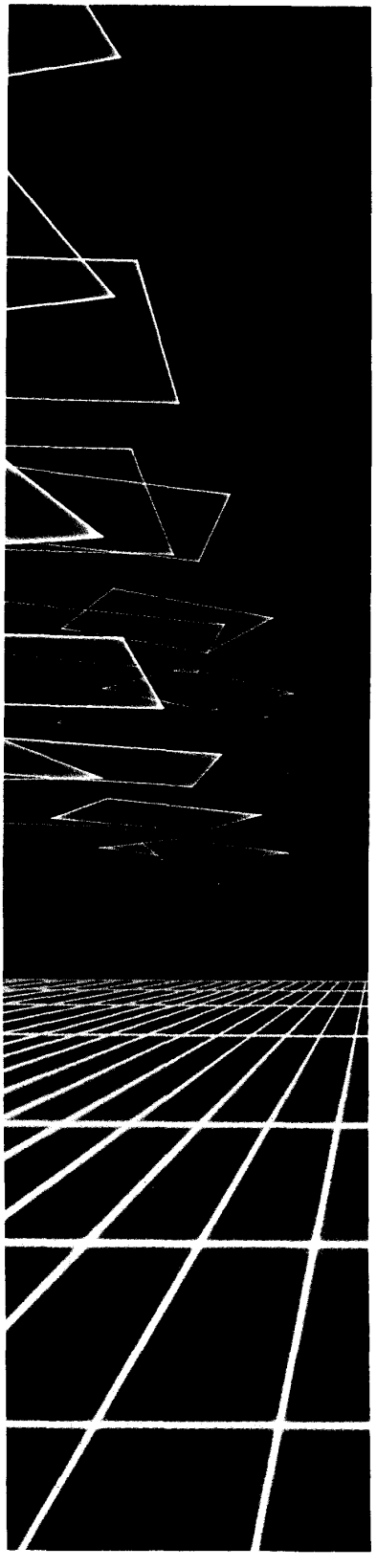

Signal Processing via Least Squares Error Modeling

JAMES A. CADZOW

Least squares error (LSE) modeling is widely employed in a variety of interdisciplinary applications involving the parametric modeling of empirical data. The signal model to be considered in this paper is composed of a linear combination of *basis signals* which are chosen so as to reflect the basic nature believed to characterize the data being modeled. In the general modeling problem herein considered, the basis signals are dependent on a set of real parameters which are to be selected so that the signal model best approximates the data in a LSE sense. Nonlinear programming algorithms for computing the optimum parameter selection are presented in which emphasis is placed on computational efficiency considerations.

The paper's development is formulated in a vector space setting and uses such fundamental vector space concepts as inner products, the range and null space of matrices, orthogonal vectors and the generalized Gram-Schmidt orthogonalization procedure. A running set of representative signal processing examples are presented to illustrate the theoretical concepts as well as point out the utility of LSE modeling. These examples include the modeling of empirical data as a sum of complex exponentials and sinusoids, linear prediction, linear recursive identification and direction finding.

In quantitatively based disciplines, investigators are often confronted with the task of constructing parametric models which approximate the behavior of empirically gathered data. If the data fit is sufficiently accurate, it is anticipated that the parametric model will provide further insight into the characterization of the phenomenon from which the measured data came. The measured data takes the form of a sequence of numbers

$$y(1),\ y(2),\ \cdots,\ y(N) \qquad (1)$$

The integer ordering argument designates the place within the data sequence where a particular element falls. The finite number of data points N reflects the fact that in any practical experiment one has access to only a finite number of measurements. In many applications, the data elements are sequentially recorded in a uniform manner so that the interval between contiguous data measurements is

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constant. This constant increment might correspond to a time interval, a distance length or a temperature increment with the nature of this interval being dictated by the specific application under consideration.

SIGNAL MODEL

Based on physical laws hypothesized as characterizing the phenomenon being observed, the investigator is often able to postulate a mathematical model that is to be used for subsequent analysis. In many important applications, this model takes the form of a linear combination of *basis signals* $\{s_m(n)\}$ as specified by

$$\hat{y}(n) = \sum_{m=1}^{M} a_m s_m(n) \quad for \ 1 \le n \le N \quad (2)$$

The scalars a_m appearing in our model correspond to the amplitudes of the basis signals. The elements of the basis signals used in our model are initially considered fixed and correspond to fundamental behaviors believed to characterize the empirical data. In later Sections, however, we shall allow the elements to be dependent on a set of parameters. For example, a sinusoidal basis signal $s(n)=sin(\omega n+\theta)$ would be dependent on the frequency (i.e., ω) and phase (i.e., θ) parameters. The ability to prudently choose these parameters will enable a greater degree of flexibility in accurately modeling empirical data.

For the case in which the basis signal elements are considered fixed, the data modeling problem corresponds to selecting the basis signal amplitudes so that data model (2) best represents the measured data (1) in some sense. Good modeling practice generally dictates that the number of basis signals M be made much smaller than the number of data elements although this requirement is relaxed in some applications. The effectiveness of signal model (2) is clearly dependent on the user making a prudent choice of the constituent set of basis signals. The basis signals constituting this signal model can be all of the same type (e.g., sinusoids) or be made up of a variety of different signal types. The following example illustrates one of the most widely employed fixed basis signal data models used to represent empirical data.

Example 1: *Let us consider an application in which it is desired to obtain a frequency representation of empirical data. The discrete Fourier transform (DFT) plays a prominent role in this regard. The DFT can be formulated in the above signal model representation where the model's basis signals are the complex sinusoids specified by*

$$s_m(n) = e^{j2\pi(m-1)(n-1)/M}$$
$$for \quad 1 \le m \le M \quad and \quad 1 \le n \le N$$

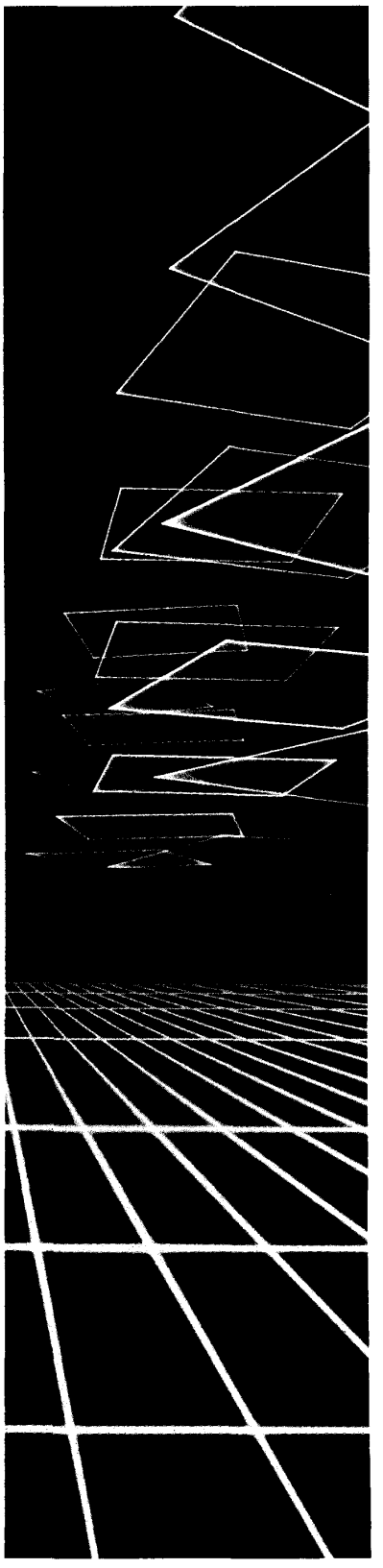
In this DFT application, the number of basis signals is chosen to equal the data's length(i.e., M = N). The complex sinusoid amplitudes can therefore be chosen so as to give an exact data representation. As is shortly shown, these amplitudes will correspond to the Fourier coefficients of the data being modeled.

MODELING CRITERION

The quality of mathematical model (2) is measured by how closely $\hat{y}(n)$ approximates the empirical data $y(n)$ being modeled. The modeling error is defined to be the difference of these two signals (i.e., $e(n)=y(n) - \hat{y}(n)$). Our objective in the fixed basis signal case now being considered is to select the basis signal amplitudes so as to drive this modeling error as close to zero as possible. A number of measures exist for quantifying how well this objective is being met. The following *modeling criterion* provides a useful means for measuring modeling fidelity

$$f(a_1, \cdots, a_M) = \sum_{n=1}^{N} w(n)|y(n) - \hat{y}(n)|^p = \sum_{n=1}^{N} w(n)|y(n) - \sum_{m=1}^{M} a_m s_m(n)|^p \quad (3)$$

This modeling criterion has been expressed as an explicit function of the basis signal amplitudes so as to emphasize this dependency. The $w(n)$ scalars here appearing are positive constants that are used to weight the relative importance of the individual error elements. In most applications a *unit weighting* is used in which case $w(n) \equiv 1$. The superscript p is a nonnegative constant which normally lies in the interval $[1, \infty]$ and is chosen so as to achieve certain objectives. For instance, if it is believed that some of the data elements $y(n)$ are unreliable,



the choice p=1 will tend to put relatively small weight on these so called *data outliers*. On the other hand, if one wishes to minimize the largest model residual the selection p=∞ achieves that goal.

In what is to follow, we shall use the *weighted squared error criterion* selection which corresponds to p=2. There are primarily three reasons for using this particular criterion: (i) the minimization of a sum of squared error criterion usually leads to acceptably good data models, (ii) this criterion has been thoroughly studied and many insightful theorems are available for its analysis, and, (iii) in applications where ideal data is corrupted by additive Gaussian noise, the LSE model often corresponds to the *maximum likelihood estimate.*

LSE Modeling: Fixed Basis Signals

We now analyze the important class of least squares error (LSE) modeling problems described above. A vector space representation of the given empirical data and the basis signals provides a particularly convenient mathematical means for carrying out this study. In what is to follow, we shall adhere to vector space representations. In this approach, the N x 1 *empirical data vector* and *basis signal vectors*, respectively, are given by

$$\underline{y}=\begin{bmatrix} y(1) \\ y(2) \\ \cdot \\ \cdot \\ \cdot \\ y(N) \end{bmatrix}, \qquad \underline{s}_m=\begin{bmatrix} s_m(1) \\ s_m(2) \\ \cdot \\ \cdot \\ \cdot \\ s_m(N) \end{bmatrix} \qquad \text{for } 1 \le m \le M \quad (4)$$

We have here adopted the convention of denoting vectors by underlined lower case letters.

In many applications, the elements of the data vector and the basis signal vectors are real valued. These vectors are therefore said to be contained in the vector space of real N-tuples as denoted by R^N. In other important applications, however, the vector elements may be complex numbers. For instance, the basis signals in the DFT are seen to be composed of complex sinusoids as shown in Example 1. In such cases, the constituent vectors are said to be contained in the vector space of complex N-tuples as designated by C^N. To treat the real and complex data cases in a common setting, we hereafter consider the constituent vectors to be contained in C^N. All the theoretical results to follow are therefore expressed in terms of a complex data model. These theoretical results are directly applicable to real data modeling applications, however, by the simple process of replacing any *complex conjugate vector transposition operator* (i.e, *) that appears by the *vector transpose operator* (i.e, T). In taking this approach, we simultaneously treat the cases of modeling real and complex valued data.

We now formulate the signal modeling problem in a vector space setting. In accordance with relationship (2) and vector representations (4), the *signal model vector* is given by

$$\hat{\underline{y}} = \sum_{m=1}^{M} a_m \underline{s}_m = S\underline{a} \quad (5)$$

In this expression, $\underline{a}$ designates the M x 1 *signal amplitude vector* whose components correspond to the a_m amplitudes of the individual basis signals while the N x M *composite basis signal matrix* S has as its columns the M basis signal vectors comprising the signal model, that is

$$S=\left[\underline{s}_1 \vdots\ \underline{s}_2 \vdots\ \ldots \vdots\ \underline{s}_M\right] \quad (6)$$

The quality of signal model (5) is measured by how closely $\hat{\underline{y}}$ approximates the empirical data vector $\underline{y}$. The degree of closeness is controlled by the signal amplitude vector which is to be chosen so as to minimize the *modeling error* vector as governed by

$$\underline{e} = \underline{y} - S\underline{a} \quad (7)$$

In the spirit of criterion (3) with p=2, we shall use the following functional for measuring modeling fidelity (or smallness of error)

$$f(\underline{a}) = \left[\underline{y}-S\underline{a}\right]^* W\left[\underline{y}-S\underline{a}\right] \quad (8)$$

where the N x N weighting matrix W is positive definite and Hermitian (i.e., $W^*=W$), and, the asterisk symbol designates the operation of complex conjugation. The weighting matrix is often chosen to be equal to the identity matrix (i.e., $W=I_N$) thereby giving rise to the *unweighted data modeling* case. For purposes of generality, however, we shall retain the general model weighting measure specified in this criterion.

In certain applications, the components of the basis signal amplitude vector may be constrained. The nature of these constraints are dependent on the specific application at hand. Whatever the case, the minimizing selection provides a data model that is most compatible with the given data in a weighted least squares error sense. For the purposes of presentation simplicity, we hereafter consider the signal amplitudes to be unconstrained. In the unconstrained case, a necessary condition for minimizing criterion (8) is that the derivatives of this criterion with respect to the signal amplitude components all be zero. Allowing for the possibility of complex signal amplitudes, it is a simple matter to show that this necessary condition takes on the form spelled out in the following theorem.

Theorem 1: A necessary and sufficient condition that a signal amplitude vector minimize squared error criterion (8) is that it satisfy the system of consistent "normal equations"

$$S^* W S \underline{a}^{\circ} = S^* W \underline{y} \qquad (9)$$

Moreover, any solution to this system of normal equations results in the squared error criterion taking on the value

$$f(\underline{a}^{\circ}) = \underline{y}^* W \underline{y} - \underline{y}^* WS \left[S^* WS \right]^{\dagger} S^* W \underline{y} \qquad (10)$$

*where [S*WS]† designates the pseudo inverse of matrix S*WS. In the set of optimum amplitude vectors specified by relationship (9), the minimum Euclidean norm (i.e., $\underline{a}^*\underline{a}$) is given by*

$$\underline{a}^{\circ}_{\min} = \left[S^* WS \right]^{\dagger} S^* W \underline{y} \qquad (11)$$

*Finally, all solutions to normal equations (9) may be expressed as a sum of this minimum Euclidean norm solution and any vector contained in the null space of matrix S*WS.*[1]

NORMALIZED MODELING CRITERION

Theorem 1 provides a convenient closed form representation for the optimum signal model. The next question posed is that of quantifying exactly how well the data model represents the empirical data. We cannot simply employ the minimum value (10) assumed by the squared error criterion, since it explicitly depends on the size of the empirical data. If the empirical data is large, it is possible that this criterion's minimum value will itself be relatively large even though the optimum model provides an excellent representation of the empirical data. Conversely, this criterion can take on a small value in those applications where the empirical data elements are small even though the data model renders a poor representation of the empirical data. Clearly, what is needed is a criterion which is independent of empirical data size. This is readily achieved by simply dividing the original criterion (8) by the weighted energy in the data being represented to give rise to the *normalized modeling criterion*

$$f_{norm}(\underline{a}) = \frac{\left[\underline{y} - S\underline{a} \right]^* W \left[\underline{y} - S \right] \underline{a}}{\underline{y}^* W \underline{y}} \qquad (12)$$

It is apparent that the minimum value of this normalized functional must always lies in the interval [0,1]. This is apparent since this functional takes on the value one for the choice $\underline{a}=\underline{o}$. We may therefore measure the fidelity of our model by determining how close to zero the normalized functional is. Values close to zero indicate good data modeling while values close to one suggest poor data modeling.

A variety of important signal processing applications can be posed in the form of a weighted LSE problem in which the basis signals are known a priori. We may therefore employ the results of Theorem 1 to treat this class of problems. The next two sections provide two important applications that can be characterized in this fashion.

Linear Prediction

The operation of *linear prediction* constitutes one of the more important applications in which the basis signal vectors are known a priori. When employing the concept of linear prediction, it is tacitly assumed that the underlying idealized data being analyzed possess the following attribute

- LINEAR PREDICTION PROPERTY: every contiguous subsequence of length $p+1$ formed from the data *y(1), y(2),..., y(N)*, satisfies a fixed linear homogeneous relationship.

It is well known that the only data sequences which possess this property are expressible as a linear combination of p or fewer exponential type signals. Thus, when employing linear prediction to data sequences, we are in effect invoking an exponential signal model.

To measure the propensity of noise contaminated non-idealized data to be modeled as a linear combination of p exponentials, it is therefore useful to appeal to the above property. In particular, we seek to find a set of *prediction coefficients* $a_1,...,a_p$ so that the *linear prediction sequence* generated according to

$$\hat{y}(n) = a_1 y(n-1) + a_2 y(n-2) + \ldots + a_p y(n-p) \quad \text{for } p+1 \le n \le N \qquad (13)$$

best approximates *y(n)* in some sense. In effect, we are attempting to use the most recent p values of the data under analysis to *predict* the next data point in a linear fashion. If a set of prediction coefficients can be found such that $\hat{y}(n) \equiv y(n)$, then the homogeneous relationship as described in the above property is satisfied.

From relationship (13), it is clear that the prediction sequence $\hat{y}(n)$ is an implicit function of the prediction coefficients. The interval $p+1 \le n \le N$ identifying this signal model is equal to the largest set of indices for which the right side of relationship (13) is comprised of data elements that are wholly contained in the data (1) under analysis. If each set of $p+1$ contiguous data elements is strongly linearly related in a consistent fashion, it then follows that the parameters a_m of this linear relationship can be selected so that the *prediction error sequence* as defined by

$$e(n) = y(p+n) - \hat{y}(p+n) \quad \text{for } 1 \le n \le N-p \qquad (14)$$

[1] The null space of S*WS is formally specified by $N(S^*WS) = \{\underline{a} : S^*WS\underline{a} = \underline{o}\}$.

can be made as small as possible.

It is possible to formulate the linear prediction problem in the format of the data modeling problem treated in the last section. This is achieved by inserting linear prediction formula (13) into relationship (14) and putting this result into the following equivalent vector format

$$\begin{bmatrix} e(1) \\ e(2) \\ \cdot \\ \cdot \\ \cdot \\ e(N-p) \end{bmatrix} = \begin{bmatrix} y(p+1) \\ y(p+2) \\ \cdot \\ \cdot \\ \cdot \\ y(N) \end{bmatrix} - \begin{bmatrix} y(p) & y(p-1) & \dots & y(1) \\ y(p+1) & y(p) & \dots & y(2) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y(N-1) & y(N-2) & \dots & y(N-p) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_p \end{bmatrix} \tag{15}$$

or more compactly as

$$\underline{e} = \underline{y} - Y\underline{a} \tag{16}$$

In this relationship, $\underline{e}$ is the *(N-p)* x 1 *prediction error vector*, $\underline{y}$ is the *(N-p)* x 1 vector whose k^{th} element is *y(k+p)*, and $\underline{a}$ is the *p* x 1 *predictor coefficient vector*. Similarly, Y is the associated *(N-p)* x *p* Toeplitz structured *data matrix* whose $(n,m)^{th}$ element is specified by *y(p+n-m)*. The data matrix Y plays the same role as the composite basis signal matrix S of the preceding Section. Thus, the columns of the data matrix correspond to the basis signal vectors.

Our objective is to select the prediction coefficient vector $\underline{a}$ so as to minimize the sum of the squared prediction errors. In accordance with the results of Theorem 1, the minimum norm prediction coefficient vector is given by

$$\underline{a}''_{\min} = \left[Y^*WY\right]^{\dagger} Y^*W\underline{y} \tag{17}$$

Furthermore, the sum of squared errors for this set of prediction errors is

$$\underline{e}^*_{\min} W \underline{e}_{\min} = \underline{y}^*W\underline{y} - \underline{y}^*WY\left[Y^*WY\right]^{\dagger} Y^*W\underline{y} \tag{18}$$

It is readily shown that the matrix $Y[Y^*WY]^{\dagger}Y^*W$ appearing in this sum of squared error criterion is a projection operator whose range space is the equal to the range space of Y.

LSE Modeling of Noise Contaminated Sinusoidal Data

In the next example, we shall consider an application in which the underlying data (1) is believed to be composed of a noise contaminated sum of sinusoids with known frequencies. To test the validity of this conjecture, investigators have often appealed to the discrete Fourier transform. We shall approach this problem, however, from the signal modeling approach taken in the last section. Specifically, the basis signals used in our model are comprised of M complex valued sinusoid vectors each having structure

$$\underline{s}_m = \begin{bmatrix} 1 \\ e^{j\omega_m} \\ e^{j2\omega_m} \\ \cdot \\ \cdot \\ \cdot \\ e^{j(N-1)\omega_m} \end{bmatrix} \qquad for\ \ 1 \le m \le M \tag{19}$$

In this set of basis signals, the sinusoidal frequencies $\omega_1,\dots,\omega_M$ are taken to be distinct and selected to lie in the interval $[0, 2\pi]$. If the data being modeled is real, it is necessary that the frequencies chosen occur in conjugate pairs (i.e., ω_k and $2\pi-\omega_k$).

Due to the makeup of the basis signals (19), it follows that the corresponding composite basis signal matrix will have a Vandermonde structure, that is

$$S = \begin{bmatrix} 1 & 1 & \dots & 1 \\ e^{j\omega_1} & e^{j\omega_2} & \dots & e^{j\omega_M} \\ e^{j2\omega_1} & e^{j2\omega_2} & \dots & e^{j2\omega_M} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ e^{j(N-1)\omega_1} & e^{j(N-1)\omega_2} & \dots & e^{j(N-1)\omega_M} \end{bmatrix} \tag{20}$$

Furthermore, since the M sinusoidal frequencies are taken to be distinct and to lie in the interval $[0, 2\pi]$, it follows that this composite basis signal matrix has full rank(S)=min(M,N). Theorem 1 indicates that the normal equations (9) will have a unique solution as specified by relationship (11). The quality of the resultant sinusoidal model is critically dependent on the selection of the model's sinusoid frequencies. They should be selected to closely correspond to the frequencies thought to be contained in the data. A standard procedure for accomplishing this is now briefly described.

DISCRETE FOURIER TRANSFORM

The *discrete Fourier transform* provides a convenient procedure for obtaining a sinusoidal model of empirical data. The basis signal vectors used in a DFT representation are specified by the complex sinusoidal vectors

$$\underline{s}_m = \begin{bmatrix} 1 \\ e^{j2\pi(m-1)/N} \\ e^{j4\pi(m-1)/N} \\ \cdot \\ \cdot \\ \cdot \\ e^{j2\pi(m-1)(N-1)/N} \end{bmatrix} \qquad for\ \ 1 \le m \le N \tag{21}$$

In this case, the number of basis signal vectors employed exactly equals the number of data points. It is a simple matter to show that these N basis vectors are pairwise orthogonal and they must therefore be linearly independent. Thus, the N x N composite basis signal matrix S formed from these basis signal vectors is nonsingular. This directly implies that the data vector $\underline{y}$ in the unity matrix weighting case $W = I_N$ can be exactly modeled by the linear combination (5), that is

$$\underline{a}^o = S^{-1}\,\underline{y} = \frac{1}{N} S^*\,\underline{y} \qquad (22)$$

In arriving at the above expression for the inverse of S, use has been made of the fact that the sinusoidal basis vectors are pairwise orthogonal and each has an Euclidean norm equal to $\sqrt{N}$. Upon carrying out the matrix-vector multiplication in expression (22), it is found that the unique basis signal amplitudes for which a perfect data model is obtained are given by

$$a_m^o = \frac{1}{N}\sum_{n=1}^{N} y(n)\; e^{-j2\pi\,(m-1)\,(n-1)/N} \qquad for\ 1 \le m \le N$$

These optimum amplitudes, however, are recognized as being the N point DFT coefficients associated with the empirical data (1).

LSE Modeling: Parametric Dependent Basis Signals

Up to this point, our efforts have been restricted to signal modeling applications in which the model's basis signals are considered fixed. A desirable enhancement in modeling fidelity is made possible, however, by allowing the basis signals behavior to be dependent on a set of parameters. Values are to be then assigned to these parameters so that the basis signal vectors are most compatible with the data being modeled. To illustrate this point, let us consider the sinusoidal modeling problem treated in the last section for the case in which the data is a perfect linear combination of two sinusoids. Thus, the invocation of a two sinusoidal model (i.e., M=2) is valid. Unfortunately, if the model's two sinusoidal frequencies (ω_1,ω_2) do not agree with the data's two sinusoid frequencies, then the corresponding optimum model can be poor. In allowing the sinusoidal model's frequencies to be parameters, however, it is conceptually possible to assign them the correct values and thereby attain a perfect data match. By taking this parametric approach, a more accurate modeling of the data is therefore made possible.

With these thoughts in mind, we shall now consider the modeling problem in which the basis signals are dependent on the components of a real valued P x 1 *signal parameter vector* $\underline{\theta}$. The composite basis signal matrix is therefore dependent on this signal parameter vector and is accordingly designated by $S(\underline{\theta})$. In typical applications, each basic signal is a function of only a subset of the components of $\underline{\theta}$. Whatever the case, the associated data modeling problem entails selecting this signal parameter vector and the associated signal amplitude vector so as to minimize the weighted squared error criterion

$$f(\underline{a},\underline{\theta}) = \left[\underline{y} - S(\underline{\theta})\,\underline{a}\right]^* W \left[\underline{y} - S(\underline{\theta})\,\underline{a}\right] \qquad (23)$$

Although the imposition of a basis signal parameter dependency has the desirable attribute of providing enhanced modeling capability, it also gives rise to a new difficulty. Specifically, in most applications the parameters of the basis signals enter in a highly nonlinear manner. A closed form solution for a signal parameter vector and signal amplitude vector that minimizes weighted squared error criterion (23) is therefore not feasible. This being the case, we must then appeal to *nonlinear programming methods* for minimizing this criterion. Since the computational load of nonlinear programming methods is proportional to the number of variables being optimized, it is advisable to reduce this number whenever possible.

In order to reduce the computational load of a nonlinear programming solution , we shall *decouple* the selection of an optimum composite parameter vector from an optimum signal amplitude vector. This is made possible due to the quadratic manner in which the signal amplitude vector enters criterion (23). Golub and Peryra first presented this decoupling technique using a *QR* decomposition procedure [5]. We shall achieve this same objective in a computationally more efficient fashion. The decoupling operation is composed of the two step process of

- finding an optimum composite parameter vector by minimizing criterion (23) in which the corresponding optimum signal amplitude vector is implicitly used.
- determining the associated optimum amplitude vector using a convenient closed form expression.

In the development that follows, it is assumed that the composite basis signal matrix $S(\underline{\theta})$ has full rank M. No loss of generality arises from this assumption since one can always remove any linearly dependent columns from the composite basis signal matrix until this full rank condition is satisfied. Whatever the case, the required decoupling operation is achieved by appealing to the concepts of W-orthogonality and projection matrices from linear algebra (e.g., see reference [15]).

W-ORTHOGONAL VECTORS AND PROJECTION MATRICES

The concept of W-orthogonal vectors plays a central role in the theoretical developments that follow where W is the positive-definite Hermitian matrix used in the squared error criterion being employed. The vectors $\underline{y}_1$ and $\underline{y}_2$ contained in C^N are said to be *W-orthogonal* if

their *inner product* as specified by $\underline{y}_1^* W \underline{y}_2$ is zero, that is

$$\underline{y}_1^* W \underline{y}_2 = 0 \tag{24}$$

Furthermore, if $\underline{y}_1$ and $\underline{y}_2$ are W-orthogonal vectors, it is a simple matter to show that

$$(\underline{y}_1 + \underline{y}_2)^* W (\underline{y}_1 + \underline{y}_2) = \underline{y}_1^* W \underline{y}_1 + \underline{y}_2^* W \underline{y}_2 \tag{25}$$

This readily proven identity is a generalization of the two-dimensional Pythagorean theorem.

From our perspective, the most important use of the above generalized Pythagorean theorem arises from the fact that all vectors $\underline{y} \in C^N$ can be uniquely decomposed as

$$\underline{y} = \underline{y}_1 + \underline{y}_2 \tag{26}$$

where $\underline{y}_1$ is a unique vector contained in the range space of $S(\underline{\theta})$ as designated by $\Re(S(\underline{\theta}))$ and $\underline{y}_2$ is a unique vector that is W-orthogonal to $\underline{y}_1$. These W-orthogonal vector components are readily computed by first determining the N x N *projection matrix* $P(\underline{\theta})$ whose range space is $\Re(S(\underline{\theta}))$ and whose null space is the W-orthogonal complement subspace of $\Re(S(\underline{\theta}))$ as designated by $\Re(S(\underline{\theta}))^{\perp}$. This W-orthogonal complement subspace consists of all vectors in C^N which are W-orthogonal to every vector in $\Re(S(\underline{\theta}))$. A closed form expression for this required projection matrix is given by

$$P(\underline{\theta}) = S(\underline{\theta})\left[S(\underline{\theta})^* W S(\underline{\theta})\right]^{-1} S(\underline{\theta})^* W \tag{27}$$

The dependency of this projection matrix on the parameter vector $\underline{\theta}$ has been explicitly recognized. The effect of this projection operator when applied to a general vector is shown in Fig. 1. Due to the basic behavior displayed in this figure, the projection matrix (27) is said *to be a projection on* $\Re(S(\underline{\theta}))$ *along* $\Re(S(\underline{\theta}))^{\perp}$. The two vector components will be orthogonal in the standard sense (i.e., separated by a 90° angle) for the case of a unit matrix weighting selection $W=I_N$.

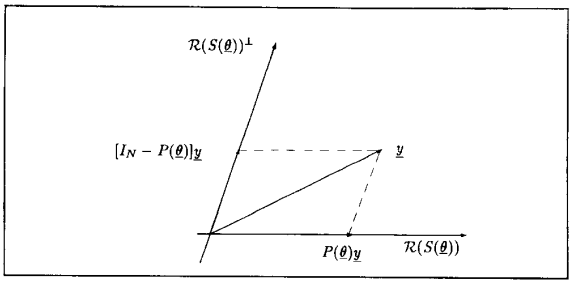


Fig. 1. Orthogonal decomposition of $\underline{y}$.

To prove that matrix (27) defines a projection matrix with the prescribed range and null spaces, we first establish that it is a projection matrix. A necessary and sufficient condition that a matrix be a projection is that it be idempotent (i.e., $P(\underline{\theta})^2 = P(\underline{\theta})$). The idempotency of matrix (27) is readily established by direct substitution. Furthermore, the range and null spaces of this matrix are readily found to be equal to $\Re(S(\underline{\theta}))$ and $\Re(S(\underline{\theta}))^{\perp}$, respectively. Thus, $P(\underline{\theta})$ is the required projection matrix. The two vectors required in decomposition (26) are therefore given by

$$\begin{aligned} \underline{y}_1 &= P(\underline{\theta})\underline{y} \\ \underline{y}_2 &= \left[I_N - P(\underline{\theta})\right]\underline{y} \end{aligned} \tag{28}$$

DECOUPLING ALGORITHM

To obtain the aforementioned parameter decoupling, we first apply the generalized Gram-Schmidt Orthogonalization procedure described in Table I to the M basis signal vectors. This procedure produces a set of vectors $q_1(\underline{\theta}),\ldots,q_M(\underline{\theta})$ which are *W-orthogonal* in the sense described by relationship (24). Moreover, these vectors span the same subspace as do the basis signal vectors $s_1(\underline{\theta}),\ldots,s_M(\underline{\theta})$ from which they were generated. To analytically capture this concept, let us introduce the associated N x M matrix $Q(\underline{\theta})$ whose columns are the $q_k(\underline{\theta})$ vectors, that is

$$Q(\underline{\theta}) = \left[q_1(\underline{\theta}) \vdots\ q_2(\underline{\theta}) \vdots\ \ldots \vdots\ q_m(\underline{\theta})\right] \tag{29}$$

From Table I, it is clear that the following matrix relationships are a direct consequence of this generalized Gram-Schmidt orthogonalization process

$$Q(\underline{\theta})^* W\, Q(\underline{\theta}) = I_N \tag{30}$$

and

$$S(\underline{\theta}) = Q(\underline{\theta})\, R(\underline{\theta}) \tag{31}$$

where I_N is the N x N identity matrix and $R(\underline{\theta})$ is a M x M nonsingular upper triangle matrix.

Step	Description of Step
1.	let $q_1(\underline{\theta}) = s_1(\underline{\theta})/\sqrt{s_1(\underline{\theta})^* W s_1(\underline{\theta})}$
2.	for $2 \le k \le M$
3.	let $\tilde{q}_k(\underline{\theta}) = s_k(\underline{\theta}) - \sum_{n=1}^{k-1} q_n(\underline{\theta})\left[q_n^*(\underline{\theta}) W s_n(\underline{\theta})\right]$
4.	let $q_k(\underline{\theta}) = \tilde{q}_k(\underline{\theta})/\sqrt{\tilde{q}_k^*(\underline{\theta}) W \tilde{q}_k(\underline{\theta})}$

Table I.
Generalized Gram-Schmidt Orthogonalization Algorithm.

Our ultimate objective is to select the signal parameter vector and signal amplitude vector so that the signal model $S(\underline{\theta})\underline{a}$ best approximates data vector $\underline{y}$ in the sense of minimizing weighted squared error criterion (23). With this is mind, we now uniquely decompose the data vector into the sum of a vector contained in the range space of $S(\underline{\theta})$ and a W-orthogonal vector. The projection operator required for this decomposition is directly obtained by substituting representation (31) into expression (27) and then using identity (30) to give

$$P(\underline{\theta}) = Q(\underline{\theta})\,Q(\underline{\theta})^{*}\,W \tag{32}$$

This required vector decomposition is therefore given by

$$\underline{y} = Q(\underline{\theta})\,Q(\underline{\theta})^{*}\,W\,\underline{y} + \left[I_N - Q(\underline{\theta})\,Q(\underline{\theta})^{*}\,W\right]\underline{y} \tag{33}$$

With this decomposition result established, the modeling error vector can be expressed as

$$\begin{aligned}\underline{y} - S(\underline{\theta})\,\underline{a} = &\left[Q(\underline{\theta})\,Q(\underline{\theta})^{*}\,W\,\underline{y} - S(\underline{\theta})\,\underline{a}\right] \\ &+ \left[I_N - Q(\underline{\theta})\,Q(\underline{\theta})^{*}\,W\right]\underline{y}\end{aligned} \tag{34}$$

in which the two vectors contained in the first rectangular brace both lie in the range space of $S(\underline{\theta})$ and the vector contained in the second rectangular brace lies in its orthogonal complement. Using the W-orthogonal property (25), it therefore follows that the weighted squared error criterion is given by

$$\begin{aligned}f(\underline{\theta},\underline{a}) = &\left[Q(\underline{\theta})\,Q(\underline{\theta})^{*}\,W\,\underline{y} - S(\underline{\theta})\,\underline{a}\right]^{*} W \left[Q(\underline{\theta})\,Q(\underline{\theta})^{*}\,W\,\underline{y}\right. \\ &\left. - S(\underline{\theta})\,\underline{a}\right] + \underline{y}^{*} W \left[I_N - Q(\underline{\theta})\,Q(\underline{\theta})^{*}\,W\right]\underline{y}\end{aligned} \tag{35}$$

Upon examination of this expression, it is seen that the weighted squared error criterion has been decomposed into two positive semi-definite terms. The second term on the right side is independent of the signal amplitude vector. On the other hand, the first term can be made equal to zero by a proper choice of the signal amplitude vector. This is a direct consequence of the fact that the range space of $Q(\underline{\theta})Q(\underline{\theta})^{*}W$ is equal to the range space of $S(\underline{\theta})$. With these thoughts in mind, the following fundamental decoupling result has been proven.

Theorem 2: *Let the N x M composite basis signal matrix $S(\underline{\theta})$ have full rank M so that it may be represented by factorization (31). It then follows that the optimum source amplitude vector for any composite source parameter vector is specified by*

$$\underline{a}^{o} = R(\underline{\theta})^{-1}\,Q(\underline{\theta})^{*}\,W\,\underline{y} \tag{36}$$

and the squared error criterion's (35) value for these optimum source amplitudes is given by

$$f(\underline{\theta},\underline{a}^{o}) = \underline{y}^{*} W\,\underline{y} - \underline{y}^{*} W\,Q(\underline{\theta})\,Q(\underline{\theta})^{*}\,W\,\underline{y} \tag{37}$$

It is important to appreciate the significance of the results described in Theorem 2. Namely, the minimization of criterion (35) that is dependent on both the signal amplitude vector $\underline{a}$ and the parameter vector $\underline{\theta}$ has been equated to that of first solving the following reduced parameter optimization problem

$$\begin{aligned}\min_{\underline{\theta},\underline{a}} f(\underline{\theta},\underline{a}) &= \min_{\underline{\theta}} f(\underline{\theta},\underline{a}^{o}) \\ &= \underline{y}^{*} W\,\underline{y} - \max_{\underline{\theta}}\, \underline{y}^{*} W\,Q(\underline{\theta})\,Q(\underline{\theta})^{*}\,W\,\underline{y}\end{aligned} \tag{38}$$

in the parameter vector $\underline{\theta}$. In this signal parameter vector minimization, the optimal amplitude vector is implicitly entered. Once an optimum parameter vector has been determined, the associated minimum Euclidean norm signal amplitude vector is generated using expression (36) here repeated

$$\underline{a}^{o} = R(\underline{\theta}^{o})^{-1}\,Q(\underline{\theta}^{o})^{*}\,W\,\underline{y} \tag{39}$$

Using this approach, we have effectively decoupled the determination of $\underline{\theta}^{o}$ from that of $\underline{a}^{o}$. This is significant since the computational requirements for solving minimization problem (38) using nonlinear programming techniques are much smaller than those required in a minimization of weighted squared error criterion (23). It is to be noted that this Golub-Pereyra decoupling procedure has been extended to include more general nonlinear separation of variables problems [12].

Newton Type Algorithmic Solutions

The most difficult phase of the decoupled optimum modeling procedure described in the previous section is normally that of solving the nonlinear optimization problem (38), here repeated:

$$f(\underline{\theta},\underline{a}^{o}) = \underline{y}^{*} W\,\underline{y} - \underline{y}^{*} W\,Q(\underline{\theta})\,Q(\underline{\theta})^{*}\,W\,\underline{y} \tag{40}$$

In effect, we are to select $\underline{\theta}$ so that the projection mapping component $Q(\underline{\theta})Q(\underline{\theta})^{*}W\underline{y}$ lies as close to the data vector $\underline{y}$ as possible. For most applications, this functional is a highly nonlinear function of $\underline{\theta}$ and we must resort to *nonlinear programming methods* to obtain a maximum.

We shall now describe a general class of descent type algorithms which can be used to iteratively minimize functional (40). The gradient vector and Jacobian matrix associated with this functional will play prominent roles in these descent algorithms. The components of the gradient vector of functional (40) are, by definition, equal

to the partial derivatives of this functional with respect to the elements of the signal parameter vector $\underline{\theta}$. Using standard differentiation, it is readily shown that this gradient vector is specified by

$$\nabla f(\underline{\theta}, \underline{a}^o) = real\left\{ J(\underline{\theta})^* W\left[\underline{y} - Q(\underline{\theta}) Q(\underline{\theta})^* W \underline{y} \right] \right\} \quad (41)$$

where the symbol *real* designates the "taking the real part of" operator. In this expression, $J(\underline{\theta})$ is the $N \times P$ *Jacobian matrix* associated with model criterion (40) as specified by

$$J(\underline{\theta}) = -\left[\frac{\partial Q(\underline{\theta}) Q(\underline{\theta})^*}{\partial \theta_1} W \underline{y} \vdots \frac{\partial Q(\underline{\theta}) Q(\underline{\theta})^*}{\partial \theta_2} W \underline{y} \vdots \cdots \right. \\ \left. \cdots \vdots \frac{\partial Q(\underline{\theta}) Q(\underline{\theta})^*}{\partial \theta_P} W \underline{y} \right] \quad (42)$$

In the class of descent algorithms to be here considered, the trial signal parameter vector at iteration k is denoted by $\underline{\theta}^{(k)}$. This vector is updated according to the formula

$$\underline{\theta}^{(k+1)} = \underline{\theta}^{(k)} \\ - \alpha_k \left[real\left\{ J(\underline{\theta}_k)^* W J(\underline{\theta}_k) + \mu I_P \right\} \right]^{-1} \nabla f(\underline{\theta}_k, \underline{a}^0) \quad (43)$$

In this expression, α_k is a positive *step size* scalar that is ideally selected to minimize $f(\underline{\theta}^{(k+1)}, \underline{a}^o)$. For our purposes, we shall only require that this step size parameter be chosen so as to satisfy the *improving condition*

$$f(\underline{\theta}^{(k+1)}, \underline{a}^o) < f(\underline{\theta}^{(k)}, \underline{a}^o)$$

It is essential that this improving condition be satisfied if update algorithm (43) is to yield a parameter vector sequence that converges to a relative minimum of functional (40).

The value assigned to the μ scalar appearing in expression (43) determines the specific descent algorithm that is implemented. For $\mu=0$, this relationship corresponds to the *Gauss-Newton* algorithm while an appropriately assigned positive value leads to the *Levenberg-Marquardt* algorithm. As the scalar μ approaches positive infinity, relationship (43) corresponds to the *method of steepest descent*. The Gauss-Newton and Levenberg-Marquardt algorithms generally provide a rapid quadratic convergent rate with an attendant moderate computational load per iteration. On the other hand, the method of steepest descent has a small computational load per iteration but its convergent rate is disappointingly slow. For most applications, the Gauss-Newton or Levenberg-Marquardt algorithms are preferable because of their rapid quadratic convergence capabilities.

STEP SIZE DETERMINATION

The step size scalar α_k is to be normally selected so that a sufficiently adequate decrease in functional value of $f(\underline{\theta}, \underline{a}^o)$ is achieved at each iteration. A simple but effective procedure which has proven successful is to evaluate the function $f(\underline{\theta}^{(k+1)}, \underline{a}^o)$ for the following sequence of decreasing step sizes

$$\alpha_k = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \left(\frac{1}{2}\right)^m, \ldots \quad (44)$$

until the first value of α_k is found for which the function $f(\underline{\theta}^{(k+1)}, \underline{a}^o)$ is less than $f(\underline{\theta}^{(k)}, \underline{a}^o)$. This step size procedure has the useful property of initially taking a full step (i.e., $\alpha = 1$) and if that step proves too large then decreasing the step until an improving parameter vector is eventually found. The use of an initial full step ensures that quadratic or superlinear convergence rates are maintained in a neighborhood of a relative minimum [3].

STOPPING CONDITIONS

In using parameter vector updating scheme (43), a means for determining when to stop the iterative process is required. In particular, a systematic procedure must be employed which terminates the algorithm whenever the parameter vector is deemed sufficiently close to the desired optimum value. There exist a variety of heuristically based measures for estimating when this condition is met. For our purposes, the algorithm is stopped whenever one (or both) of the following *stopping conditions* is satisfied.

$$(i) \left| \frac{f(\underline{\theta}^{(k)}, \underline{a}^o) - f(\underline{\theta}^{(k-1)}, \underline{a}^o)}{\underline{y}^* W \underline{y}} \right| < \varepsilon_1 \quad (45)$$

(ii) step size scalar α_k must be made smaller than ε_2 in order to improve the LSE criterion.

The scalars ε_1 and ε_2 are user selected with smaller values leading to more iterations of the algorithm before the stopping condition is triggered. It is important that these constants be selected small enough so that the algorithm stops when $\underline{\theta}_k$ is sufficiently close to an optimum, but, large enough so that an unduly large number of iterations are not employed to gain insignificant changes in the parameter vector.

Gradient and Jacobian Matrix Determination

In accordance with the comments made in the last section, our emphasis is concentrated on nonlinear programming algorithms which depend only on the

determination of the gradient and Jacobian associated with squared error functional (40). These entities are seen to depend on the partial derivatives of the vector $Q(\underline{\theta})Q(\underline{\theta})^*W\underline{y}$ relative to the elements of the parameter vector. To determine these partials, we will make use of the fact that $Q(\underline{\theta})^*Q(\underline{\theta})W$ is a projection matrix whose range space equals the range space of $S(\underline{\theta})$. It therefore follows that

$$Q(\underline{\theta})Q(\underline{\theta})^* W S(\underline{\theta}) = S(\underline{\theta}) \tag{46}$$

Upon taking the partial derivative of this identity with respect to θ_k and then rearranging terms, we have

$$\frac{\partial Q(\underline{\theta})\, Q(\underline{\theta})^*}{\partial \theta_k} W S(\underline{\theta}) = \left[I_N - Q(\underline{\theta})\, Q(\underline{\theta})^* W \right] \frac{\partial S(\underline{\theta})}{\partial \theta_k} \tag{47}$$

We next right multiply each side of this expression by $S(\underline{\theta})\,\dagger Q(\underline{\theta})Q(\underline{\theta})^* = R(\underline{\theta})^{-1}Q(\underline{\theta})^*$ and use the readily established identity $S(\underline{\theta})S(\underline{\theta})\dagger Q(\underline{\theta}) = Q(\underline{\theta})$ to obtain

$$\frac{\partial Q(\underline{\theta})\, Q(\underline{\theta})^*}{\partial \theta_k} W Q(\underline{\theta})\, Q(\underline{\theta})^* = \left[I_N - Q(\underline{\theta}) Q(\underline{\theta})^* W \right] \frac{\partial S(\underline{\theta})}{\partial \theta_k} R(\underline{\theta})^{-1} Q(\underline{\theta})^* \tag{48}$$

To complete our development we next take the partial derivative of the projection idempotent property $Q(\underline{\theta})Q(\underline{\theta})^*W = [Q(\underline{\theta})Q(\underline{\theta})^*W]^2$ to obtain

$$\frac{\partial Q(\underline{\theta})\, Q(\underline{\theta})^*}{\partial \theta_k} W = \left[\frac{\partial Q(\underline{\theta})\, Q(\underline{\theta})^*}{\partial \theta_k} W Q(\underline{\theta})\, Q(\underline{\theta})^* + Q(\underline{\theta})\, Q(\underline{\theta})^* W \frac{\partial Q(\underline{\theta})\, Q(\underline{\theta})^*}{\partial \theta_k} \right] W \tag{49}$$

The two matrices appearing within the braces on the right hand side of this identity are seen to be complex conjugates. Upon right multiplying each side of this expression by $\underline{y}$ and substituting in relationship (48), the expression for the required partial derivative is given by

$$\begin{aligned} &\frac{\partial Q(\underline{\theta})\, Q(\underline{\theta})^*}{\partial \theta_k} W \underline{y} \\ &= \left[\left[I_N - Q(\underline{\theta}) Q(\underline{\theta})^* W \right] \frac{\partial S(\underline{\theta})}{\partial \theta_k} R(\underline{\theta})^{-1} Q(\underline{\theta})^* \right] W \underline{y} \\ &\quad + \left[\left[I_N - Q(\underline{\theta}) Q(\underline{\theta})^* W \right] \frac{\partial S(\underline{\theta})}{\partial \theta_k} R(\underline{\theta})^{-1} Q(\underline{\theta})^* \right]^* W \underline{y} \end{aligned} \tag{50}$$

Kaufman has suggested a modification to the Golub-Pereyra method in which a significant easing in the computational load often arises [7]. Kaufman's approximation of the required partial derivatives is given by

$$\frac{\partial Q(\underline{\theta})Q(\underline{\theta})^*}{\partial \theta_k} W \underline{y} \approx \left[I_N - Q(\underline{\theta})^* Q(\underline{\theta}) W \right] \frac{\partial S(\underline{\theta})}{\partial \theta_k} S^{\dagger}(\underline{\theta})$$

Empirical evidence suggests that this approximation provides similar convergence characteristics as does the Golub-Pereyra method. Since the computational load is typically much smaller, the Kaufman modification is generally preferred in many applications.

Initial Parameter Value Assignment

A critical factor in the satisfactory performance of any nonlinear programming algorithm is that of generating an appropriately good set of parameter values to initiate the algorithm. If the initializing parameters are chosen sufficiently close to the global minimum, then most well behaved algorithms will produce a parameter vector sequence that converges to that minimum. On the other hand, a poor choice for the initial parameter values can cause even the best performing algorithm to produce a parameter vector sequence that converges to an inferior relative minimum. With this in mind, we shall now direct our attention to the vital issue of initial parameter value selection.

It is impossible to provide a universally effective procedure for selecting initializing parameters for the general class of LSE modeling problems. A method which has proven successful in several important LSE problems, however, is now described. It is applicable to those modeling problems for which the basis signal vectors comprising the columns of composite basis signal matrix $S(\underline{\theta})$ are each dependent on their own individual parameter vectors, that is

$$S(\underline{\theta}) = \left[\underline{s}_1(\underline{\theta}_1) \vdots \underline{s}_2(\underline{\theta}_2) \vdots \ \ldots \ \vdots \underline{s}_M(\underline{\theta}_M) \right] \tag{51}$$

where $\underline{\theta}_m$ designates the parameter vector associated with the m^{th} basis signal vector. Fortunately, many important LSE modeling problems can be described in this fashion. The proposed initializing procedure is predicated on a generalization of the *coordinate descent algorithm* employed in nonlinear programming [9]. In the coordinate descent algorithm, a function of many variables is minimized through the process of generating an improving value for one variable at a time, while holding the other variables fixed. This process is sequenced through all the variables and then repeated until the variables converge hopefully to the desired global minimum.

To adapt the coordinate descent algorithm to the LSE modeling problem in which the composite basis signal matrix takes the form (51), we shall examine the general case in which there are m basis signals where $m \leq M$. Applying the generalized Gram-Schmidt orthogonalization process described in Table I to the

basis signals $\underline{s}_1(\underline{\theta}_1), \underline{s}_2(\underline{\theta}_2), \ldots, \underline{s}_m(\underline{\theta}_m)$, there is generated the set of W-orthogonal vectors which form the m columns of the m x N matrix given by

$$Q_m(\underline{\theta}_1 \ \ldots \ \underline{\theta}_{m-1}, \underline{\theta}_m) = \left[\underline{q}_1(\underline{\theta}_1) \vdots \ \underline{q}_2(\underline{\theta}_1, \underline{\theta}_2) \vdots \ \ldots \vdots \ \underline{q}_m(\underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_m)\right] \quad \text{for } 1 \le m \le M \tag{52}$$

The first W-orthogonal vector $\underline{q}_1(\underline{\theta}_1)$ is seen to depend only on parameter vector $\underline{\theta}_1$, the second W-orthogonal vector $\underline{q}_2(\underline{\theta}_1, \underline{\theta}_2)$ to depend only on the parameter vectors $\{\underline{\theta}_1, \underline{\theta}_2\}$ and so forth. This is a direct consequence of the fact that the basis signals are each dependent on there own parameter vectors. It is instructive to decompose matrix (52) as

$$Q_m(\underline{\theta}_1, \ldots, \underline{\theta}_{m-1}, \underline{\theta}_m) = \left[Q_{m-1}(\underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_{m-1}) \vdots \ \underline{q}_m(\underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_m)\right] \quad \text{for } 1 \le m \le M \tag{53}$$

The first m - 1 columns of this matrix are seen to correspond to the first m - 1 W-orthogonal vectors generated by the generalized Gram-Schmidt orthogonalization process while the last column is generated according to

$$\underline{q}_m(\underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_m) = \frac{\left[I_N - Q_{m-1}(\underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_{m-1}) Q^*_{m-1}(\underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_{m-1}) W\right] \underline{s}_m(\underline{\theta}_m)}{\sqrt{\underline{s}_m(\underline{\theta}_m)^* W \underline{s}_m(\underline{\theta}_m)}} \quad \text{for } 1 \le m \le M \tag{54}$$

It is apparent that relationships (53) and (54) constitute a convenient implementation of the generalized Gram-Schmidt process outlined in Table I in which at the first iteration m = 1 one sets Q_o equal to the zero matrix.

We next apply the results of Theorem 2 in which the optimal signal amplitude vector is implicitly selected. The associated weighted squared error criterion is by relationship (37) given by

$$\begin{aligned} f_m(a''_1, \underline{\theta}_1, \ldots, a''_m, \underline{\theta}_m) &= \underline{y}^* W \underline{y} - \underline{y}^* W Q_m(\underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_m) Q_m(\underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_m)^* W \underline{y} \\ &= \underline{y}^* W \underline{y} - \underline{y}^* W [Q_{m-1}(\underline{\theta}_1, \ldots, \underline{\theta}_{m-1}) Q_{m-1}(\underline{\theta}_1, \ldots, \underline{\theta}_{m-1})^* \\ &\quad + \underline{q}_m(\underline{\theta}_1, \ldots, \underline{\theta}_m) \underline{q}_m(\underline{\theta}_1, \ldots, \underline{\theta}_m)^*] W \underline{y} \\ &= f_{m-1}(a''_1, \underline{\theta}_1, \ldots, a''_{m-1}, \underline{\theta}_{m-1}) \\ &\quad - \underline{y}^* W \underline{q}_m(\underline{\theta}_1, \ldots, \underline{\theta}_m) \underline{q}_m(\underline{\theta}_1, \ldots, \underline{\theta}_m)^* W \underline{y} \end{aligned} \tag{55}$$

This expression provides an iterative means for updating the squared error modeling criterion as new basis vectors are entered.

COORDINATE DESCENT ALGORITHM

The coordinate descent algorithm entails a sequential application of expressions (53)-(55) to generate a set of initial parameter vectors. At the first iteration m = 1, the vector $\underline{q}_1(\underline{\theta}_1)$ is generated using relationship (54) with $Q_o(\underline{\theta}_1)$ set equal to the zero matrix, that is

$$\underline{q}_1(\underline{\theta}_1) = \frac{\underline{s}_1(\underline{\theta}_1)}{\sqrt{\underline{s}_1(\underline{\theta}_1)^* W \underline{s}_1(\underline{\theta}_1)}} \tag{56}$$

This vector is then substituted into relationship (55) to give the amplitude optimized weighted squared error criterion's value

$$f_1(a''_1, \underline{\theta}_1) = \underline{y}^* W \underline{y} - \underline{y}^* W \underline{q}_1(\underline{\theta}_1) \underline{q}_1(\underline{\theta}_1)^* W \underline{y} \tag{57}$$

We now seek a parameter vector $\underline{\theta}_1$ that minimizes this criterion. In most applications of interest, a closed form solution for a minimizing parameter vector is not feasible. In this all to common situation, an approximation to an optimum parameter vector generally entails a direct evaluation of criterion (57) on a grid of $\underline{\theta}_1$ values believed to encompass the minimum. This entails the evaluation of vector expression (56) on this grid and then substituting these grid evaluated vector values into criterion (57). That grid point which renders this criterion its minimum is taken as a first approximation to the optimum choice for $\underline{\theta}_1$. Computational considerations usually dictate that the grid points be sparse thereby giving rise to a relatively crude approximation of the optimum parameter vector. Interpolation techniques can then be employed to improve upon this first approximation. The parameter vector obtained by this direct evaluation procedure is designated by $\underline{\theta}^o_1$. The matrix $Q_1(\underline{\theta}^o_1)$ is also formed using expression (53).

At the next step of the sequential process, the first parameter vector is held fixed at the approximate optimum value obtained at the first step (i.e, $\underline{\theta}^o_1$). We then seek the two parameter vector combination $(\underline{\theta}^o_1, \underline{\theta}_2)$ which best represents the data vector $\underline{y}$ in the minimum squared error sense. It is important to realize that the two parameter vectors obtained in this manner are generally not the optimum selection for all possible two parameter vector combinations. The process of fixing the first parameter vector and then selecting the *conditioned* optimum second parameter vector, however, typically leads to a sufficiently good suboptimum choice. Although we generally lose the possibility for obtaining an optimum parameter vector pair, this procedure of decomposing the problem into a sequence of simpler individual parameter vector optimizations is computationally more viable. With this in mind, the second step is now outlined.

The W-orthogonal vector $\underline{q}_2(\underline{\theta}^o_1, \underline{\theta}_2)$ that arises from the generalized Gram-Schmidt procedure is generated according to relationship (54) with m = 2, that is

$$q_2(\underline{\theta}''_1,\underline{\theta}_2) = \frac{[I_N - Q_1(\underline{\theta}''_1)Q_1(\underline{\theta}''_1)^*W]\,\underline{s}_2(\underline{\theta}_2)}{\sqrt{\underline{s}_2(\underline{\theta}_2)^*W\,\underline{s}_2(\underline{\theta}_2)}} \qquad (58)$$

From expression (55), the signal amplitude vector optimized weighted criterion is given by

$$f_2(a_1^0, \underline{\theta}_1^0, a_2^0, \underline{\theta}_2) = f_1(a_1^0, \underline{\theta}_1^0) - \underline{y}^*W\,\underline{q}_2(\underline{\theta}_1^0, \underline{\theta}_2)\,\underline{q}_2(\underline{\theta}_1^0, \underline{\theta}_2)^*W\,\underline{y} \qquad (59)$$

The first term on the right side is fixed and equals the minimized criterion value for a one basis signal model found at step one. It is now desired to select a value for $\underline{\theta}_2$ which minimizes the two basis signal model criterion (59). For reasons just elaborated upon in step one, a direction evaluation of this criterion will typically be required for an minimizing selection. A direct evaluation of this criterion on a grid of $\underline{\theta}_2$ values is then carried out. The approximation to this minimizing selection is here denoted by $\underline{\theta}^o{}_2$. The determination of the matrix $Q_2(\underline{\theta}^o{}_1,\underline{\theta}^o{}_2)$ using expression (53) completes the second step of the coordinate descent algorithm.

This process may be straightforwardly continued in an iterative fashion. Specifically, at the m - 1^{st} stage it is assumed that a set of m - 1 parameter vectors $\underline{\theta}^o{}_1$, $\underline{\theta}^o{}_2$, ..., $\underline{\theta}^o{}_{m-1}$ have been selected to represent the empirical data by m - 1 basis signals. It is now desired to append to this set another parameter vector $\underline{\theta}_m$ so that the enlarged set of parameter vectors best represent the data vector. To achieve this objective, we simply use expression (54) to generate $\underline{q}_m(\underline{\theta}^o{}_1, \ldots, \underline{\theta}^o{}_{m-1}, \underline{\theta}_m)$ and then evaluate criterion (55) on a grid of $\underline{\theta}_m$ values. This leads to an approximation of the minimizing value $\underline{\theta}^o{}_m$. The procedure is continued until a full set of M signal parameter vectors $\{\underline{\theta}^o{}_1, \underline{\theta}^o{}_2, \ldots, \underline{\theta}^o{}_M\}$ have been obtained to represent the given data vector. Typically, the signal parameter vectors obtained using this iterative procedure provide good initial model parameter values for many modeling problems of interest.

Exponential Signal Models

The notion of modeling empirical data as a linear combination of exponential signals is popular and widely employed. An exponential signal model may be directly obtained using the nonlinear programming approach taken in the earlier Section on "Gradient and Jacobian Matrix Determination." In particular, the N x 1 basis signal vectors comprising the composite signal matrix take the form

$$\underline{s}(r_m, \omega_m) = \begin{bmatrix} r_m e^{j\omega_m} \\ (r_m)^2 e^{j2\omega_m} \\ \cdot \\ \cdot \\ \cdot \\ (r_m)^N e^{jN\omega_m} \end{bmatrix} \quad for\ 1 \le m \le M \qquad (60)$$

where r_m and ω_m represent the *damping factor* and *frequency* parameters of this basis signal, respectively[2]. These exponential signal vectors form the columns of the composite basis signal matrix $S(\underline{\theta})$. If the data being modeled is real, it is essential that complex conjugate pairs of exponential signal vectors (60) appear in $S(\underline{\theta})$.

To implement the proposed nonlinear programming algorithms presented in the two preceding Sections, it is necessary to compute the partials derivatives of matrix $S(\underline{\theta})$ with respect to the parameters r_m and ω_m. Since these parameters only appear in the column which contains vector $\underline{s}(r_m, \omega_m)$ (and its complex conjugate for real data models), the required partials derivatives are straightforwardly given by

$$\frac{\partial \underline{s}(r_m, \omega_m)}{\partial r_m} = \begin{bmatrix} e^{j\omega_m} \\ 2r_m e^{j2\omega_m} \\ \cdot \\ \cdot \\ \cdot \\ N(r_m)^{N-1} e^{jN\omega_m} \end{bmatrix} \qquad (61)$$

$$\frac{\partial \underline{s}(r_m, \omega_m)}{\partial \omega_m} = \begin{bmatrix} jr_m e^{j\omega_m} \\ j2(r_m)^2 e^{j2\omega_m} \\ \cdot \\ \cdot \\ \cdot \\ jN(r_m)^N e^{jN\omega_m} \end{bmatrix} \qquad (62)$$

With these partial derivatives available, we are in a position to implement the nonlinear programming algorithms of the earlier Section on "Gradient and Jacobian" methods to find the best exponential signal model.

Example 2: *To illustrate the effectiveness of the exponential modeling algorithm, we shall now treat the special case of identifying two complex sinusoids in additive noise. For this purpose, we shall use the example treated by Kumaresan and Tufts in testing their sinusoidal identification method [18]. In particular, the data consisted of the sum of two unit amplitude complex sinusoids*

$$y(n) = e^{j2\pi(0.50n+0.125)} + e^{j2\pi 0.52n} + \omega(n) \qquad (63)$$
$$for \quad 1 \le n \le 25$$

with frequencies f_1=0.50 and f_2=0.52 corrupted by additive zero mean white Gaussian noise w(n) whose real and imaginary components each have variance σ^2. The sinusoid signal-to-noise ratios (SNR) are each specified by 10 $LOG_{10}(1/2\ \sigma^2)$. One hundred inde-

[2] For purposes of presentation simplicity, we here employ only a first order exponential model. Basis signals of higher order (e.g., nz^n) can be incorporated in a similar fashion.

pendent runs of data sequence (63) were generated for each of the signal-to-noise ratios of 0, 5, 10, 15, 20, 25 and 30 dB.

These data sets were used to statistically test the effectiveness of the Gauss-Newton algorithm implementation of a two sinusoid signal model (i.e., $r_1=r_2=1$) in which the unit weighting $W=I_N$ is used. In this testing, the parameter initializing scheme of the preceding Section is employed to obtain initial frequency estimates $(\hat{f}_1^{(0)}, \hat{f}_2^{(0)})$ for each of the one hundred data sets. The grid of frequency values used in this scheme is taken to be $\omega_k = 2\pi k/N$, for $k = 0,1,...,N-1$. It is readily shown that this initializing scheme is equivalent to taking the DFT of the data and choosing the two frequencies that are associated with the two largest Fourier coefficient magnitudes. Using these initial frequency estimates, the Gauss-Newton algorithm with step size selection (44) is employed to generate refined frequency estimates. The algorithm was deemed to have converged when one of the two stopping conditions (45) is first met. The stopping condition scalars were chosen as $\varepsilon_1 = 10^{-10}$ and $\varepsilon_2 = (1/2)^9$. Stopping condition (ii) invariably triggered the algorithm's termination with this convergence typically requiring four iterations.

The quality of the Gauss-Newton generated estimates is compared to that achieved by the Kumaresan-Tufts sinusoidal identification algorithm [18]. In using the KT maximum likelihood behaved method, their suggested optimal prediction filter order L = 18 is chosen. Also in accordance with their method, the two KT frequency estimates are taken to correspond to the angles of the two roots of the KT prediction error filter polynomial that lie closest to the unit circle. The goodness of the frequency estimates obtained using the LSE modeling and KT algorithms is measured by the sum of squared error criterion

$$c_m = \frac{1}{100}\sum_{k=1}^{100}\left[f_m - \hat{f}_m^{(k)}\right]^2 \qquad for\ m=1,2 \qquad (64)$$

for each of the two frequency estimates.

Statistics relating to the results of the one hundred trial runs at each of the SNRs tested are displayed in Table II and in Figs. 2 and 3. Table II lists the sample means of the frequency estimates obtained with the initial frequency estimates, the Gauss-Newton algorithm and the Kumaresan-Tufts method. For all SNR's tested except 0 dB, the LSE modeling algorithms provided virtual unbiased frequency estimates. The statistics arising from the frequency parameter error criterion (64) are displayed in Figs. 2 and 3 in reciprocal fashion along with the associated Cramer-Rao bound (see reference [11]). From these results it is apparent that the parametric sinusoidal model algorithm outperforms the Kumaresan-Tufts method at all SNRs exceeding 0 dB with both approaching the Cramer-Rao bound performance at higher SNRs. Furthermore, the proposed initial parameter selection procedure provided sufficiently good initial frequency estimates in this application.

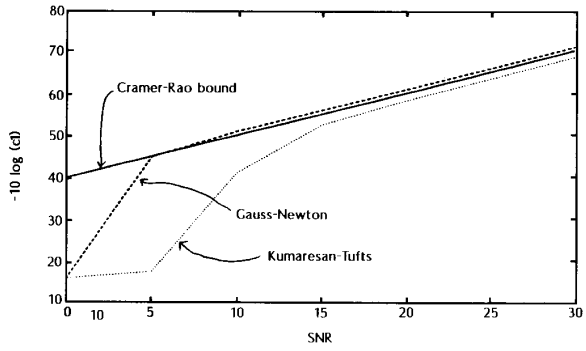


Fig. 2. Reciprocal of the squared frequency error estimates for f_1.

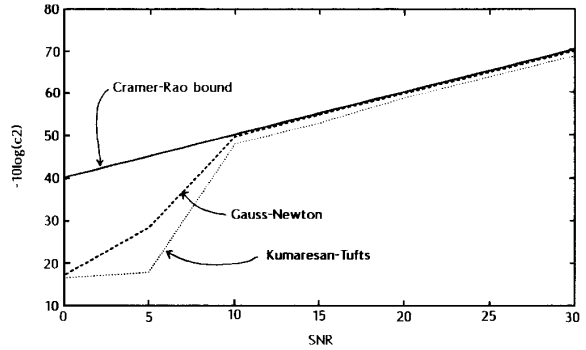


Fig. 3. Reciprocal of the squared frequency error estimates for f_2.

Linear Recursive System Identification

In modeling empirical data, linear recursive systems often provide a particularly powerful means for obtaining useful representations. When using this approach, it is generally assumed that the empirical data is specified by a two-tuple sequence

$$(x_k, y_k) \qquad for\ \ 1 \le k \le N \qquad (65)$$

In standard system identification applications, x_k and y_k represent the measured *excitation* and *output* sequences of the phenomenon being modeled, respectively. On the other hand, in applications where it is desired to approximate a single measured data sequence y_k as a linear combination of damped complex exponentials, the fictitious excitation sequence x_k is taken to be a unit-impulse sequence applied at $k=1$. This is in recognition of the fact that the unit-impulse response of a linear recursive system is equal to a linear combination of damped complex exponentials. An example of this specific application is shortly given.

With the above thoughts in mind, it is now desired to model the two-tuple data (65) by the following linear recursive system of order *(p,q)*

$$\hat{y}_n(\underline{a},\underline{b}) = -\sum_{k=1}^{p} a_k \hat{y}_{n-k}(\underline{a},\underline{b}) + \sum_{k=0}^{q} b_k x_{n-k} \qquad (66)$$
$$for\ \ 1 \le n \le N$$

We have here explicitly expressed the system response $\hat{y}_n(\underline{a},\underline{b})$ as a function of the feedback parameter vector $\underline{a}$

SNR (dB)	f_1^{INIT}	f_2^{INIT}	f_1^{GN}	f_2^{GN}	f_1^{KT}	f_2^{KT}
0	0.4291	0.5890	0.4283	0.5900	0.4562	0.5402
5	0.5050	0.5254	0.5000	0.5284	0.4806	0.5348
10	0.5067	0.5151	0.4999	0.5207	0.4982	0.5197
15	0.5076	0.5138	0.5000	0.5204	0.4994	0.5197
20	0.5081	0.5132	0.5000	0.5202	0.4998	0.5199
25	0.5081	0.5131	0.5000	0.5201	0.4999	0.5199
30	0.5081	0.5131	0.5000	0.5201	0.4999	0.5200

Table II.
Sample Means of Frequency Estimates.

with components a_k and the feedforward parameter vector $\underline{b}$ with components b_k. This is done so as to emphasize this dependency. In using this recursive model, it is tacitly assumed that data sequences x_k and y_k are identically zero for $n \le 0$. If this is not the case in a given application, then one simply reduces the indices over which modeling expression (66) holds to *[p+1,N]* so that each term on the right side is available. For purposes of presentation brevity, we shall not pursue this latter possibility. A straightforward modification of the approach now taken, however, can be made to accommodate for this possibility.

Our objective is to select the linear recursive model's parameters a_k and b_k so that the model's response $\underline{\hat{y}}_n(\underline{a},\underline{b})$ best matches the given data $\underline{y}_n$ in the LSE sense. The minimization problem to be considered is therefore given by

$$\min_{\underline{a},\underline{b}} \left[\underline{y} - \underline{\hat{y}}(\underline{a},\underline{b}) \right]^* W \left[\underline{y} - \underline{\hat{y}}(\underline{a},\underline{b}) \right] \qquad (67)$$

where $\underline{y}$ and $\underline{\hat{y}}(\underline{a},\underline{b})$ are each N x 1 vectors whose elements correspond to the given data (65) and the model response (66), respectively. In order to formulate this problem in the fashion presented in previous sections, we introduce the *auxiliary linear recursive system* as specified by

$$\tilde{y}_n(\underline{a}) = x(n) - \sum_{k=1}^{p} a_k \tilde{y}_{n-k}(a) \qquad for\ 1 \le n \le N \qquad (68)$$

where the auxiliary model's response has been expressed as an explicit function of the feedback parameter vector $\underline{a}$. It is a simple matter to show that the response of linear recursive system (66) is given by the linear combination

$$\hat{y}_n(\underline{a},\underline{b}) = b_0 \tilde{y}_n(\underline{a}) + b_1 \tilde{y}_{n-1}(\underline{a}) + \ldots + b_q \tilde{y}_{n-q}(\underline{a}) \qquad (69)$$

We may now represent relationship (69) in the vector format

$$\underline{\hat{y}}(\underline{a},\underline{b}) = \begin{bmatrix} \tilde{y}_1(\underline{a}) & 0 & \ldots & 0 & 0 \\ \tilde{y}_2(\underline{a}) & \tilde{y}_1(\underline{a}) & \ldots & 0 & 0 \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ \tilde{y}_q(\underline{a}) & \tilde{y}_{q-1}(\underline{a}) & \ldots & \tilde{y}_1(\underline{a}) & 0 \\ \tilde{y}_{q+1}(\underline{a}) & \tilde{y}_q(\underline{a}) & \ldots & \tilde{y}_2(\underline{a}) & \tilde{y}_1(\underline{a}) \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ \tilde{y}_N(\underline{a}) & \tilde{y}_{N-1}(\underline{a}) & \ldots & \tilde{y}_{N+1-q}(\underline{a}) & \tilde{y}_{N-q}(\underline{a}) \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ b_q \end{bmatrix}$$

or more compactly as

$$\underline{\hat{y}}(\underline{a},\underline{b}) = S(\underline{a})\,\underline{b} \qquad (70)$$

The N x $(q+1)$ matrix $S(\underline{a})$ has a Toeplitz lower triangular structure with response entries $\tilde{y}_k(\underline{a})$.

The task of selecting the linear recursive model's parameters so that its response is most compatible to the given data observations (65) has now been posed in a manner consistent with that of the general nonlinear LSE problem previously treated. In particular, we wish to select the linear recursive system's parameter vectors $\underline{a}$ and $\underline{b}$ so as to minimize the squared error criterion

$$f(\underline{a},\underline{b}) = \left[\underline{y} - S(\underline{a})\,\underline{b} \right]^* W \left[\underline{y} - S(\underline{a})\,\underline{b} \right] \qquad (71)$$

In terms of the modeling notations employed in the earlier Section on "Parametric Dependent Basis Signals", the signal parameter vector and signal amplitude vector correspond to the feedback parameter vector $\underline{a}$ and the feedforward parameter vector $\underline{b}$, respectively. We may now use the factorization approach taken in that Section to obtain the reduced dimensioned criterion. In particular, the factorization of $S(\underline{a})$ as specified by

$$S(\underline{a}) = Q(\underline{a})\,R(\underline{a}) \qquad (72)$$

is computed. It follows from the results of the "Parametric Dependent" Section that squared error criterion (71) can be expressed to that of minimizing the reduced dimensioned functional

$$f(\underline{a},\underline{b}^o) = \underline{y}^* W \underline{y} - \underline{y}^* W Q(\underline{a}) Q(\underline{a})^* W \underline{y} \quad (73)$$

Our task is to then select the p x 1 feedback parameter vector $\underline{a}$ so as to minimize this functional. Once an optimum $\underline{a}^o$ has been determined, the corresponding minimum Euclidean norm selection for the feedforward parameter vector $\underline{b}$ is from expression (39) given by

$$b^o = R(\underline{a}^o)^{-1} Q(\underline{a}^o)^* W \underline{y} \quad (74)$$

The determination of an optimum linear recursive model has therefore been reduced to minimizing squared error criterion (73) by a proper choice of the feedback vector $\underline{a}$. If the Newton type algorithms described in an earlier Section are to be used, it is necessary to determine the Jacobian matrix associated with the weighted squared error criterion. This entails determining the partial derivatives of $\underline{Q}(\underline{a})\underline{Q}(\underline{a})^*\underline{y}$ with respect to the a_k coefficients. In accordance with relationship (50), the key requirement is to determine the partial derivatives $\partial S(\underline{a})/\partial\, a_k$.

DETERMINATION OF $\partial S(\underline{a})/\partial a_m$

The determination of the matrices $\partial S(\underline{a})/\partial a_m$ are obtained by taking the partial derivatives of the auxiliary system response elements $\tilde{y}_n(\underline{a})$ which comprise matrix $S(\underline{a})$ as is evident from expression (70). With this in mind, the partial derivatives of auxiliary system relationship (68) with respect to a_m are specified by

$$\frac{\partial \tilde{y}_n(\underline{a})}{\partial a_m} = -\tilde{y}_{n-m}(\underline{a}) - \sum_{k=1}^{p} b_k \frac{\partial \tilde{y}_{n-k}(\underline{a})}{\partial a_m} \quad \text{for } 1 \le n \le N \quad (75)$$

Upon examination of this expression, the required partial derivatives can be interpreted as being the response of the linear system with transfer function

$$H(z) = \frac{-1}{S(z)} = \frac{-1}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_p z^{-p}} \quad (76)$$

to the excitation $\tilde{y}_{n-m}(\underline{a})$. For example, the partial derivatives $\partial \tilde{y}_n(\underline{a})/\partial a_1$ for $1 \le n \le N$ are obtained by using the configuration shown in Fig. 4. This response sequence forms the first column of matrix $\partial S(\underline{a})/\partial a_1$. The remaining columns of this matrix are simply down shifted versions of the first column, that is

$$\frac{\partial S(\underline{a})}{\partial a_1} = \begin{bmatrix} \frac{\partial \tilde{y}_1(\underline{a})}{\partial a_1} & 0 & \ldots & 0 & 0 \\ \frac{\partial \tilde{y}_2(\underline{a})}{\partial a_1} & \frac{\partial \tilde{y}_1(\underline{a})}{\partial a_1} & \ldots & 0 & 0 \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ \frac{\partial \tilde{y}_q(\underline{a})}{\partial a_1} & \frac{\partial \tilde{y}_{q-1}(\underline{a})}{\partial a_1} & \ldots & \frac{\partial \tilde{y}_1(\underline{a})}{\partial a_1} & 0 \\ \frac{\partial \tilde{y}_{q+1}(\underline{a})}{\partial a_1} & \frac{\partial \tilde{y}_q(\underline{a})}{\partial a_1} & \ldots & \frac{\partial \tilde{y}_2(\underline{a})}{\partial a_1} & \frac{\partial \tilde{y}_1(\underline{a})}{\partial a_1} \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ \frac{\partial \tilde{y}_N(\underline{a})}{\partial a_1} & \frac{\partial \tilde{y}_{N-1}(\underline{a})}{\partial a_1} & \ldots & \frac{\partial \tilde{y}_{N+1-q}(\underline{a})}{\partial a_1} & \frac{\partial \tilde{y}_{N-q}(\underline{a})}{\partial a_1} \end{bmatrix} \quad (77)$$

The remaining partial derivatives matrices $\partial S(\underline{a})/\partial a_m$ are related to matrix $\partial S(\underline{a})/\partial a_1$ in a straightforward manner. In particular, relationship (75) indicates that matrix $\partial S(\underline{a})/\partial a_m$ is obtained by simply downshifting each column of $\partial S(\underline{a})/\partial a_1$ by m positions and putting zeros in the vacated first m rows while dropping the last m rows of $\partial S(\underline{a})/\partial a_1$.

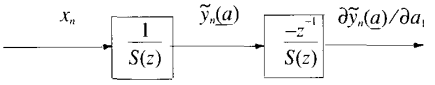


Fig. 4. Determination of partial derivatives.

INITIAL PARAMETER VALUES

As indicated earlier, a critical point in the successful use of any nonlinear programming algorithm is that of generating a good set of parameters to initiate the algorithm. If these initiating parameters are poorly selected, then the resulting parameter vector sequence may converge to a poor relative minimum or it may not converge at all. With this in mind, we now describe a systematic method first presented by Shanks [14] that typically generates acceptably good initial recursive system parameters. It is predicated on the assumption that the system response (66) accurately approximates y_n. If this is true, it follows that by replacing $\hat{y}_n(\underline{a},\underline{b})$ by y_n in relationship (66), the associated model error sequence

$$\varepsilon_n = \sum_{k=0}^{p} a_k y_{n-k} - \sum_{k=0}^{q} b_k x_{n-k} \qquad \text{for } 1 \le n \le N \quad (78)$$

should be sufficiently close to zero where $a_o = 1$. We shall use this hypothesis to obtain a set of parameter values to initiate a nonlinear programming algorithm.

We first formulate the model error equations (78) in the familiar vector format

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_{p+1} \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{p+1} \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{bmatrix} + \begin{bmatrix} 0 & 0 & \ldots & 0 \\ y_1 & 0 & \ldots & 0 \\ \cdot & \cdot & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ y_p & y_{p-1} & \ldots & y_1 \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ y_{N-1} y_{N-2} & \ldots & y_{N-p} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_p \end{bmatrix} - \begin{bmatrix} x_1 & o & 0 & \ldots & 0 \\ x_2 & x_1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ x_{q+1} & x_q & x_{q-1} & \ldots & x_1 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ x_N & x_{N-1} & x_{N-2} & \ldots & x_{N-q} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_q \end{bmatrix} \quad (79)$$

This residual expression can be compactly represented as

$$\underline{\varepsilon} = \underline{y} - [\,-Y \vdots X\,] \begin{bmatrix} \underline{a} \\ \underline{b} \end{bmatrix} = \underline{y} - S \begin{bmatrix} \underline{a} \\ \underline{b} \end{bmatrix} \quad (80)$$

where matrices X and Y are specified in relationship (79). A good set of initial values for the parameter vectors $\underline{a}$ and $\underline{b}$ can generally be obtained by minimizing the sum of squared residuals $\underline{\varepsilon}^*\underline{\varepsilon}$. The required minimum Euclidean norm solution is given by

$$\begin{bmatrix} \underline{a} \\ \underline{b} \end{bmatrix} = S^{+}\underline{Y} \tag{81}$$

where S^{+} designates the pseudo inverse of S.

Example 3: *The above modeling approach is now used to find the best recursive approximation of the causal unit-impulse response component of an ideal low-pass filter. Iterative procedures for solving this class of problems have been advanced by several authors (e.g., see references [1], [6], [17]. Using the inverse Fourier transform of the ideal zero phase low-pass filter with cutoff frequency ω_c, the causal component is specified by*

$$h_n = \begin{cases} 0 & n<0 \\ \dfrac{\omega_c}{\pi} & n=0 \\ \dfrac{\sin(n\omega_c)}{n\pi} & n\geq 1 \end{cases} \tag{82}$$

We shall now approximate this ideal unit-impulse response behavior by a linear recursive filter of order p = q = 6. Since this ideal response is of infinite length, computational considerations dictate that only a truncated version of this ideal sequence be approximated. The length of the truncated should be chosen suitably large so that it contains most of energy of the infinite length sequence.

With these thoughts in mind, the recursive system's parameters are to be selected so that its unit-impulse response (i.e., $x_n = \delta_n$) best approximates h_n over the interval $0 \leq n \leq 127$. The Shank's initializing procedure (81) is used to obtain initial parameter values. The refined recursive parameters generated by the Gauss-Newton algorithm with unit weighting $W=I_N$ are given in Table III. It took 15 iterations for the Gauss-Newton algorithm to converge to these values. The algorithm was deemed to have converged when stopping condition (45i) is first met with $\varepsilon_i = 10^{-10}$.

Plots of the truncated unit-impulse response for the ideal filter and the linear recursive filter of order (6,6) are shown in superimposed fashion in Fig. 5. They are virtually indistinguishable in this plot. The normalized sum of squared error criterion (12) is found to equal 0.0057 for this optimum filter quantifying the close match observed in Fig. 5. Clearly, this low order linear recursive filter provides an excellent approximation of the ideal behavior.

It is readily established that twice the real part of the Fourier transform of the untruncated ideal unit-impulse response (82) equals the frequency response of the ideal low pass filter. Plots of twice the real part of the Fourier transform of the truncated ideal unit-impulse response and the recursive system's frequency response are displayed in Fig. 6. This truncation operation is seen to yield only a close approximation of the ideal low pass frequency response behavior. Upon increasing the length of the truncated ideal unit-impulse response significantly beyond 128, a better approximation may be obtained. This may require increasing the order of the recursive system above (6,6).

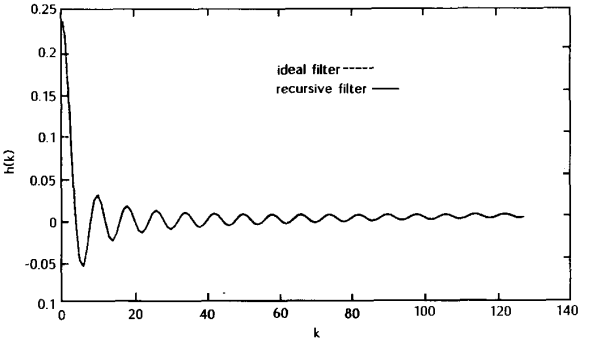


Fig. 5. Unit-impulse response of ideal (dashed line) and recursive (solid line) system model.

Multiple Data Set Modeling Problem

In many signal processing applications, one is given a sequence of N x 1 measurement vectors of the form

$$\underline{y}^{(k)} = \begin{bmatrix} y_1^{(k)} \\ y_2^{(k)} \\ \cdot \\ \cdot \\ \cdot \\ y_N^{(k)} \end{bmatrix} \quad \text{for } 1 \leq k \leq K \tag{83}$$

For example, in array signal processing these vectors

k	0	1	2	3	4	5	6
a_k		-3.6166	36.3769	-6.5215	4.0849	-1.4561	0.2388
b_k	0.2500	-0.6799	0.9416	-0.6998	0.3027	-0.0554	0.0067

Table III.
Coefficients of Optimum LSE Recursive Low Pass Filter.

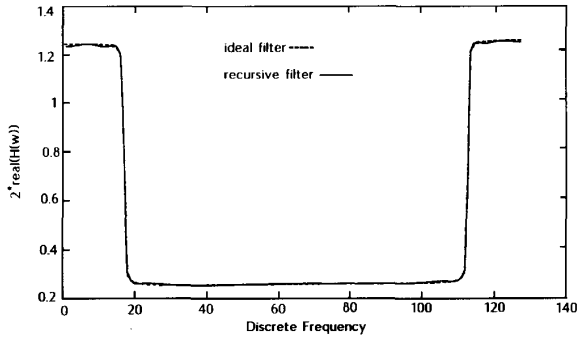


Fig. 6. Twice the "real part" of the Fourier transform of the ideal unit-impulse response (dashed), and of the recursive system model's frequency response (solid).

would correspond to sampled values of the N sensors comprising an array. Whatever the case, it is desired to represent each of these data sequences as a linear combination of the column vectors of the same N x M matrix $S(\underline{\theta})$ whose elements are dependent on a P x 1 real parameter vector $\underline{\theta}$. The associated modeling problem corresponds to finding a parameter vector $\underline{\theta}$ and the K x 1 amplitude vectors $\underline{a}_k$ which render a minimum to the weighted squared error function

$$f(\underline{\theta}, \underline{a}_1, \ldots, \underline{a}_K) = \sum_{k=1}^{K} [\underline{y}^{(k)} - S(\underline{\theta})\underline{a}_k]^* W [\underline{y}^{(k)} - S(\underline{\theta})\underline{a}_k] \qquad (84)$$

A minimizing choice for the above function is readily obtained in a fashion similar to that taken in previous sections. In particular, the Gram-Schmidt generated representation for $S(\underline{\theta})$ is first made as spelled out by relationship (31). It then follows that the optimum selection of $\underline{\theta}$ is obtained by minimizing the functional

$$f(\underline{\theta}, \underline{a}_1^0, \ldots, \underline{a}_K^0) = \sum_{k=1}^{K} \left[\underline{y}^{(k)*} W \underline{y}^{(k)} - \underline{y}^{(k)*} W Q(\underline{\theta}) Q(\underline{\theta})^* W \underline{y}^{(k)} \right] \qquad (85)$$

Once the optimum choice $\underline{\theta}^o$ has been determined, the corresponding optimum values for the amplitude vectors are then given by

$$\underline{a}_k^0 = R(\underline{\theta}^0)^{-1} Q(\underline{\theta}^0)^* W \underline{y}^{(k)} \qquad \text{for} 1 \le k \le K \qquad (86)$$

The nonlinear programming techniques described earlier (i.e., the Newton method and the parameter initialization procedure) may now be employed for minimizing function (85). For example, it is readily shown that the gradient vector associated with this functional is, according to expression (41), given by

$$\nabla f(\underline{\theta}, \underline{a}^0) = real\left\{ \sum_{k=1}^{K} J(\underline{\theta})^* W \left[\underline{y}^{(k)} - Q(\underline{\theta}) Q(\underline{\theta})^* W \underline{y}^{(k)} \right] \right\} \qquad (87)$$

Moreover, the KN x P Jabobian matrix associated with this residual vector is readily obtained. In particular, using the same approach taken in the earlier Section on Newton type algorithms, the Jacobian matrix required to implement the Gauss-Newton or Levenberg-Marquardt algorithms is given by

$$J(\underline{\theta}) = -\begin{bmatrix} \frac{\partial Q(\underline{\theta}) Q(\underline{\theta})^*}{\partial \theta_1} W \underline{y}^{(1)} & \cdots & \frac{\partial Q(\underline{\theta}) Q(\underline{\theta})^*}{\partial \theta_p} W \underline{y}^{(1)} \\ \vdots & & \vdots \\ \frac{\partial Q(\underline{\theta}) Q(\underline{\theta})^*}{\partial \theta_1} W \underline{y}^{(K)} & \cdots & \frac{\partial Q(\underline{\theta}) Q(\underline{\theta})^*}{\partial \theta_p} W \underline{y}^{(K)} \end{bmatrix} \qquad (88)$$

Direction Finding

A natural application of the general problem considered in the previous section is that of *direction finding*. The author has applied this approach to solve the direction finding problem using a LSE procedure (see reference [2]). A brief description of this solution procedure is now given. In the direction finding problem, there exists an array of N sensors located at the points $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_N$ in real three dimensional space. It is assumed that M plane (or spherical) waves are incident on this array and it is desired to use sampled values of the induced sensor signals to estimate the direction-of-arrival of the incident wave fronts. For this purpose, the data vector set (83) corresponds to K samples of the N sensor signals. The so-called N x 1 *snapshot* vector $\underline{y}^{(k)}$ has as its components the $y_n^{(k)}$ which correspond to the k^{th} sample of the n^{th} sensor signal.

For narrowband incident plane waves whose center frequency are each ω_o, it is well known that in the noise free case the K snapshot vectors associated with this problem are each equal to a linear combination of the M *steering vectors* associated with each of the incident plane waves[3]. The steering vector for a plane wave is specified by

$$\underline{s}(\underline{\theta}) = \begin{bmatrix} e^{j\omega_0 \kappa(\underline{\theta})' \underline{z}_1 / c} \\ e^{j\omega_0 \kappa(\underline{\theta})' \underline{z}_2 / c} \\ \cdot \\ \cdot \\ \cdot \\ e^{j\omega_0 \kappa(\underline{\theta})' \underline{z}_N / c} \end{bmatrix} \qquad (89)$$

where ω_o designates the center frequency of the incident narrowband source and c is the medium's velocity of propagation. In this expression, the 3 x 1 vector $\kappa(\underline{\theta})$ corresponds to the unit direction of an incident source as specified by

$$\kappa(\underline{\theta}) = \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) \\ \sin(\theta_1)\cos(\theta_2) \\ \sin(\theta_2) \end{bmatrix} \qquad (90)$$

with θ_1 and θ_2 being the azimuth and elevation angles, respectively, of the incident plane wave.

For this direction-of-arrival problem, the N x M matrix $S(\underline{\theta})$ employed in the earlier Section on "Gradient and

[3] An analogous characterization can be made for broadband plane wave by working in the spectral domain.

Jacobians" has the M steering vectors associated with each incident plane wave as its columns, that is

$$S(\underline{\theta}) = \left[\underline{s}(\underline{\theta}_1) \vdots \ \underline{s}(\underline{\theta}_2) \vdots \ \ldots \vdots \ \underline{s}(\underline{\theta}_M) \right] \qquad (91)$$

where $\underline{\theta}_m$ designates the 2 x 1 vector whose components are the azimuth angle $\theta_m(1)$ and elevation angle $\theta_m(2)$ corresponding to the m^{th} incident plane wave. It is a simple matter to show that the partial derivatives needed to compute the associated Jacobian matrix are directly determined from

$$\frac{\partial \underline{s}(\underline{\theta}_m)}{\partial \theta_m(1)} = j \frac{\omega_o}{c} \begin{bmatrix} \left\{-z_1(1) f_{sc} + z_1(2) f_{cc}\right\} e^{L z_1} \\ \cdot \\ \cdot \\ \cdot \\ \left\{-z_N(1) f_{sc} + z_N(2) f_{cc}\right\} e^{L z_N} \end{bmatrix} \qquad (92)$$

and

$$\frac{\partial \underline{s}(\underline{\theta}_m)}{\partial \theta_m(2)} = j \frac{\omega_o}{c} \begin{bmatrix} \left\{-z_1(1) f_{cc} - z_1(2) f_{ss} + z_1(3) f_c\right\} e^{L z_1} \\ \cdot \\ \cdot \\ \cdot \\ \left\{-z_N(1) f_{cc} - z_N(2) f_{ss} + z_N(3) f_c\right\} e^{L z_N} \end{bmatrix} \qquad (93)$$

where the four trigonometric functions f and the exponent L are defined as follows:

$f_{sc} = \sin(\theta_m(1))\cos(\theta_m(2))$ $\quad f_{cc} = \cos(\theta_m(1))\cos(\theta_m(2))$

$f_{ss} = \sin(\theta_m(1))\sin(\theta_m(2))$ $\quad f_c = \cos(\theta_m(2))$

$L = e^{j\omega \kappa(\theta)'/c}$

Example 4: *To illustrate the effectiveness of LSE techniques for generating direction-of-arrivals, let us consider the special two-dimensional case in which the array sensors are located in the (x,y) plane at the points*

$\underline{z}_1 = [0,-287.5]'$ $\underline{z}_2 = [0,-187.5]'$ $\underline{z}_3 = [0,-87.5]'$ $\underline{z}_4 = [0, 87.5]'$
$\underline{z}_5 = [0, 187.5]'$ $\underline{z}_6 = [0, 287.5]'$ $\underline{z}_7 = [287.5, 0]'$ $\underline{z}_8 = [187.5, 0]'$
$\underline{z}_9 = [87.5, 0]'$ $\underline{z}_{10} = [-87.5, 0]'$ $\underline{z}_{11} = [-187.5, 0]'$ $\underline{z}_{12} = [-287.5, 0]'$

The sensor locations are here expressed in units of feet. Furthermore, let there be two (M=2) unit power narrowband plane waves incident on this array with center frequency 14.85 MHz in which the velocity of propagation is taken to be 3 x 10^8 meters/second. The plane waves travel parallel to the (x,y) plane with direction of arrivals

$$\underline{\theta}_1^{(1)} = 20^0; \ \underline{\theta}_2^{(1)} = 0^0 \qquad \underline{\theta}_1^{(2)} = 23^0; \ \underline{\theta}_2^{(2)} = 0^0 \qquad (94)$$

The snapshot data is generated according to

$$\underline{y}^{(k)} = a_1^{(k)} \underline{s}(\underline{\theta}^{(1)}) + a_2^{(k)} \underline{s}(\underline{\theta}^{(2)}) + \underline{\eta}^{(k)} \qquad (95)$$

For our purpose, the $a_k(n)$ amplitudes characterizing the snapshot data are taken to be random zero mean, unit variance Gaussian processes. Moreover, $\eta^{(k)}$ represents a N x 1 additive independent zero mean Gaussian noise vector. One hundred independent trial runs were made at each of the SNRs of -5,0,5,10,15, and 20 dB. The number of snapshots used in each trial run was set at K = 100. The two incident sources are said to be incoherent if the two Gaussian sequences $a_1(n)$ and $a_2(n)$ are independent and coherent if these sequences have unit correlation. The latter coherence condition is here met by letting $a_1^{(k)} = a_2^{(k)}$. Source coherency plays an important role in direction-of-arrival estimation methods. Some algorithms perform well for incoherent sources but not well for coherent sources (e.g., MUSIC [13]).

The general coordinate descent method described in the Section on "Initial Parameter Value Assignment" was used to generate initial angle of arrival estimates required in the Gauss-Newton algorithm. It is to be noted that this initialization procedure for the direction finding problem was first developed by Ziskind and Wax [19]. Using these initial angle estimates, the Gauss-Newton algorithm with step size selection (44) is used to generate refined DOA angle estimates. The algorithm was deemed to have converged when one of the two stopping conditions (45) is first met. The stopping condition scalars were chosen as $\varepsilon_1 = 10^{-10}$ and $\varepsilon_2 = (1/2)^9$. Stopping condition (ii) invariably triggered the algorithm's termination with this convergence typically requiring three to four iterations. The results of the trial runs are summarized in Fig. 7 for the case of incoherent sources. In this figure, the estimates achieved with the MUSIC algorithm as well as the Cramer-Rao bound are also shown. From these

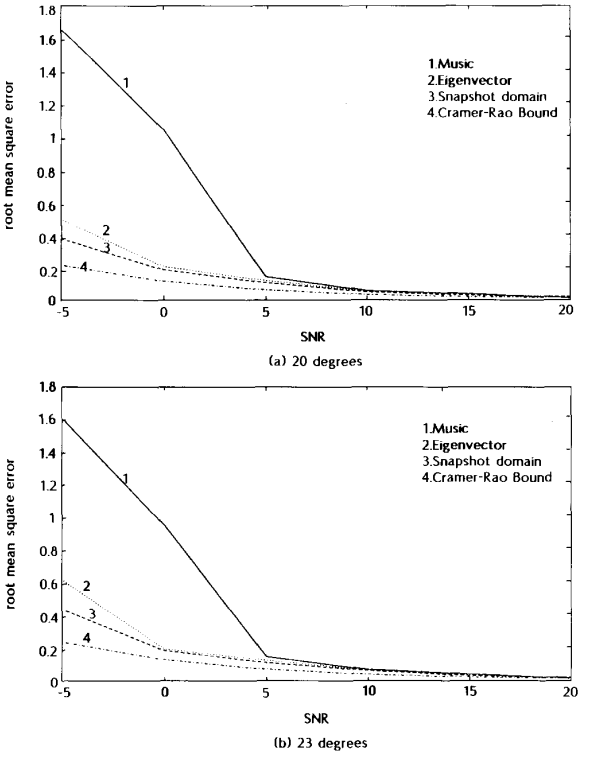


Fig. 7. Root mean squared error of azimuth angle-of-arrival estimates for LSE modeling and MUSIC algorithms: Incoherent Case. 1 = Music; 2 = Eigenvector; 3 = Snapshot domain; 4 = Cramer-Rao bound.

results it is seen that the LSE modeling method provides better performance than MUSIC. In fact, the LSE modeling approach has a performance that approaches the Cramer-Rao bound for the snapshot data. Also shown in this figure are the estimates achieved when an eigenvector LSE modeling approach is taken (see reference [2]).

When the two sources were coherent, the MUSIC algorithm was unable to resolve the two sources for any of the one hundred trial runs. The LSE modeling method, however, was able to achieve a resolution for each run. The root mean squared error behavior for these two frequencies estimates are summarized in Fig. 8 for the coherent case. Not surprisingly, the incoherent case estimates are generally superior to the coherent case estimates.

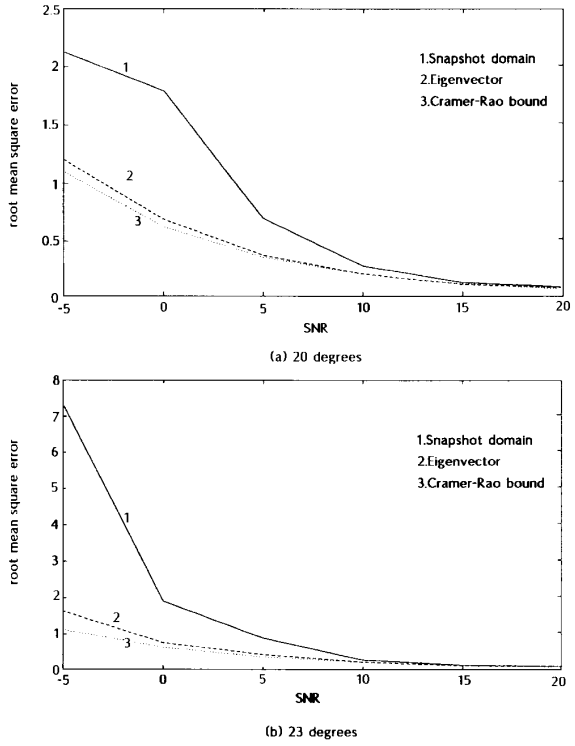


Fig. 8. Root mean squared error of azimuth angle-of-arrival estimates for LSE modeling and MUSIC algorithms: Coherent Case. 1 = Snapshot domain; 2 = Eigenvector; 3 = Cramer-Rao bound.

Conclusion

A systematic LSE modeling procedure for approximating empirically obtained data has been presented. In the approach taken, the data is to be approximated by a linear combination of basis signal vectors constituting a composite signal matrix $S(\underline{\theta})$. These basis signals are user selected and reflect the given application at hand. LSE algorithms for obtaining the optimal choice for the parameter vector $\underline{\theta}$ were developed. Although the Gauss-Newton type algorithm occupied center stage in this development, a variety of other algorithms could have been employed in the same model setting. This is illustrated by the Rosenbrock algorithm as briefly described in the Box. Of critical importance in any of these nonlinear programming techniques was the development of an effective procedure for generating initial parameter estimates. One such method was described in the Section on "Initial Parameters" for an important class of modeling problems. The utility of the LSE modeling approach and the initial parameter estimation procedure were illustrated by several examples.

The Rosenbrock Algorithm

As indicated previously, the most computationally intense component of the proposed LSE modeling approach is usually that of minimizing the weighted squared error functional

$$f(\underline{\theta}, \underline{a}^0) = \underline{y}^* W \underline{y} - \underline{y}^* W Q(\underline{\theta})^* Q(\underline{\theta}) W \underline{y} \qquad (96)$$

Although we have used Gauss-Newton type algorithms for achieving this minimization, other distinctly different algorithms could have been used instead. To demonstrate this point, the little-known Rosenbrock algorithm is now briefly described. This algorithm has the valuable attribute of only requiring evaluations of functional (96) and not its gradient. This can be important in those applications where either a closed formed expression for the gradient is unavailable or where the gradient's evaluation is computationally expensive. When the number of model parameters is modest, it has been the author's experience that the Rosenbrock algorithm performs as effectively as does the Gauss-Newton algorithm on the class of modeling problems considered in this paper.

A description of a typical iteration of the Rosenbrock method for minimizing squared error criterion (96) is now given (see reference [8] for a more thorough description). The following entities will be required

- an initial set of P pairwise P x 1 orthonormal *direction vectors* denoted by $\underline{d}_1^{(0)}, \underline{d}_2^{(0)}, \ldots, \underline{d}_p^{(0)}$. These vectors can be chosen to be the standard basis vector in R^P.
- a set of P step size scalars s_k associated with directions $\underline{d}_k$.
- multiplying scalars $\alpha > 1$ and $0 < \beta < 1$ that are used to adaptively adjust the step sizes. Rosenbrock has suggested the choice $\alpha = 3$ and $\beta = 0.5$.

The current value of the nonlinear signal parameter vector is denoted by $\underline{\theta}^{(k)}$. A step of the Rosenbrock algorithm is said to be *successful* if the perturbed value of functional (96) is less than or equal to its unperturbed value (i.e., $f[\underline{\theta}^{(k)} + s_k \underline{d}_k, \underline{a}^0) \geq f(\underline{\theta}^{(k)}, \underline{a}^0)$). With these preliminaries completed, a standard iteration of the Rosenbrock algorithm is outlined in Table IV.

The Rosenbrock algorithm has proven to be useful on a diversified set of modeling problems. Its utility is primarily based on the requirement that at each iteration improvements are made in each of P orthogonal directions. Furthermore, the first perturbation vector $\underline{d}_1$ is selected to point in the direction of total improvement made at the previous iteration (see step 9). This direction reflects, in some sense, the local direction in which functional (96) decreases most rapidly.

1. Using an appropriate algorithm (e.g., see "Initial Parameter Value Assignment"), compute values for the initial signal parameter vector $\underline{\theta}^{(1)}$ and set $k = 1$.
2. Enter the initial perturbation directions and step size scalars.
3. A step s_k in the direction $\underline{d}_k$ is taken from the prevailing signal vector.
4. If $f(\underline{\theta}^{(k)} + s_k\underline{d}_k, \underline{a}^o) \leq f(\underline{\theta}^{(k)}, \underline{a}^o)$, a *success* is recorded, the step size s_k is multiplied by α and the perturbed parameter vector $(\underline{\theta}^{(k)} + s_k\underline{d}_k)$ replaces the prevailing parameter vector.
5. If $f(\underline{\theta}^{(k)} + s_k\underline{d}_k, \underline{a}^o) > f(\underline{\theta}^{(k)}, \underline{a}^o)$, a *failure* is recorded, the step size is multiplied by $-\beta$, and the prevailing signal parameter is retained.
6. If $k < P$, set $k = k + 1$ and repeat Steps 4 - 6.
7. Repeat Steps 3 - 6 until at least one success followed by a failure is recorded in each of the P directions. When this condition is met, set the prevailing signal parameter vector to $\underline{\theta}^{(k+1)}$.
8. Select a new set of orthonormal direction vectors. The critically important first direction vector $\underline{d}_1$ is selected to be a unit length vector pointing in the direction $\underline{\theta}^{(k+1)} - \underline{\theta}^{(k)}$. The remaining $P - 1$ orthonormal direction vectors can be selected in an arbitrary fashion (e.g., see [8]).
9. If a user-prescribed stopping condition is satisfied, the algorithm is terminated. Otherwise, return to Step 3.

Table IV.
Steps of the Rosenbrock Algorithm.

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James A. Cadzow is a Centennial Professor in the Electrical Engineering Department at Vanderbilt University. He received the B.S. and M.S. degrees in Electrical Engineering from SUNY at Buffalo and the Ph.D. from Cornell University. His research interests include parametric modeling, signal enhancement, direction finding, signal and system identification, and algorithm synthesis. Professor Cadzow has authored over one hundred refereed papers, written four text books and is an IEEE Fellow. He has also been a professor of electrical engineering at Arizona State University, Virginia Polytechnic Institute, and SUNY at Buffalo.

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