mathcal{P})$. Orthonormal bases for the observable (o) and unobservable ($\bar{o}$) subspaces can be found from the right singular vectors of the the observability matrix $\mathcal{P}$.

$$y = \mathcal{P}x = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} x = U \begin{bmatrix} \sigma_{o,1} & & & \\ & \ddots & & \\ & & \sigma_{o,n_o} & \\ & & & 0 \end{bmatrix} \begin{bmatrix} \hat{B}_o^{\mathsf{T}} \\ \hat{B}_{\bar{o}}^{\mathsf{T}} \end{bmatrix} x \tag{11}$$

where the dimension of the controllable space is n_c and the dimension of the observable space is n_o. The Kalman decomposition determines bases for the subspaces (co), ($c\bar{o}$), ($\bar{c}o$), and ($\bar{c}\bar{o}$), having n_{co}, $n_{c\bar{o}}$, $n_{\bar{c}o}$, and $n_{\bar{c}\bar{o}}$ columns. ($n_{co} + n_{c\bar{o}} + n_{\bar{c}o} + n_{\bar{c}\bar{o}} = n$)

To put a state-space realization into Kalman decomposition form, we need a transformation matrix of the form

$$T = \left[\hat{B}_{co} \ , \ \hat{B}_{c\bar{o}} \ , \ \hat{B}_{\bar{c}o} \ , \ \hat{B}_{\bar{c}\bar{o}}\right] , \tag{12}$$

where $\hat{B}_{co}$ is a $n \times n_{co}$ matrix whose columns form an orthonormal basis for the controllable and observable subspaces, $\hat{B}_{c\bar{o}}$ is a $n \times n_{c\bar{o}}$ matrix whose columns form an orthonormal basis for the controllable and unobservable spaces, etc. Thus, to transform our system into Kalman decomposition form, we need to find the bases for the four different spaces.

We begin by finding $\hat{B}_{co}$ from $\hat{B}_c$, $\hat{B}_o$ and $\hat{B}_{\bar{o}}$, which can be found from $\mathcal{Q}$ and $\mathcal{P}$ by Eq's. (10) and (11). Consider some vector x_{co} that is fully within the controllable and unobservable subspaces. because x_{co} is within both the controllable and observable subspaces, it can be expressed as a linear combination of the basis vectors of x_c and x_o:

$$\begin{aligned} x_{co} &= a_1\hat{B}_{c,1} + \ldots + a_{n_c}\hat{B}_{c,n_c} &= \hat{B}_c a \\ &= b_1\hat{B}_{o,1} + \ldots + b_{n_o}\hat{B}_{o,n_o} &= \hat{B}_o b. \end{aligned} \tag{13}$$

where

$\hat{B}_{c,i}$: ith column of $\hat{B}_c$,
$\hat{B}_{o,i}$: ith column of $\hat{B}_o$,
a : $n_c \times 1$ vector of coefficients,
b : $n_o \times 1$ vector of coefficients.

Note that a and b are *not* arbitrary; they must be selected such that x_{co} is within both the controllable space and the observable space. Additionally, a and b come in unique pairs and form their own subspaces, both of dimension n_{co}. If we consider the subspace of a to be spanned by the vectors $\{a_1, \ldots, a_{n_{co}}\}$ and the subspace of b to be spanned by $\{b_1, \ldots, b_{n_{co}}\}$, then we can write the basis of x_{co} as

$$\hat{B}_{co} = \left[\hat{B}_c a_1 \ \cdots \ \hat{B}_c a_{n_{co}}\right] = \hat{B}_c \hat{B}_a \tag{14}$$

or

$$\hat{B}_{co} = \left[\hat{B}_o b_1 \ \cdots \ \hat{B}_o b_{n_{co}}\right] = \hat{B}_o \hat{B}_b, \tag{15}$$

where $\hat{B}_a$ is a $n_c \times n_{co}$ matrix of the basis vectors for the subspace formed by the possible a vectors, and $\hat{B}_b$ is a $n_o \times n_{co}$ matrix of the basis vectors for the subspace formed by the possible b vectors. (Note that this yields correct dimensions for $\hat{B}_{co}$.)

Knowing that the bases $\hat{B}_o$ and $\hat{B}_{\bar{o}}$ are orthogonal complements,

$$\hat{B}_{\bar{o}}^{\mathsf{T}} \hat{B}_o = 0 \ , \tag{16}$$

and front-multiplying

$$\hat{B}_{co} = \hat{B}_c \hat{B}_a = \hat{B}_o \hat{B}_b \tag{17}$$

by $\hat{B}_{\bar{o}}^{\mathsf{T}}$, we obtain

$$\hat{B}_{\bar{o}}^{\mathsf{T}} \hat{B}_c \hat{B}_a = 0 \tag{18}$$

In other words, the basis for the coefficients a is in the null space of $\hat{B}_{\bar{o}}^{\mathsf{T}} \hat{B}_c$.

$$\hat{B}_a = \mathcal{N}([\hat{B}_{\bar{o}}^{\mathsf{T}} \hat{B}_c]). \tag{19}$$

and a basis $\hat{B}_{co}$ is

$$\hat{B}_{co} \ = \ \hat{B}_c \cdot \mathcal{N}([\hat{B}_{\bar{o}}^{\mathsf{T}} \hat{B}_c]). \tag{20}$$

In a completely analogous way,

$$\begin{aligned} \hat{B}_{c\bar{o}} &= \hat{B}_c \cdot \mathcal{N}([\hat{B}_o^{\mathsf{T}} \hat{B}_c]) && (21) \\ \hat{B}_{\bar{c}o} &= \hat{B}_{\bar{c}} \cdot \mathcal{N}([\hat{B}_{\bar{o}}^{\mathsf{T}} \hat{B}_{\bar{c}}]) && (22) \\ \hat{B}_{\bar{c}\bar{o}} &= \hat{B}_{\bar{c}} \cdot \mathcal{N}([\hat{B}_o^{\mathsf{T}} \hat{B}_{\bar{c}}]) && (23) \end{aligned}$$

The coordinate transformation

$$x = \left[\hat{B}_{co} \ , \ \hat{B}_{c\bar{o}} \ , \ \hat{B}_{\bar{c}o} \ , \ \hat{B}_{\bar{c}\bar{o}}\right] \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} \tag{24}$$

transforms the subspaces (co), $(c\bar{o})$, $(\bar{c}o)$, and $(\bar{c}\bar{o})$ into the state x. The resulting state-space realization is:

$$\frac{d}{dt}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u \tag{25}$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} D \end{bmatrix} u \tag{26}$$