Balanced Model Reduction

CEE 629. System Identification

Duke University, Fall 2017

1 Balanced Realization

A balanced realization is a realization for which the controllability gramian Q and the observability gramian P both equal a diagonal matrix of the Hankel singular values, Σ. In other words, the state covariance matrix of pulse or white-noise responses equals the output covariance matrix of free output responses, and they are both diagonal matrices.

To transform a realization into a balanced form, we require the $n \times n$ coordinate transformation matrix T from balanced coordinates x_b to the given coordinates x ($x = Tx_b$) such that the observability and controllability gramians are diagonal and equal.

Substituting $TA_bT^{-1} = A$, $T^{-\mathsf{T}}A_b^{\mathsf{T}}T^{\mathsf{T}} = A^{\mathsf{T}}$, $C_bT^{-1} = C$, and $TB_b = B$ into Lyapunov equations,

$$0 = A^{\mathsf{T}}PA - P + C^{\mathsf{T}}C \quad \text{and} \quad 0 = AQA^{\mathsf{T}} - Q + BB^{\mathsf{T}} \ , \tag{1}$$

and relating the resulting Lyapunov equations to

$$0 = A_b^{\mathsf{T}}\Sigma A_b - \Sigma + C_b^{\mathsf{T}}C_b \quad \text{and} \quad 0 = A_b\Sigma A_b^{\mathsf{T}} - \Sigma + B_bB_b^{\mathsf{T}} \ , \tag{2}$$

we find that the condition for the coordinate transformation into balanced form is

$$T^{\mathsf{T}}PT = T^{-1}QT^{-\mathsf{T}} = \Sigma = \mathsf{diag}(\sigma_1, \cdots, \sigma_n) \ , \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0. \tag{3}$$

A sketch of the solution of equation (3) for T follows.

$$\begin{aligned}
(T^{\mathsf{T}}PT)(T^{-1}QT^{-\mathsf{T}}) &= \Sigma^2 \\
T^{\mathsf{T}}P(TT^{-1})QT^{-\mathsf{T}} &= \Sigma^2 \\
T^{\mathsf{T}}PQT^{-\mathsf{T}} &= \Sigma^2 \\
R^{\mathsf{T}}R &= Q \text{ ... the Cholesky decomp of } Q \\
T^{\mathsf{T}}(R^{-1}R)P(R^{\mathsf{T}}R)T^{-\mathsf{T}} &= \Sigma^2 \\
T^{\mathsf{T}}R^{-1}(RPR^{\mathsf{T}})RT^{-\mathsf{T}} &= \Sigma^2 \\
(RT^{-\mathsf{T}})^{-1}(RPR^{\mathsf{T}})(RT^{-\mathsf{T}}) &= \Sigma^2 \\
RPR^{\mathsf{T}} &= U\Sigma^2U^{\mathsf{T}} \text{ ... the eigenvalue decomp of } RPR^{\mathsf{T}} \\
(RT^{-\mathsf{T}})^{-1}(U\Sigma^2U^{\mathsf{T}})(RT^{-\mathsf{T}}) &= \Sigma^2 \\
V\Sigma^2V^{-1} &= \Sigma^2 \text{ ... define } V \\
V &= (RT^{-\mathsf{T}})^{-1}U \text{ ... define } V \\
V &= T^{\mathsf{T}}R^{-1}U \\
V^{-1} &= U^{\mathsf{T}}RT^{-\mathsf{T}} \\
VU^{\mathsf{T}} &= T^{\mathsf{T}}R^{-1} \text{ ... solve for } T \text{ in terms of } V \\
VU^{\mathsf{T}}R &= T^{\mathsf{T}} \\
R^{\mathsf{T}}UV &= T
\end{aligned} \tag{4}$$

To determine V,

$$\begin{aligned}
T^{-1}QT^{\mathsf{T}} &= \Sigma \\
(V^{-1}U^{\mathsf{T}}R^{-\mathsf{T}})\; Q\; (R^{-1}UV^{-1}) &= \Sigma \text{ ... substitue } T \text{ in terms of } V \\
V^{-1}U^{\mathsf{T}}R^{-\mathsf{T}}R^{\mathsf{T}}RR^{-1}UV^{-1} &= \Sigma \text{ ... substitute } R^{\mathsf{T}}R = Q \\
V^{-1}U^{\mathsf{T}}(R^{-\mathsf{T}}R^{\mathsf{T}})(RR^{-1})UV^{-1} &= \Sigma \\
V^{-1}(U^{\mathsf{T}}U)V^{-1} &= \Sigma \\
V^{-1}V^{-1} &= \Sigma \\
V^{-2} &= \Sigma \\
V &= \Sigma^{-1/2}
\end{aligned}$$

So,

$$T = R^{\mathsf{T}}U\Sigma^{-1/2} \tag{5}$$
$$T^{-1} = \Sigma^{1/2}U^{\mathsf{T}}R^{-\mathsf{T}} \tag{6}$$

To compute this coordinate transformation matrix,

1. Solve the Lyapunov equations for the controllability gramian Q and the observability gramian P.
2. Compute the Cholesky decomposition of Q, $R^{\mathsf{T}}R = Q$.
3. Compute the eigenvalue decomposition of $RPR^{\mathsf{T}} = U\Sigma U^{\mathsf{T}}$. This gives the Hankel singular values, Σ.
4. Evaluate equations (5) and (6)

In a balanced model the first state x_1 is the most observable and the most controllable. The last state x_n is the least so. The reduced balanced realization simply retains the first r states.

$$A_r = A_b(1:r, 1:r); \qquad B_r = B_b(1:r, :); \qquad C_r = C_b(:, 1:r).$$

2 Example system in complex modal coordinates

linear dynamics: $\dot{x} = Ax + Bu$

system output: $y = Cx$

parameter values:

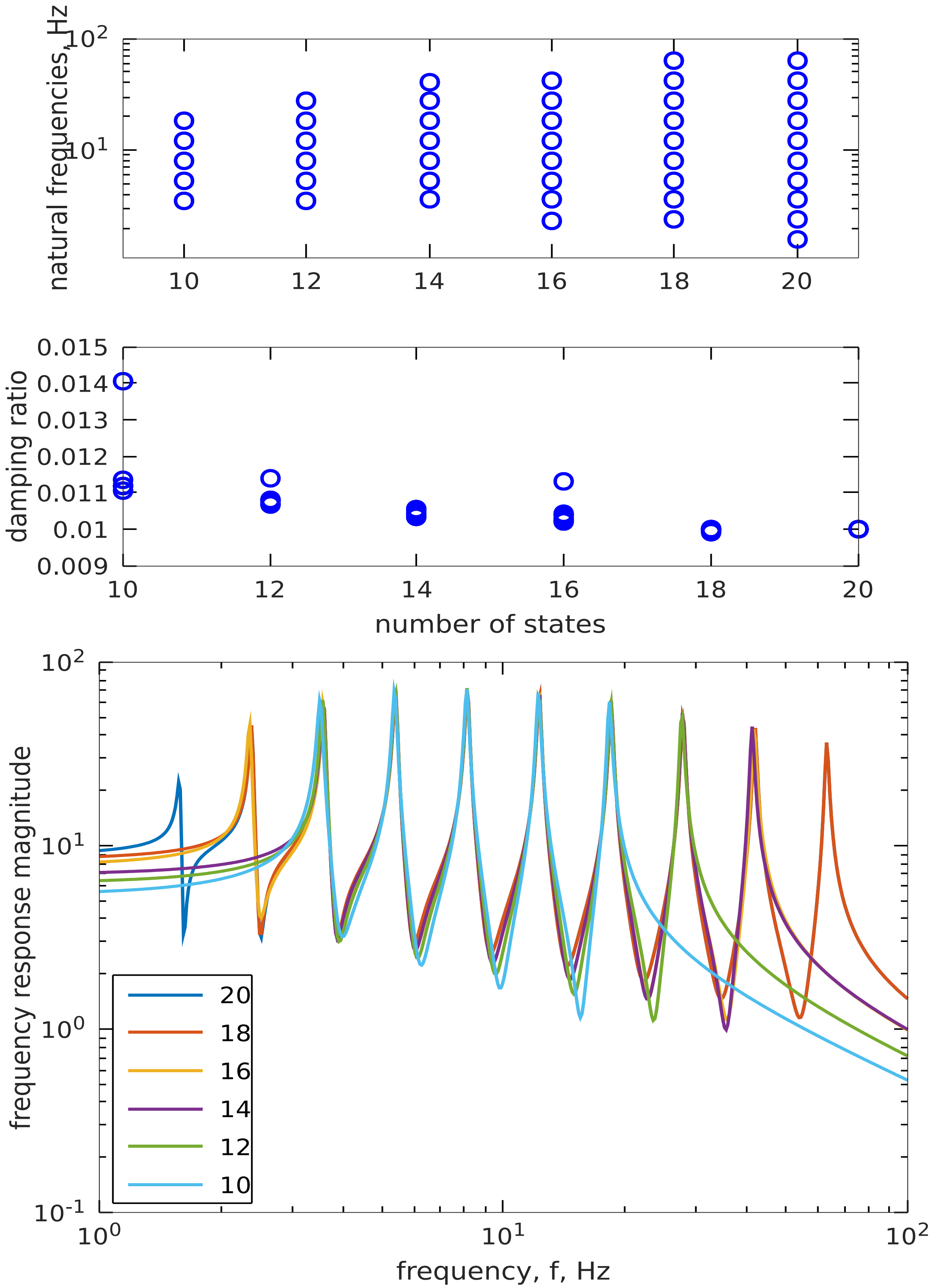
$$A = \mathsf{diag}\left(\begin{bmatrix} p_i & \\ & p_i^* \end{bmatrix}\right) \qquad A \in \mathbb{C}^{n\times n} \qquad p_i = \sigma_i + \sqrt{-1}\omega_i$$

$$B = \begin{bmatrix} & \vdots & \\ - & b_i & - \\ - & b_i^* & - \\ & \vdots & \end{bmatrix} \qquad B \in \mathbb{C}^{n\times m} \qquad b_i = i + i\sqrt{-1}$$

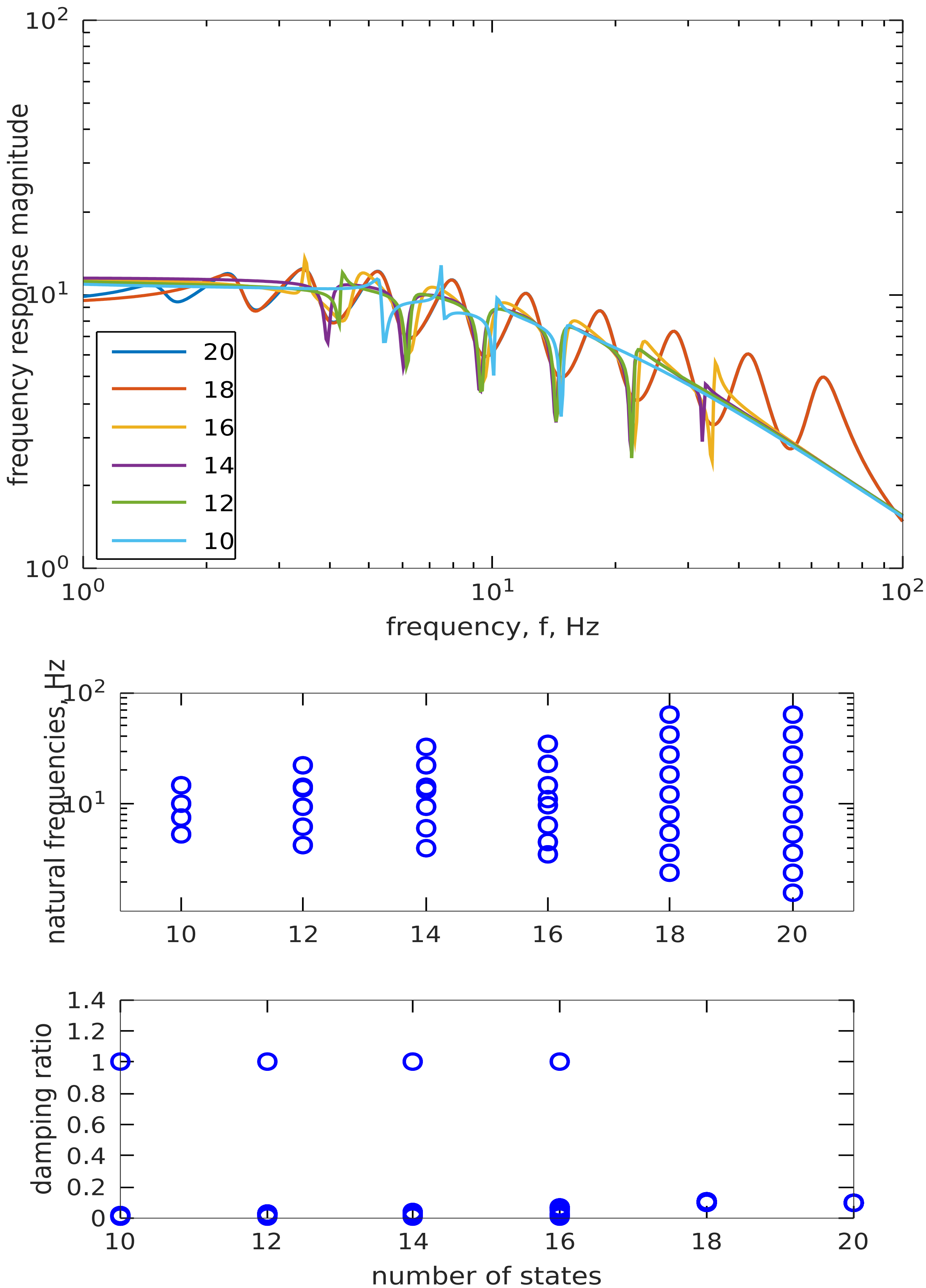
$$C = \begin{bmatrix} & | & | & \\ \cdots & c_j & c_j^* & \cdots \\ & | & | & \end{bmatrix} \qquad C \in \mathbb{C}^{l\times n} \qquad c_j = 1 + j\sqrt{-1}$$

$$D = 0 \qquad D \in \mathbb{C}^{l\times m}$$

2.1 Lightly Damped Dynamics $\sigma_i = \omega_i/100$



2.2 Heavily Damped Dynamics $\sigma_i = \omega_i/10$



bal_real.m

```
function [ Ab, Bb, Cb, G, T, Ti ] = bal_real( A, B, C )
% [ Ab, Bb, Cb, G, T, Ti ] = bal_real( A, B, C )
% balanced realization for a continuous time system
%
%   input      A,B,C       a continuous-time state-space realization
%
%  output      Ab,Bb,Cb    the balanced continuous-time state-space realization
%              G           the balanced observability and controllability gramians
%              T           coordinate transformation matrix
%              Ti          inverse of coordinate transformation matrix

  P = liap ( A , C );   % solve left  Liapunov eq'n    A'*P + P*A  + C'*C  = 0
  Q = liap ( A , B );   % solve right Liapunov eq'n    A *Q + Q*A' + B *B' = 0
  R = chol(Q);          % right Cholesky factor of Q ... R'*R = Q

  [U,G2] = eig(R*P*R'); % diagonalization of R*P*R'

% the balanced observability and controllability gramian
  G = sqrt(real(diag(G2)));

% the coordinate transformation matrix ...
  T   = R'*U * diag(1./sqrt(G));

% the inverse of the coordinate transformation matrix ...
  Ti = diag(sqrt(G)) * U' * inv(R');

% sorted grammian ...
  [G,idx] = sort(G,'descend');

% determine the state re-sorting matrix, S    X_sorted  =  S * X
  n = length(A);
  S = zeros(n);
  for i=1:n
      S(i,idx(i)) = 1;
  end
  Ti = S*Ti;     T   = T/S;       % reorganize the transformation matrices

  Ab = Ti*A*T;                    % balanced dynamics matrix
  Bb = Ti*B;                      % balanced  input  matrix
  Cb =     C*T;                   % balanced  output  matrix

% ------------------------------------------------------- H.P. Gavin 2013-10-18
```

balreal_test.m

```matlab
% balreal_test.m --- test balanced model reduction

% Specify an LTI system in modal coordinates
 r = 1;          % number of inputs
 m = 1;          % number of outputs
 n = 20;         % number of internal states (even) ... complex-conjugate poles

% generate a random LTI system with under-damped dynamics in diagonalized form

  z = 0.1;    % Heavily Damped
  z = 0.01;   % Lightly Damped

  wn = 2*pi*logspace(0.2,1.8,10);

 [A,B,C,D] = modalLTI(wn,z,m,r,'unif');

 damp(A)                            % check the nat'l frequencies and damping ratios

 [Ab,Bb,Cb, G, T, Ti] = bal_real ( A, B, C );

 w   = 2*pi*logspace(0,2,500);
 nw  = length(w);
 mag = zeros(nw,n/4);
 pha = zeros(nw,n/4);

 eig_vals = NaN(n,n/4+1);
 for iter = 1:n/4+1

        r = n-2*iter+2;           % remaining number of states

        Ar = Ab(1:r,1:r);        Br = Bb(1:r,:);          Cr = Cb(:,1:r);

        eig_vals(1:r,iter) = eig(Ar);

        [mag(:,iter),pha(:,iter)] = mybode(Ar,Br,Cr,D,1,w);

end

% ------------------------------------------------------------------------ Plots

figure(1)
 clf
  loglog(w/2/pi, mag)
   legend('20','18','16','14','12','10',3);
   legend('location','southwest')
   xlabel('frequency, f, Hz')
   ylabel('frequency response magnitude')

figure(2)
 clf
  subplot(211)
   semilogy(n-[0:2:n/2], abs(eig_vals)/2/pi, 'ob','MarkerSize',5);
    axis([ n/2-1 n+1 1.1 1e2])
    ylabel('natural frequencies, Hz')
  subplot(212)
   plot(n-[0:2:n/2], -real(eig_vals)./abs(eig_vals+eps), 'ob','MarkerSize',5);
%    axis([ n/2-1 n+1 0 1.2*max(max(-real(eig_vals)./abs(eig_vals+eps))) ])
    xlabel('number of states')
    ylabel('damping ratio')

% ------------------------------------------------------------------------ balreal_test
```