A brief motivation to study Subspace System Identification

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What is System Identification?

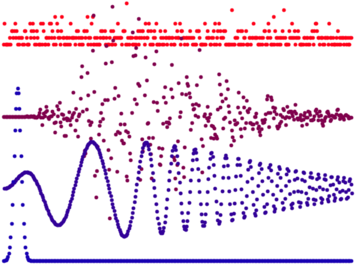
What is System Identification?

- the practice of **extracting models from measurements**
- a sub-set of parameter estimation specific to **dynamical systems**
- **modern** methods (1990's to today) (e.g., ERA and SSID)
  - **linear systems theory**
  - **geometric** perpsective of projections
    ... as opposed to least sqaures
  - leverages bases of linear vector spaces computed from **matrix decompositions** e.g., QR (LQ) and SVD
  - a **projection** of matrices of data onto space of models
  - characterized by **direct** matrix operations
    ... *no iterative guess-and-check*
  - multi-input, multi-output (**MIMO**) systems
  - the **order** of the system is contained in the data
    ... **high order** models (a dozen or more states)

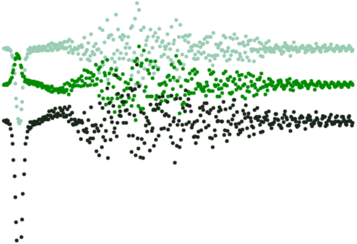
Find a dynamic model from within the measurements

Measurements

sequence of inputs $u(k)$



sequence of outputs $y(k)$



Models

- MIMO FIR filters

$$y(k) = \sum_i H(i) u(k-i)$$

- state-space models

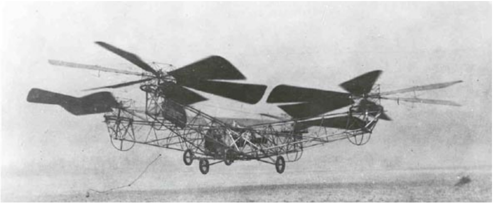
$$x(k+1) = A\ x(k) + B\ u(k) + w(k)$$
$$y(k) = C\ x(k) + D\ u(k) + v(k)$$

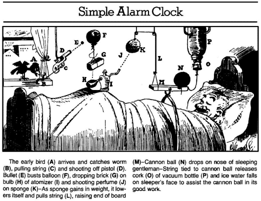
- transfer functions

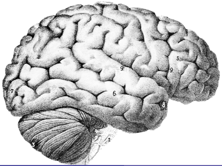
$$\bar{y}(z) = \bar{H}(z)\ \bar{u}(z)$$

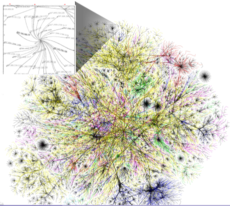
- nonlinearities . . . restricted to . . .
  $u(k) = f(\hat{u}(k))$
  $\hat{y}(k) = g(y(k))$
  . . . $x(k+1)$ is linear in $x(k)$

dynamical systems









Geometric interpretation of ordinary least squares — fit a two-term polynomial to three data points

- model equation $y_{model} = a_1 x + a_2 x^2$,
- using values of the independent variable (error free, by assumption) $x^T = [0.6, -0.7, 0.9]$ ... (and $(x^2)^T = [0.36, 0.49, 0.81]$)
- and using values of the dependent variable (measured, with some error) $y_{data}^T = [0.2, 0.3, 1.2]$
- model basis ... $y_{model} = [x, x^2]a = Xa$,
  where the columns of X form a basis of the model.
- what are the OLS estimates for model parameters, a_1, a_2?
- ordinary least squares parameter estimates ... $\hat{a} = [X^T X]^{-1}(X^T y_{data})$
- ordinary least squares model ...
  $y_{model} = X\hat{a} = (X[X^T X]^{-1} X^T) y_{data} = P_X^{\perp} y_{data}$
  where $P_X^{\perp}$ is the orthogonal projection matrix onto the basis spanned by the columns of X. (It's idempotent.)
  - **The orthogonal projection of the data onto the space of models minimizes the Euclidean length of $y_{data} - y_{model}$.**
  - **The shortest distance from a point to a plane.**

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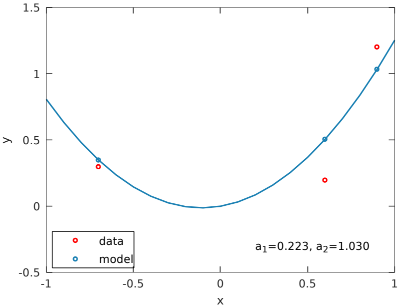
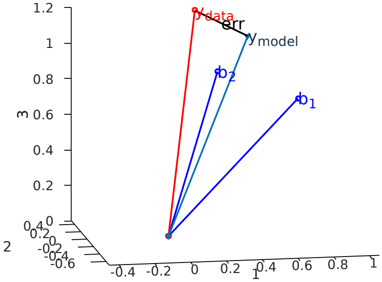
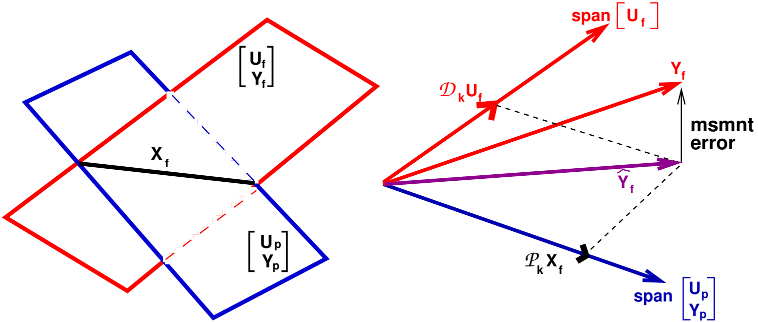




projections and the identification of LTI system dynamics

- The outputs y are linear in the states x and the inputs u. $y = Cx + Du$
- But the states are not measured!
- This is the fundamental problem in LTI system ID.
- How do we compute a state sequence corresponding to measured u and y?

oblique projection of data onto space of models



Requirements

- persistent measured input . . .
  or an (assumed) persistent (unmeasured) noise disturbance
- the system responds persistently to persistent inputs
- the outputs do not feed back to inputs
  (there's a way around this)

What could possibly go wrong?

…these problems are cousins …

- excessive measurement noise and/or process noise
- nearly linearly-dependent bases (ill-conditioning)
- over-parameterization
- non-linearity …Hammerstein systems

What to do about it.

. . . we will start the term examining some of these methods . . .

- pre-conditioning data (filtering, detrending) . . . but be careful!
- ℓ_2 regularization . . . bias and covariance
- ℓ_1 regularization (as a QP)
- rank reduction and total least squares
- *structured* rank reduction . . . a lot of opportunity here, I think.
- . . . nonlinearity . . . ??

What good is an identified LTI system ... in an arbitrary basis?

- What is x? **... It's a good question!**
- If you don't know what the identified state sequence x physically corresponds to, can you do *anything* with the model? ... e.g., control?
- Maybe prediction is enough.
- Maybe you know the mechanisms at play within the system ... physics-based modeling.
- estimate (a few) physical parameters, e.g., L, C, R, K, M, C, from the identified LTI system dynamics, e.g., $p_j = -\zeta_j \omega_{nj} + i\omega_{dj}$
- usually a *non-linear* least squares problem
  Levenberg-Marquardt, etc.
  but with *vastly reduced* spaces of parameters and data!

System ID seems so essential! Why did it take until now to figure out how to do it right?

Lineage

- Jacopo Riccati (1676 - 1754)
- Alexandre-Theophile Vandermonde (1735 - 1796)
- Carl Friedrich Gauss (1777 - 1855)
- Camille Jordan (1838 - 1922)
- Hermann Hankel (1839 - 1873)
- Ferdinand Georg Frobenius (1849 - 1917)
- Andrey Markov (1856 - 1922)
- Aleksandr Lyapunov (1857 - 1918)
- Issai Schur (1875 - 1941)
- Otto Toeplitz (1881 - 1940)
- Norbert Wiener (1894 - 1964)
- Alston Scott Householder (1904 - 1993)
- Rudolf E. Kálmán (1930 - 2016)
- Gene H. Golub (1932 - 2007)

Subspace System Identification

Deterministic Subspace System Identification

discrete-time LTI in state-space

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + w(k)\,, \quad x \in \mathbb{R}^n\,, \quad u \in \mathbb{R}^r \\ y(k) &= C\,x(k) + D\,u(k) + v(k)\,, \quad y \in \mathbb{R}^m \end{aligned}$$

disturbance - noise covariance

$$E\left[\begin{pmatrix} w(p) \\ v(p) \end{pmatrix}^T \begin{pmatrix} w(q)^T & v(q)^T \end{pmatrix}\right] = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta_{pq}$$

apply to sequences

$$\begin{bmatrix} x(i+1)\cdots x(i+j) \\ y(i) \quad \cdots y(i+j-1) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(i)\cdots x(i+j-1) \\ u(i)\cdots u(i+j-1) \end{bmatrix} + \begin{bmatrix} w(i)\cdots w(i+j-1) \\ v(i)\cdots v(i+j-1) \end{bmatrix}$$

$$\begin{bmatrix} X_{i+1,j} \\ Y_{i,j} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_{i,j} \\ U_{i,j} \end{bmatrix} + \begin{bmatrix} W_{i,j} \\ V_{i,j} \end{bmatrix}$$

So, a realization A, B, C, D may be obtained from input / output sequences with a direct matrix decomposition,
if a state vector sequence $[x(i) \ \cdots \ x(i+j)]$ is known.

Subspace ID : notation and data matrices

Substitute state equation into output equation, recursively, to obtain

$$\begin{bmatrix} y(i) & \cdots & y(i+j-1) \\ y(i+1) & \cdots & y(i+j) \\ \vdots & & \vdots \\ y(i+k-1) & \cdots & y(i+j+k-2) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} \begin{bmatrix} x(i) & \cdots & x(i+j-1) \end{bmatrix}$$

$$+ \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ CA^{k-2} & CA^{k-3}B & CA^{k-4}B & \cdots & D \end{bmatrix} \cdot \begin{bmatrix} u(i) & \cdots & u(i+j-1) \\ \vdots & & \vdots \\ u(i+k-1) & \cdots & u(i+j+k-2) \end{bmatrix} +$$

$$Y_{i,j,k} = \mathcal{P}_k X_{i,j} + \mathcal{D}_k U_{i,j,k} + V_{i,j,k}$$

$Y_p \equiv Y_{1,j,k}$	$U_p \equiv U_{1,j,k}$	$X_p \equiv X_{1,j}$	past I/O data and states
$Y_f \equiv Y_{k+1,j,k}$	$U_f \equiv U_{k+1,j,k}$	$X_f \equiv X_{k+1,j}$	future I/O data and states

$$\begin{aligned} Y_p &= \mathcal{P}_k X_p + \mathcal{D}_k U_p + V_p \\ Y_f &= \mathcal{P}_k X_f + \mathcal{D}_k U_f + V_f \end{aligned}$$

Subspace ID : three necessary assumptions

1. The input is persistent — $U_{i,j,k}$ has full row rank
2. The states respond persistently — $X_{i,j}$ has full row rank
3. There is no feedback $u \underset{\not\Leftarrow}{\Longrightarrow} y$ (there are ways around this) — $span(X_{i,j}) \cap span(U_{i,j,k}) \in \{\emptyset\}$

So,

$$rank \begin{bmatrix} U_p \\ Y_p \end{bmatrix} = kr + n$$

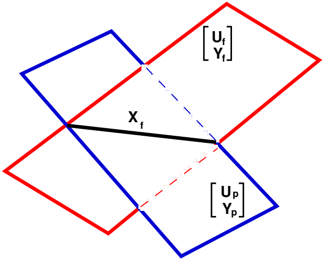
and any input / output sequence

$$[\tilde{u}(i) \ \cdots \ \tilde{u}(i+j-1)] \ , \ [\tilde{y}(i) \ \cdots \ \tilde{y}(i+j-1)]$$

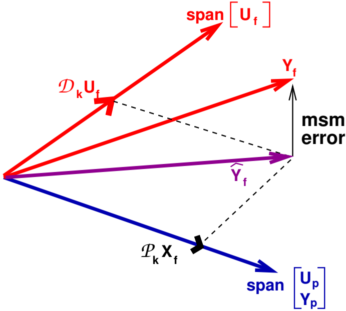
is in the basis of

$$\begin{bmatrix} U_p \\ Y_p \end{bmatrix}$$

Subspace ID : main lemma



$$span\,[X_f] = span \begin{bmatrix} U_p \\ Y_p \end{bmatrix} \cap span \begin{bmatrix} U_f \\ Y_f \end{bmatrix}$$



X_f lies in the span of $\begin{bmatrix} U_p \\ Y_p \end{bmatrix}$

$\hat{Y}_f = \mathcal{P}_k X_f + \mathcal{D}_k U_f$

$\mathcal{P}_k X_f$ lies in span $\begin{bmatrix} U_p \\ Y_p \end{bmatrix}$, and span $\begin{bmatrix} U_f \\ Y_f \end{bmatrix}$,

$\mathcal{P}_k X_f$ is the *oblique projectction* of Y_f into $\begin{bmatrix} U_p \\ Y_p \end{bmatrix}$ parallel to U_f.

Subspace ID : computation

Compute the oblique projectcion via LQ decomposition

$$\begin{bmatrix} U_f \\ \begin{bmatrix} U_p \\ Y_p \end{bmatrix} \\ Y_f \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$

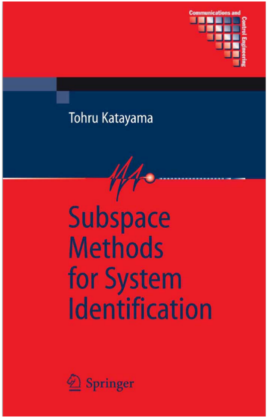
$$\mathcal{P}_k X_f = L_{32} \, L_{22}^{-1} \begin{bmatrix} U_p \\ Y_p \end{bmatrix}$$

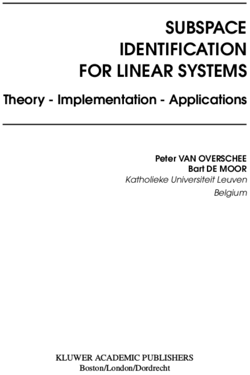
Compute the state sequence via SVD

$$\begin{aligned} \mathcal{P}_k X_f &= U \Sigma V^T = U \Sigma^{1/2} T \cdot T^{-1} \Sigma^{1/2} V^T \\ \mathcal{P}_k &= U_n \Sigma_n^{1/2} T \\ X_f &= T^{-1} \Sigma_n^{1/2} V_n^T = \begin{bmatrix} x(k+1) & \cdots & x(k+j) \end{bmatrix} \end{aligned}$$

The state sequence is thus obtained, but in an arbitrary basis. . . . n4sid

primary reference texts for this course




SUBSPACE IDENTIFICATION FOR LINEAR SYSTEMS

Theory - Implementation - Applications

Peter VAN OVERSCHEE
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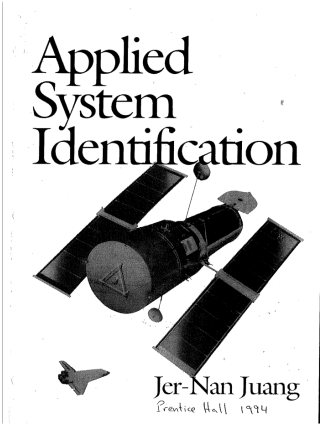


- Katayama, Tohru, *Subspace Methods for System Identification,* Springer, 2005.
- Van Overschee, Peter, and De Moore, Bart, *Subspace Identification for Linear Systems,* Kluwer, 1996.
- Juang, Jer-Nan, *Applied System Identification,* Prentice Hall, 1994.

Some of the other topics in this course

Total Least Squares

- solve the over-determined noisy system

$$y \approx Xa$$

- errors in y only . . . o.l.s.

$$y + \tilde{y} = Xa$$

- errors in y **and** X . . . t.l.s.

$$y + \tilde{y} = \left[X + \tilde{X}\right] a$$

- the t.l.s. problem

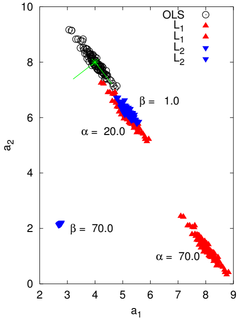
$$\min \left\| \left[\tilde{X} \ \tilde{y}\right] \right\|_F^2 \quad \text{such that} \quad y + \tilde{y} = \left[X + \tilde{X}\right] a$$

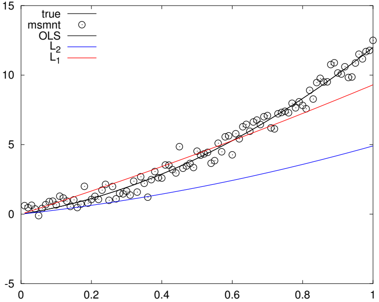
- Eckart-Young thm.

OLS, ℓ_1, and ℓ_2 regularization

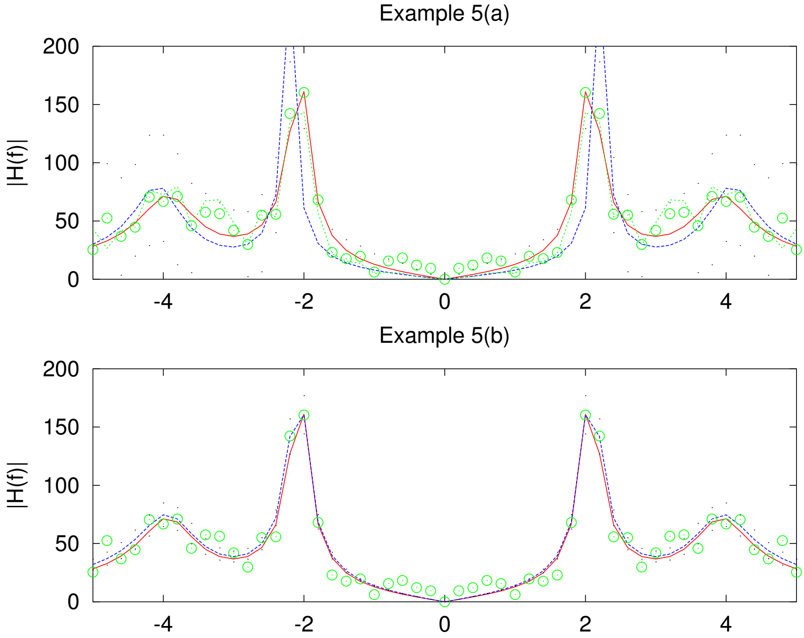
$$\hat{y}(x; a_1, a_2) = a_1 x + a_2 x^2 \qquad y = \hat{y} + \eta \quad \sigma_\eta = 0.5$$

OLS criterion	$\min_a \|\|y - Xa\|\|_2^2$
ℓ_1 regularization criterion	$\min_a \|\|y - Xa\|\|_2^2 + \alpha \|\|a\|\|_1$
ℓ_2 regularization criterion	$\min_a \|\|y - Xa\|\|_2^2 + \beta \|\|a\|\|_2$





Nonlinear Least Squares - Levenberg Marquardt



Singular Spectrum Analysis

- rank reduction of a Hankel matrix of a time series

$$Y = \begin{bmatrix} y_1 & y_2 & \cdots & \cdots & y_{j-1} & y_j \\ y_2 & y_3 & \cdots & \cdots & y_j & y_{j+1} \\ \vdots & \vdots & & & \vdots & \vdots \\ y_{k-1} & y_k & \cdots & \cdots & y_{k+j-3} & y_{k+j-2} \\ y_k & y_{k+1} & \cdots & \cdots & y_{k+j-2} & y_{k+j-1} \end{bmatrix}$$

- truncated SVD expansion

$$Y_n = U_n \Sigma_n V_n^T$$

- reconstruct a Hankel form ...
  average cross-diagonals (??) ...or *structured* low-rank approximation
- extract a reduced basis from a very noisy measurement

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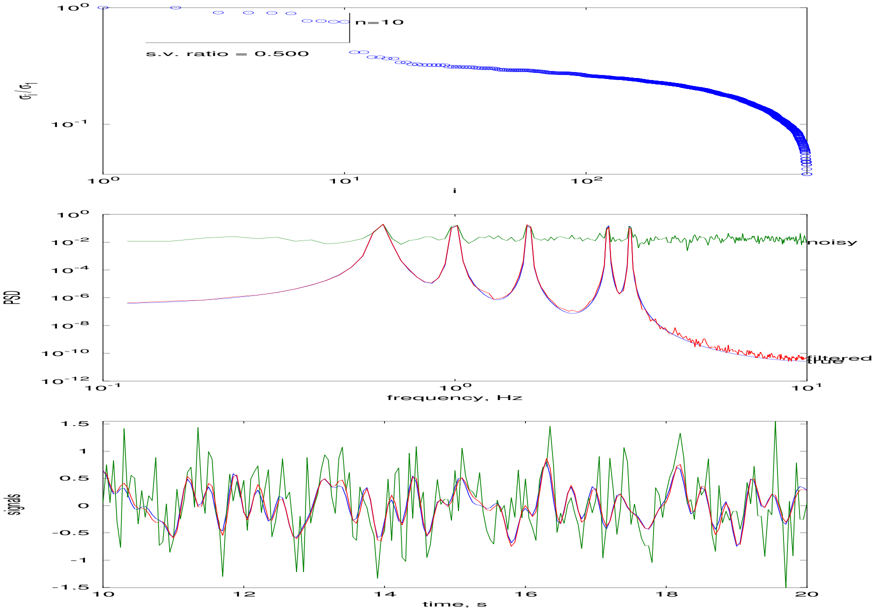


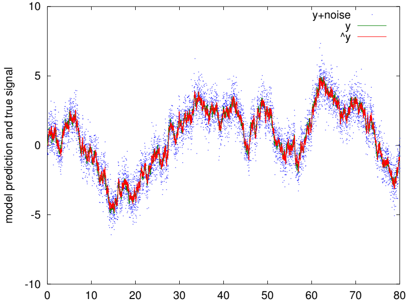
MIMO Wiener Filtering

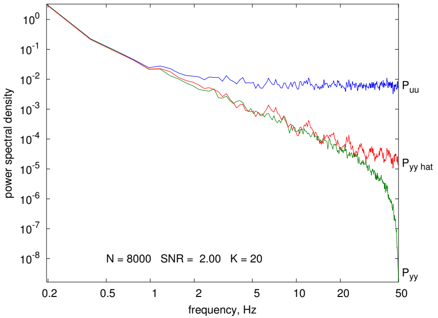
- Estimate the (matrix-valued) Wiener coefficients (Markov parameters) $H(i)$

$$y(k) = \sum_i H(i)\, u(k-i)$$

- filter a noisy measurement from broad-band input-output data
- recover a reduced basis from a very noisy measurement
- identify the impuse response of a MIMO system from noisy I/O data

Wiener filtering





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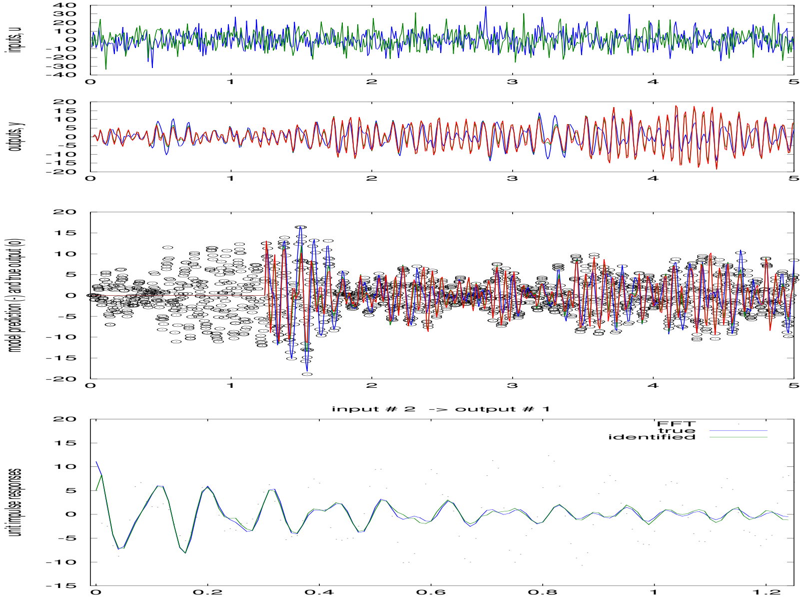

preliminaries needed for subspace identification

- linear vector spaces; decompositions; truncation; projections; ordinary least squares
- total least squares; SSA and Wiener filters
- nonlinear least squares; SQP; and ℓ_1 regularization
- linear time invariant systems, Kalman filter; controllability and observability; Lyapunov equations; balanced realizations and model reduction;

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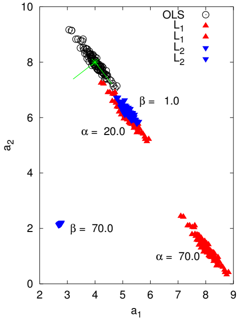

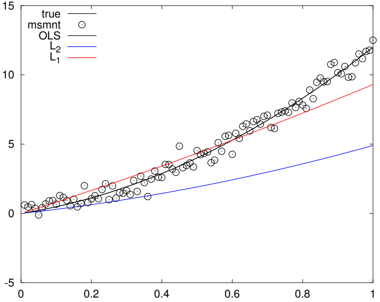

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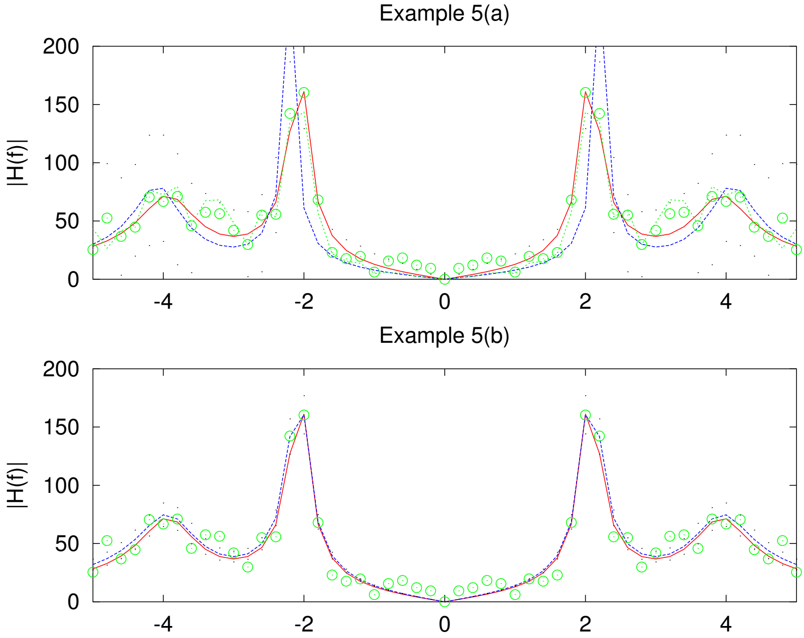

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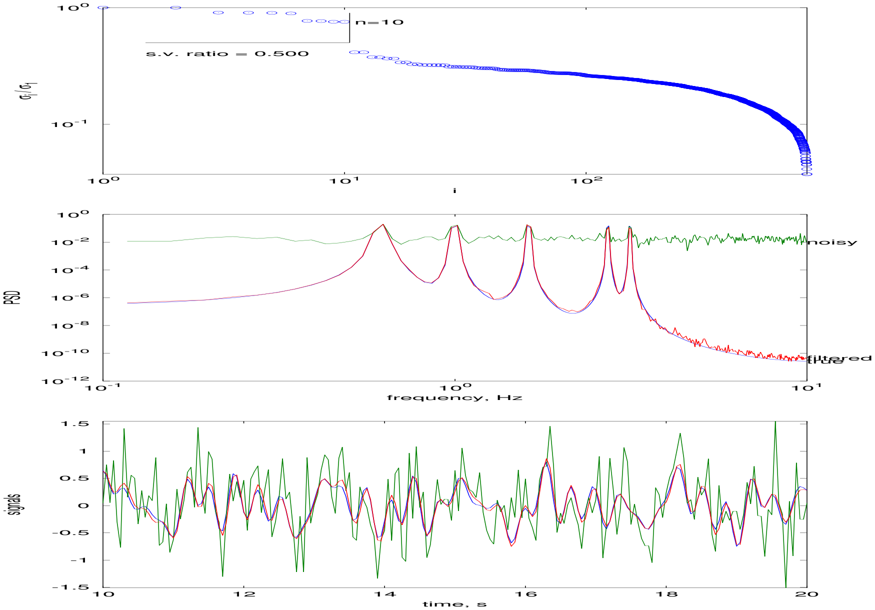

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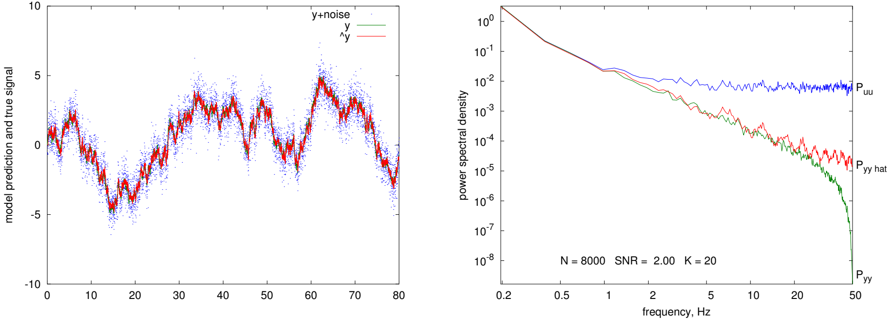

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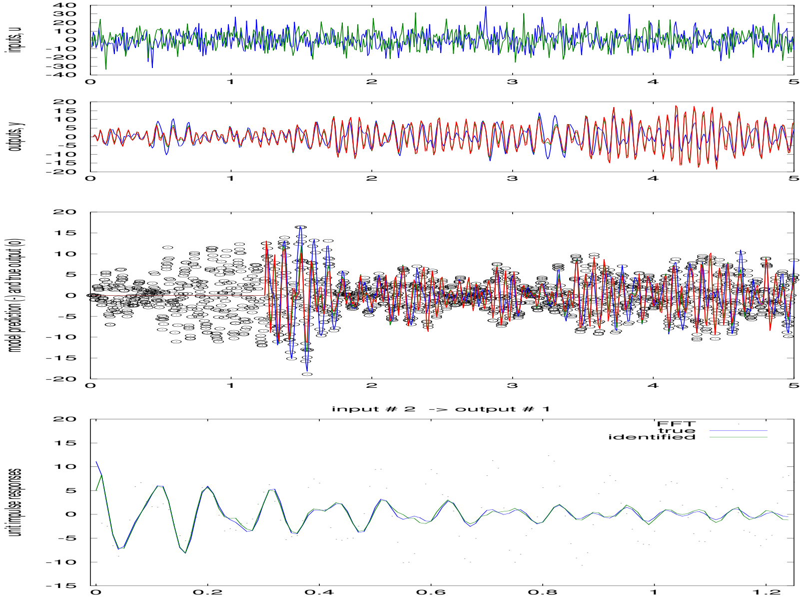

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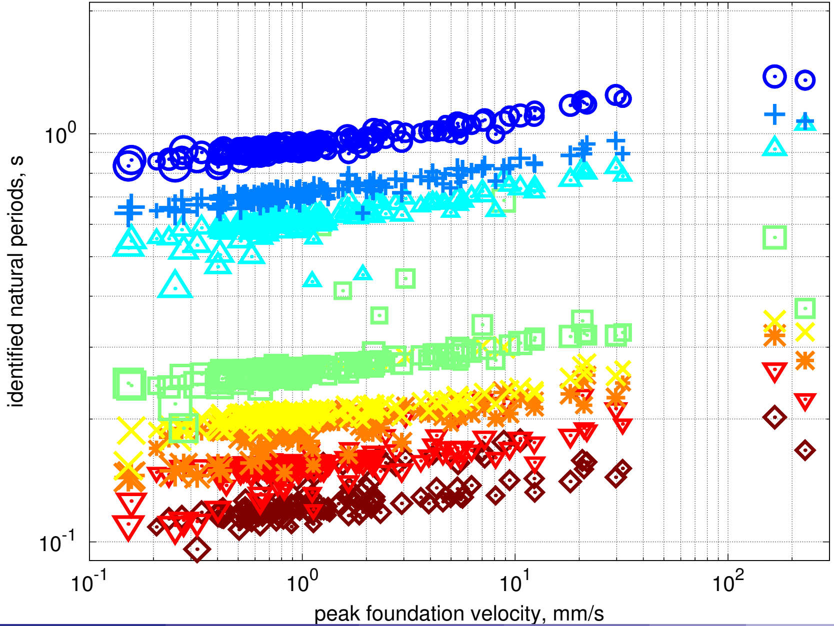
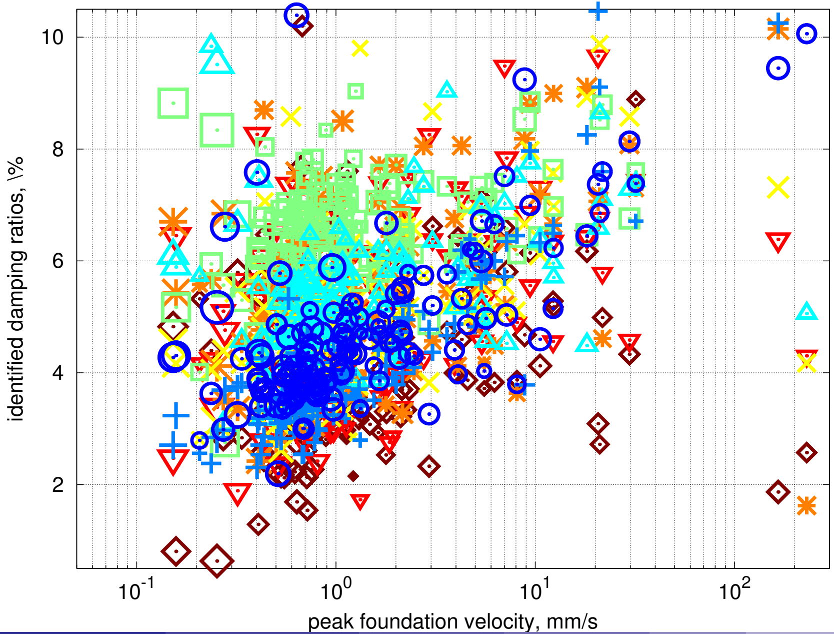
Modeling Field Data

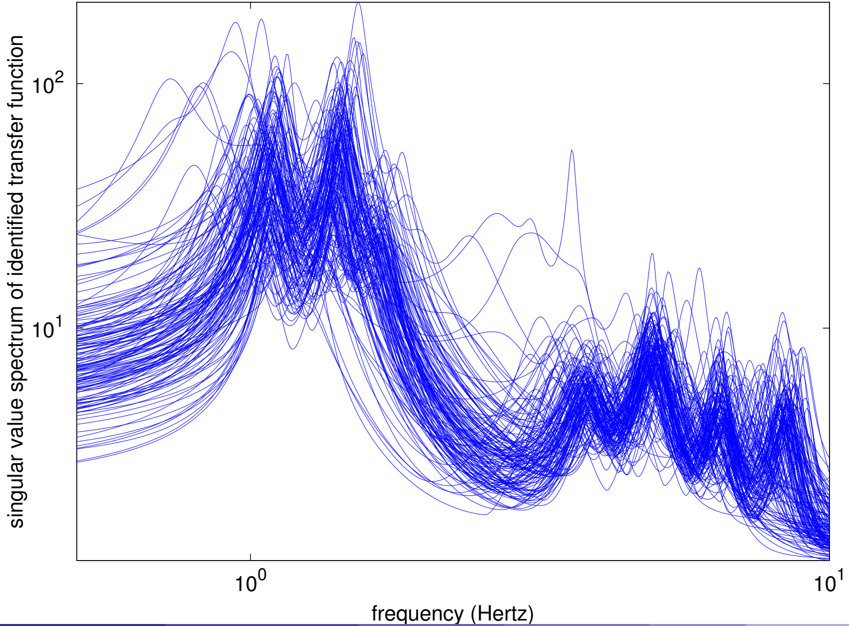
System Identification - Natural Periods



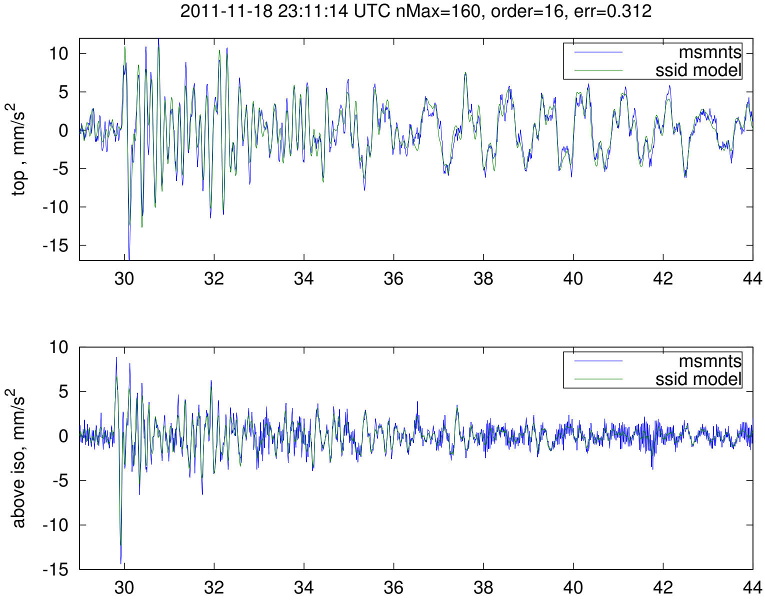
System Identification - Damping Ratios



System Identification - Frequency Responses

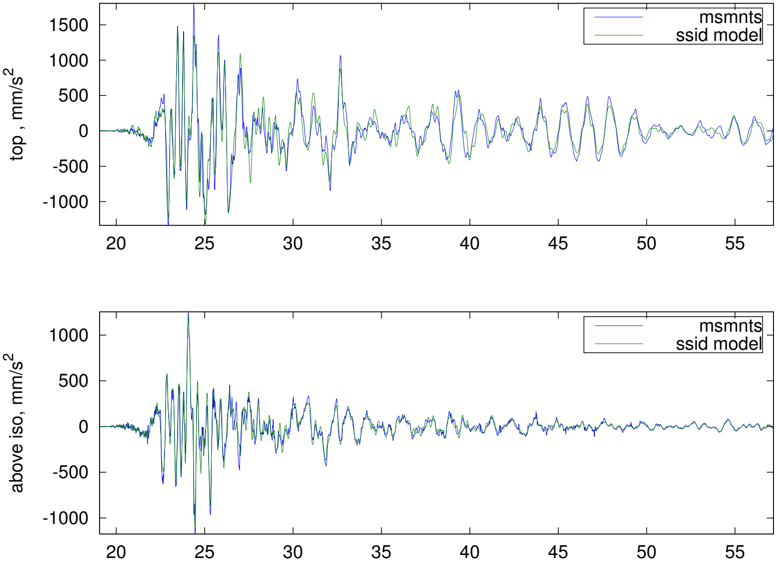


System Identification - modeling low-amplitude behavior



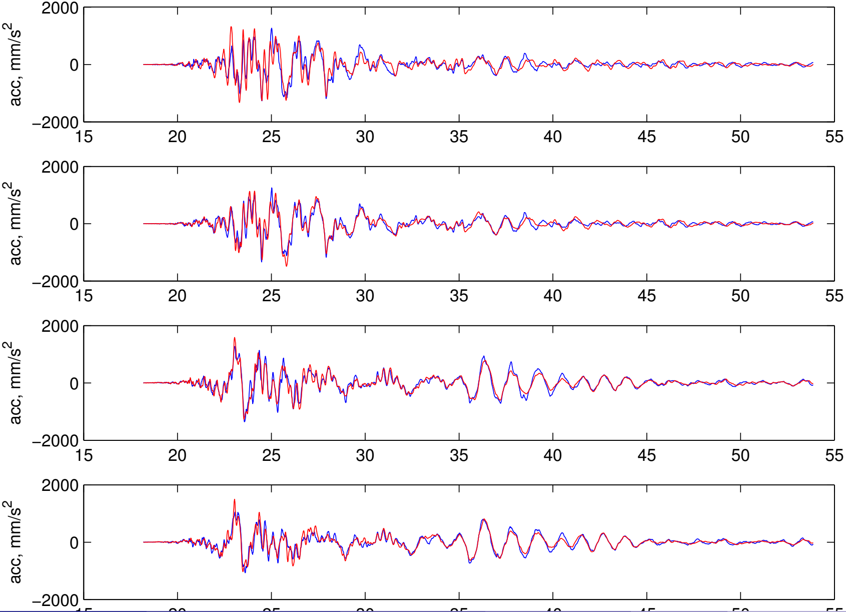
System Identification - modeling high-amplitude behavior

2011-12-23 02:17:47 - M 6.00 nMax=160, order=16, err=0.424



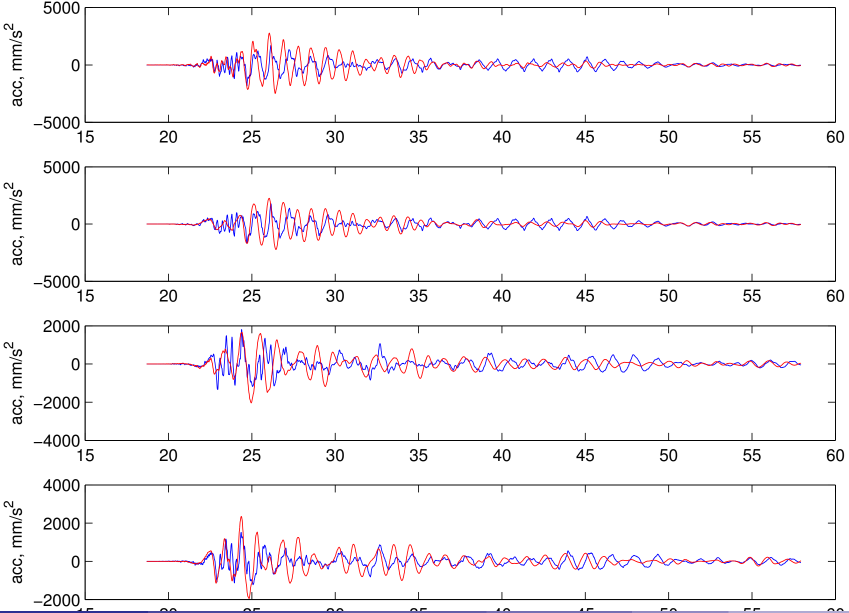
System Identification - Response Prediction

Record #114 Parameters Modeling Record #103 Output



System Identification - Response Prediction

Record #132 Parameters Modeling Record #114 Output



System Identification and Response Prediction