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Extracting Physical Parameters of Mechanical Models From Identified State-Space Representations

In this study a new solution for the identification of physical parameters of mechanical systems from identified state space formulations is presented. With the proposed approach, the restriction of having a full set of sensors or a full set of actuators for a complete identification is relaxed, and it is shown that a solution can be achieved by utilizing mixed types of information. The methodology is validated through numerical examples, and conceptual comparisons of the proposed methodology with previously presented approaches are also discussed. [DOI: 10.1115/1.1483836]

1 Introduction

System identification, in the most general sense, can be described as the identification of the conditions and properties of mathematical models that aspire to represent real phenomena in an adequate fashion. The choice of such models is very much dependent on the type of application one considers. In finite element formulations, identification of physical parameters generally refers to the identification of the mass, damping, and stiffness parameters in the second order matrix differential equations. A possible approach is to identify these parameters directly from experimental dynamic data (see, for example, the works of Agbabian et al. [1] and Smyth et al. [2]). However, the most widely employed approach consists in identifying the modal parameters of the system, and to use them to update a pre-existing finite element model. Some of the noteworthy efforts and discussions in this direction are those of Ewins [3], Mottershead and Friswell [4], Berman [5], Baruch [6,7], and Beck and Katafygiotis [8].

The identification of the parameters in a first-order differential equation formulation has also received considerable attention, as evidenced by the works of Ibrahim and Mikulcik [9], Ibrahim [10], Vold et al. [11], Juang and Pappa [12], Juang et al. [13,14], and Luş et al. [15,16]. However, if one starts with a state-space model, and tries to identify the parameters of the second order model, issues such as nonuniqueness of the solution have to be considered, making such an "inverse" problem quite complex.

Usually, the modal parameters required for updating structural models are the undamped (normal) modal parameters, whereas when one works with the first-order formulation, the identified modal parameters are complex, and correspond, in some sense, to the damped modal parameters of the second-order formulation. Therefore, the retrieval of the undamped modal parameters from the identified complex modes also constitutes an important problem, and the study by Sestieri and Ibrahim [17] presents a well-documented discussion. One assumption often employed is that the vibrational modes of the second-order model are uncoupled (modal damping). In this case, arguably the most often employed method to retrieve the undamped modal parameters is the so-called standard technique (e.g., see Imregun and Ewins [18], Ibrahim [19], and Alvin [20]). It is well known, however, that this approximation loses its validity when the system under consideration is highly coupled. To overcome this limitation, many authors have focused their attention on how to retrieve the undamped modal parameters from complex modal parameters for the case of general damping. Some of the most noteworthy discussions include the works of Ibrahim [19], Alvin and Park [21], Zhang and Lallement [22], Yang and Yeh [23], Alvin et al. [24], Tseng et al. [25,26], Chen et al. [27], and Balmès [28].

Taking the inverse problem one step further, one might be interested in directly obtaining the parameters of the second-order model. When one tries to retrieve the second order parameters from the identified state-space model, various methodologies impose different restrictions on the number of sensors and actuators employed, assuming that all the modes of the structure have been successfully identified. The most restrictive requirement is that of having as many sensors and actuators as the number of identified modes, which was discussed by Yang and Yeh [23]. Later on (Alvin and Park [21]) this requirement was improved upon by requiring that only the number of sensors should be equal to the number of identified modes, with one co-located sensor–actuator pair. A further generalization was presented by Tseng et al. [25,26] for the case when the number of actuators is equal to the number of second-order modes, providing the most general solution available for a full set of actuators or sensors, with one co-located sensor–actuator pair.

In this study, we further improve on the requirement concerning the number of sensors and actuators. Based on some concepts previously discussed by Sestieri and Ibrahim [17], and Balmès [28], it is shown that the physical parameters of the second order model can be obtained by using the solution of a symmetric complex eigenvalue problem. The minimum requirement for the proposed methodology is that all the degrees-of-freedom should con-

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tain *either* a sensor *or* an actuator, with at least one co-located sensor–actuator pair. It should be noted that this solution implicitly contains the solutions for the cases with full set of sensors and/or full set of actuators, and therefore, the approach discussed in this study provides a more general solution for the inverse problem.

2 Symmetric Formulation of a First-Order Dynamic Model

One of the most well-known linear time invariant models for dynamical systems is undoubtedly the matrix form of Newton's second law of motion written for discretized spatial domains, i.e.,

$$\mathcal{M}\ddot{\mathbf{q}}(t)+\mathcal{L}\dot{\mathbf{q}}(t)+\mathcal{K}\mathbf{q}(t)=\mathcal{B}\mathbf{u}(t)$$

$$\mathbf{y}(t)=\begin{bmatrix}\mathcal{C}_p\mathbf{q}(t)\\ \mathcal{C}_v\dot{\mathbf{q}}(t)\\ \mathcal{C}_a\ddot{\mathbf{q}}(t)\end{bmatrix} \tag{1}$$

where $\mathbf{q}(t)$ indicates the vector of the (generalized) nodal displacements, with ($\dot{}$) and ($\ddot{}$) representing, respectively, the first and second-order derivatives with respect to time. The vector $\mathbf{u}(t)$, of dimension $r\times 1$, is the input vector containing r external excitations acting on the system while $\mathbf{y}(t)$ represents the measurement vector, which may contain any combination of nodal displacements, velocities, and/or accelerations. For an N-degree-of-freedom system, $\mathcal{M}\in\Re^{N\times N}$, $\mathcal{L}\in\Re^{N\times N}$, and $\mathcal{K}\in\Re^{N\times N}$ are the symmetric positive definite mass, damping, and stiffness matrices, respectively, while $\mathcal{B}\in\Re^{N\times r}$ is the input matrix. The matrix $[\mathcal{C}_p^T\mathcal{C}_v^T\mathcal{C}_a^T]^T\in\Re^{m\times N}$ represents the output matrix that may incorporate position, velocity, and acceleration measurements, with m denoting the total number of outputs.

By defining a state vector $\mathbf{z}(t)=[\mathbf{q}(t)^T\dot{\mathbf{q}}(t)^T]^T$, the equations of motion in (1) can be conveniently written as

$$\begin{bmatrix}\mathcal{L} & \mathcal{M}\\ \mathcal{M} & \mathbf{0}\end{bmatrix}\dot{\mathbf{z}}(t)+\begin{bmatrix}\mathcal{K} & \mathbf{0}\\ \mathbf{0} & -\mathcal{M}\end{bmatrix}\mathbf{z}(t)=\begin{bmatrix}\mathcal{B}\\ \mathbf{0}\end{bmatrix}\mathbf{u}(t) \tag{2a}$$

$$\mathbf{y}(t)=[\mathcal{C}_p\ \ \mathbf{0}]\mathbf{z}(t) \tag{2b}$$

where, for ease of exposition, we have considered only position measurements in the output equation of Eqs. (2). However, the following results are true for any type of measurements (positions, velocities, or accelerations), and the generalization to velocity and acceleration measurements will be discussed in detail in a subsequent section. The advantage of rewriting Eqs. (1) into Eqs. (2) is that now the associated eigenvalue problem is kept symmetric and can be written in a matrix form as

$$\begin{bmatrix}\mathcal{L} & \mathcal{M}\\ \mathcal{M} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\psi}\boldsymbol{\Lambda}\end{bmatrix}\boldsymbol{\Lambda}=\begin{bmatrix}-\mathcal{K} & 0\\ \mathbf{0} & \mathcal{M}\end{bmatrix}\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\psi}\boldsymbol{\Lambda}\end{bmatrix} \tag{3}$$

where $\boldsymbol{\psi}_{N\times 2N}=[\boldsymbol{\psi}_1\boldsymbol{\psi}_2\ldots\boldsymbol{\psi}_{2N}]$ is the matrix containing the eigenvectors of the complex eigenvalue problem

$$(\lambda_i^2\mathcal{M}+\lambda_i\mathcal{L}+\mathcal{K})\psi_i=0$$

and $\boldsymbol{\Lambda}_{2N\times 2N}$ is the diagonal matrix of the complex eigenvalues $\lambda_i=\sigma_i\pm j\omega_i$ (with $j=\sqrt{-1}$). When all the modes of the structure are underdamped, all the eigenvalues appear in complex conjugate pairs, i.e., they can be ordered such that $\lambda_{2i-1}=\lambda_{2i}^*$ with $i=1,2,\ldots,N$, where the superscript (*) denotes complex conjugate. This implies that the complex eigenvectors have the similar property that $\psi_{2i-1}=\psi_{2i}^*$ for $i=1,2,\ldots,N$. In general, these eigenvectors can be arbitrarily scaled; however, if the scaling is chosen such that (see Sestieri and Ibrahim [17] and Balmès [28])

$$\boldsymbol{\psi}^T\mathcal{M}\boldsymbol{\psi}\boldsymbol{\Lambda}+\boldsymbol{\Lambda}\boldsymbol{\psi}^T\mathcal{M}\boldsymbol{\psi}+\boldsymbol{\psi}^T\mathcal{L}\boldsymbol{\psi}=\mathbf{I}$$

$$\boldsymbol{\Lambda}\boldsymbol{\psi}^T\mathcal{M}\boldsymbol{\psi}\boldsymbol{\Lambda}-\boldsymbol{\psi}^T\mathcal{K}\boldsymbol{\psi}=\boldsymbol{\Lambda}$$

or in matrix form

$$\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\psi}\boldsymbol{\Lambda}\end{bmatrix}^T\begin{bmatrix}\mathcal{L} & \mathcal{M}\\ \mathcal{M} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\psi}\boldsymbol{\Lambda}\end{bmatrix}=\mathbf{I} \tag{4a}$$

$$\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\psi}\boldsymbol{\Lambda}\end{bmatrix}^T\begin{bmatrix}\mathcal{K} & \mathbf{0}\\ \mathbf{0} & -\mathcal{M}\end{bmatrix}\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\psi}\boldsymbol{\Lambda}\end{bmatrix}=-\boldsymbol{\Lambda} \tag{4b}$$

then, for a *proportionally damped system*, the real and imaginary parts of the components of these complex eigenvectors are equal in magnitude.

Once the symmetric eigenvalue problem (Eqs. (4)) has been solved, we can now conveniently rewrite Eqs. (2) by using the transformation $\mathbf{z}(t)=[\boldsymbol{\psi}^T(\boldsymbol{\psi}\boldsymbol{\Lambda})^T]^T\boldsymbol{\zeta}(t)$ so that

$$\dot{\boldsymbol{\zeta}}(t)=\boldsymbol{\Lambda}\boldsymbol{\zeta}(t)+\boldsymbol{\psi}^T\mathcal{B}\mathbf{u}(t) \tag{5a}$$

$$\mathbf{y}(t)=\mathcal{C}_p\boldsymbol{\psi}\boldsymbol{\zeta}(t). \tag{5b}$$

For ease of exposition, let us indicate with $\mathbf{M}(k,:)$ and $\mathbf{M}(:,l)$ the kth row and the ith column, respectively, of a generic matrix $\mathbf{M}$. The equations of motion rewritten in form (5) have the important property that, for a generic ith degree-of-freedom that contains a co-located sensor–actuator pair,

$$\mathcal{C}_p(i,:)\boldsymbol{\psi}=[\boldsymbol{\psi}^T\mathcal{B}(:,i)]^T, \tag{6}$$

and this property will be of great use (1) for determining and scaling the eigenvectors, and (2) for developing the concept of input-output equivalence, as presented in detail in Section 4.

3 Identification of the Physical Parameters of the System

The proposed identification algorithm consists of two well-defined phases: (1) the determination of a first-order model of the system, and (2) the transformation of such an identified model into a second-order model.

From general input-output data, it is possible to identify a state space realization in some arbitrary basis, and such a realization can be expressed as

$$\dot{\mathbf{x}}(t)=\mathbf{A}_C\mathbf{x}(t)+\mathbf{B}_C\mathbf{u}(t)$$

$$\mathbf{y}(t)=\mathbf{C}_C\mathbf{x}(t)+\mathbf{D}_C\mathbf{u}(t) \tag{7}$$

where now $\mathbf{A}_C\in\Re^{2N\times 2N}$, $\mathbf{B}_C\in\Re^{2N\times r}$, $\mathbf{C}_C\in\Re^{m\times 2N}$, and $\mathbf{D}_C\in\Re^{m\times r}$ are continuous time system matrices. In this study, an ERA/OKID based approach, as discussed by Juang et al. [13,14] and Luş et al. [15,16], was used for the identification of the discrete time system matrices (namely the matrices $\boldsymbol{\Phi}$, $\boldsymbol{\Gamma}$, $\mathbf{C}$, and $\mathbf{D}$), and these discrete time matrices were converted to their continuous time counterparts using the zero-order hold assumption. By considering the transformation $\mathbf{x}=\boldsymbol{\varphi}\boldsymbol{\theta}$, the continuous time system of Eqs. (7) can also be written in modal coordinates as

$$\dot{\boldsymbol{\theta}}(t)=\boldsymbol{\Lambda}\boldsymbol{\theta}(t)+\boldsymbol{\varphi}^{-1}\mathbf{B}_C\mathbf{u}(t) \tag{8a}$$

$$\mathbf{y}(t)=\mathbf{C}_C\boldsymbol{\varphi}\boldsymbol{\theta} \tag{8b}$$

where the matrix $\boldsymbol{\Lambda}$ contains the continuous time eigenvalues of the identified state space model, and $\boldsymbol{\varphi}$, of order $2N\times 2N$, is the matrix of the corresponding eigenvectors. The matrix $\mathbf{D}_C$ has been omitted in Eq. (8*b*) because it is independent of coordinate transformations. It is noteworthy that in the system of Eqs. (8), the products $\boldsymbol{\varphi}^{-1}\mathbf{B}_C$ and $\mathbf{C}_C\boldsymbol{\varphi}$ appear; these products impose a strong limitation on the order of the second-order model to be identified, whose dimensions are now constrained either by the number of actuators, or by the number of sensors (Tseng et al. [25,26]).

If the first-order system of Eqs. (7) was identified using data that actually came from the second-order model of Eq. (1), the models represented by Eqs. (5) and (8) are different models of the same system. Therefore, we look for a transformation matrix, $\mathcal{T}$, that relates these two representations, i.e.:

$$\mathcal{T}^{-1}\boldsymbol{\Lambda}\mathcal{T}=\boldsymbol{\Lambda} \tag{9a}$$

$$\mathcal{T}^{-1}\boldsymbol{\varphi}^{-1}\mathbf{B}_C=\boldsymbol{\psi}^T\mathcal{B} \tag{9b}$$

$$\mathbf{C}_C\boldsymbol{\varphi}\mathcal{T}=\mathcal{C}_p\boldsymbol{\psi}. \tag{9c}$$

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If there are no repeated roots, it is easy to show that the transformation matrix is diagonal, i.e., $\mathcal{T}=\mathrm{diag}(t_1,t_2,\ldots,t_{2N})$ and its values are complex conjugate. By examining Eqs. (9), it is clear that the matrix $\mathcal{T}$ has a twofold effect: (1) to transform the eigenvectors from those of a nonsymmetric eigenvalue problem to those of a symmetric eigenvalue problem, and (2) to properly scale such eigenvectors. Here we discuss the identification of this transformation matrix $\mathcal{T}$ and the eigenvectors ψ when there are no repeated roots, and the input and output matrices ($\mathcal{B}$ and $\mathcal{C}_p$, respectively) of the finite element model are known. These input and output matrices are assumed to contain binary information, i.e., in the case of the input matrix $\mathcal{B}$, the coefficient in the ith row ($i=1,2,\ldots,N$) and jth column ($j=1,2,\ldots,r$) of $\mathcal{B}$ is 1 if the jth actuator is placed on the ith degree-of-freedom and this coefficient is 0 if the jth actuator is not placed on the ith degree-of-freedom. Similarly, the coefficient in the ith row ($i=1,2,\ldots,m$) and jth column ($j=1,2,\ldots,N$) of the output matrix $\mathcal{C}_p$ is 1 if the ith sensor is placed on the jth degree-of-freedom and this coefficient is 0 if the ith sensor is not placed on the jth degree-of-freedom.

To present the proposed methodology in a concise manner, let us assume that the input and output matrices of both representations (in Eqs. (9*b*) and (9*c*)) have been expanded to incorporate all the degrees-of-freedom. This is most easily achieved by incorporating columns of zeros in the input matrices ($\mathbf{B}_C$ and $\mathcal{B}$) and rows of zeros in the output matrices ($\mathbf{C}_C$ and $\mathcal{C}_p$) for the degrees-of-freedom that are either not excited or not measured. Furthermore, assume that these input and output matrices have been arranged so that the ith column of the input matrix corresponds to the ith degree-of-freedom (and hence there will be a column of zeros if there is no actuator placed on the ith degree-of-freedom), and similarly, the ith row of the output matrix corresponds to the ith degree-of-freedom (a row of zeros if there is no sensor on the ith degree-of-freedom). Now the previous transformation Eqs. (9) can be written in an ''expanded'' form as

$$\mathcal{T}^{-1}\Lambda\mathcal{T}=\Lambda \tag{10a}$$

$$\mathcal{T}^{-1}\varphi^{-1}\mathbf{B}_C^E=\psi^T\mathcal{B}^E \tag{10b}$$

$$\mathbf{C}_C^E\varphi\mathcal{T}=\mathcal{C}_p^E\psi \tag{10c}$$

where $\mathbf{B}_C^E$, $\mathcal{B}^E$, $\mathbf{C}_C^E$, and $\mathcal{C}_p^E$ are the expanded versions of the matrices $\mathbf{B}_C$, $\mathcal{B}$, $\mathbf{C}_C$, and $\mathcal{C}_p$, respectively.

The identification of the transformation matrix $\mathcal{T}$ and the properly scaled complex eigenvectors ψ can be investigated by studying a general limit case, since it can be shown that the case of full set of sensors and the case of full set of actuators are special cases of the general approach. Let us assume that each degree-of-freedom contains *either* an actuator *or* a sensor, with one degree-of-freedom containing a co-located sensor–actuator pair (hence $r+m=N+1$). With the notation introduced in Section 2, if the co-located sensor–actuator pair is at the ith degree-of-freedom the well-known co-location requirement can be written as

$$\mathcal{C}_p^E(i,:)\psi=(\psi^T\mathcal{B}^E(:,i))^T. \tag{11}$$

Using the co-location requirement, the transformation matrix $\mathcal{T}$ can be evaluated from Eqs. (10*b*), (10*c*), and Eq. (11) as

$$\mathbf{C}_C^E(i,:)\varphi\mathcal{T}=(\mathcal{T}^{-1}\varphi^{-1}\mathbf{B}_C^E(:,i))^T;$$
$$\mathbf{C}_C^E(i,:)\varphi\mathcal{T}^2=(\varphi^{-1}\mathbf{B}_C^E(:,i))^T. \tag{12}$$

Since the matrix $\mathcal{T}$ is diagonal, each t_i ($i=1,2,\ldots,2N$) can be uniquely determined from Eq. (12). Once these scaling factors are obtained, what is left to be determined is the complex eigenvector matrix ψ.

The information pertaining to a certain degree of freedom is embedded either in the input matrix or in the output matrix. Going back to Eqs. (10), the output matrices in Eq. (10*c*) essentially contain information about only m degree-of-freedom (with $m<N$). If there is a sensor at the kth degree-of-freedom then the kth row of the matrix ψ can be evaluated using Eq. (10*c*), i.e.,

$$\psi(k,:)=\mathbf{C}_C^E(k,:)\varphi\mathcal{T}. \tag{13}$$

On the other hand, if there is no sensor at the kth degree-of-freedom then $\mathcal{C}_p^E(k,:)\psi=\mathbf{0}_{1\times 2N}$. However, if a degree-of-freedom is instrumented with either a sensor or an actuator, the kth row of the matrix ψ can be evaluated using Eq. (10*b*) as

$$\psi(k,:)=(\mathcal{T}^{-1}\varphi^{-1}\mathbf{B}_C^E(:,k))^T. \tag{14}$$

Clearly, this argument is valid for all the N degrees-of-freedom and so all the rows of the matrix ψ can be evaluated. It should be noted that, for the ith degree-of-freedom that contained the co-located sensor–actuator pair, one can use either Eq. (13) or Eq. (14), since they lead to the same result by the co-location requirement in Eq. (11).

If there is a full set of sensors ($\mathrm{rank}(\mathcal{C}_p)=N$, $\mathcal{C}_p^E\equiv\mathcal{C}_p$, and $\mathbf{C}_C^E\equiv\mathbf{C}_C$), or a full set of actuators ($\mathrm{rank}(\mathcal{B})=N$, $\mathcal{B}^E\equiv\mathcal{B}$, and $\mathbf{B}_C^E\equiv\mathbf{B}_C$), the scaling factors are still evaluated from Eq. (11). Once the scaling factors are evaluated, one can identify the complex eigenvector matrix ψ using

$$\mathcal{C}_p^{-1}\mathbf{C}_C\varphi\mathcal{T}=\psi \tag{15}$$

when there is a full set of sensors, or

$$\mathcal{T}^{-1}\varphi^{-1}\mathbf{B}_C\mathcal{B}^{-1}=\psi^T \tag{16}$$

when there is a full set of actuators. Clearly, these two cases can be regarded as special cases of the general formulation presented in this section.

Once the properly scaled eigenvector matrix ψ is evaluated, the mass, damping, and stiffness matrices of the finite element model can be obtained using the orthogonality conditions in Eqs. (4). As discussed in Balmès [28], algebraic manipulations on Eqs. (4) leads to the following identities:

$$\begin{bmatrix}\mathcal{L} & \mathcal{M}\\ \mathcal{M} & \mathbf{0}\end{bmatrix}^{-1}=\begin{bmatrix}0 & \mathcal{M}^{-1}\\ \mathcal{M}^{-1} & -\mathcal{M}^{-1}\mathcal{L}\mathcal{M}^{-1}\end{bmatrix}=\begin{bmatrix}\psi\\ \psi\Lambda\end{bmatrix}\begin{bmatrix}\psi\\ \psi\Lambda\end{bmatrix}^T=\begin{bmatrix}\psi\psi^T & \psi\Lambda\psi^T\\ \psi\Lambda\psi^T & \psi\Lambda^2\psi^T\end{bmatrix} \tag{17a}$$

$$\begin{bmatrix}\mathcal{K} & \mathbf{0}\\ \mathbf{0} & -\mathcal{M}\end{bmatrix}^{-1}=\begin{bmatrix}\mathcal{K}^{-1} & \mathbf{0}\\ \mathbf{0} & -\mathcal{M}^{-1}\end{bmatrix}=-\begin{bmatrix}\psi\\ \psi\Lambda\end{bmatrix}\Lambda^{-1}\begin{bmatrix}\psi\\ \psi\Lambda\end{bmatrix}^T=-\begin{bmatrix}\psi\Lambda^{-1}\psi^T & \psi\psi^T\\ \psi\psi^T & \psi\Lambda\psi^T\end{bmatrix}. \tag{17b}$$

In order for Eqs. (17) to be valid, it is necessary that

$$\mathcal{M}=(\psi\Lambda\psi^T)^{-1},\quad \mathcal{L}=-\mathcal{M}\psi\Lambda^2\psi^T\mathcal{M}, \tag{18a}$$

$$\mathcal{K}=-(\psi\Lambda^{-1}\psi^T)^{-1},\quad \psi\psi^T=\mathbf{0} \tag{18b}$$

and Eqs. (18) provide the required expressions for the mass, damping and stiffness matrices of the second-order model of the system.

3.1 Observations. There is a sign choice for the square roots when one solves for the scaling factors in $\mathcal{T}$ (see Eqs. (12)); however, this does not have any effect on the identified mass, damping, and stiffness matrices. To investigate this point, first let us note that a sign change in the scaling factor t_i causes a sign

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change in the ith complex mode ψ_i. This sign change in ψ_i has no effect on the mass matrix, since the expression for the mass matrix can be written as

$$\begin{aligned}\mathcal{M}=(\psi\lambda\psi^T)^{-1}&=(\lambda_1\psi_1\psi_1^T+\lambda_2\psi_2\psi_2^T+\ldots+\lambda_{2N}\psi_{2N}\psi_{2N}^T)^{-1}\\&=(\lambda_1(-\psi_1)(-\psi_1^T)+\lambda_2(-\psi_2)(-\psi_2^T)+\ldots\\&\quad+\lambda_{2N}(-\psi_{2N})(-\psi_{2N}^T))^{-1}\end{aligned}\tag{19}$$

and this expression is clearly invariant under a sign change for any of the complex eigenvectors. Analogous arguments can be used to show that the damping matrix $\mathcal{L}$ and the stiffness matrix $\mathcal{K}$ are also invariant under a sign change for the ψ_i ($i=1,2,\ldots,2N$).

On the other hand, a change in the ordering of the rows of the complex eigenvector matrix ψ changes the final form of the mass, damping, and stiffness matrices in the sense that two different ordering schemes lead to two different sets that differ only by the arrangement of rows and columns. In fact, if we consider the expression in Eq. (19) for the mass matrix, an interchange between the kth and lth rows of ψ clearly leads to an interchange between the kth and lth rows and columns of the mass matrix. However, this rearrangement also takes place in the damping and the stiffness matrices. In conclusion, this nonuniqueness is equivalent to the reordering of the degrees-of-freedom in the representation of Eq. (1).

In the foregoing discussion, it was assumed that there was only one co-located sensor–actuator pair, but in general, it is possible to have more co-located sensors and actuators. These extra conditions are redundant if the system is noise free, i.e., the scaling factors obtained by investigating one co-located sensor–actuator pair also satisfies the co-location requirement of any other co-located sensor–actuator pair. However, in the presence of noise, it might be best to proceed with a least-squares approach to obtain the entries of the matrix $\mathcal{T}$ (for a thorough investigation on the effects of noise on the proposed approach, the reader is referred to the work of Luş [29]).

If instead of displacement measurements one uses velocity or acceleration measurements, the output equation in Eqs. (5) can be rewritten as

- for velocity measurements:

$$\mathbf{y}(t)=[\mathbf{0}\ \ \mathcal{C}_v]\begin{bmatrix}\psi\\ \psi\Lambda\end{bmatrix}\zeta(t)=\mathcal{C}_v\psi\Lambda\zeta(t)\tag{20}$$

- for acceleration measurements:

$$\mathbf{y}(t)=[\mathbf{0}\ \ \mathcal{C}_a]\begin{bmatrix}\psi\\ \psi\Lambda\end{bmatrix}\zeta(t)=\mathcal{C}_a\psi\Lambda^2\zeta(t)+\mathcal{C}_a\psi\Lambda\psi^T\mathcal{B}\mathbf{u}(t).\tag{21}$$

Clearly, these changes lead to some alterations in Eq. (9c), according to the type of measurements used:

$$\mathbf{C}_C\varphi\mathcal{T}=\mathcal{C}_v\psi\Lambda\quad\text{for velocity measurements}$$

$$\mathbf{C}_C\varphi\mathcal{T}=\mathcal{C}_a\psi\Lambda^2\quad\text{for acceleration measurements.}$$

Analogous to the output matrix $\mathcal{C}_p$, the output matrices $\mathcal{C}_v$ and $\mathcal{C}_a$ also contain binary information (as discussed in Section 3). Therefore, all we have to do to use the algorithms and discussions of Section 3 is to use $\mathbf{C}_C\varphi\Lambda^{-1}$ in Eq. (9c) for velocity measurements or $\mathbf{C}_C\varphi\Lambda^{-2}$ in the case of acceleration measurements. It is noteworthy that, in the case of acceleration measurements, only the first term enters in the identification process while the second term, independent of the transformation matrix, needs to be accounted only for simulation purposes.

In general, one can possibly use all types of measurements simultaneously, and in that case each row of the matrix $\mathbf{C}_C\varphi$ must be handled separately with regards to the changes discussed above. Once appropriate alterations are made according to the type of sensor one uses, the formulations and discussions presented in Section 3 remain unchanged.

4 Concept of Input-Output Equivalence

The formulation presented in this study has one main advantage over previous studies, in the sense that the methodology presented here has more general theoretical implications about the number of sensors or actuators that can be used in dynamic testing. In order to clarify this point, let us consider an N degree-of-freedom system. By taking the Laplace transform of Eqs. (5) and by combining the two transformed equations, it is possible to obtain an expression that relates the input transform vector, $\mathbf{U}(s)$, and the output transform vector, $\mathbf{Y}(s)$, as

$$\mathbf{Y}(s)=\mathcal{C}_p\psi[s\mathbf{I}-\Lambda]^{-1}\psi^T\mathcal{B}\mathbf{U}(s)=\mathcal{C}_p\mathbf{H}(s)\mathcal{B}\mathbf{U}(s)\tag{22}$$

where the matrix $\mathbf{H}(s)$, of dimension $N\times N$, represents the transfer function matrix of the system. The complete knowledge of $\mathbf{H}(s)$ would allow one to determine the response of the system at any point for an arbitrary input applied at any degree-of-freedom creating a complete predictive model of the system. Hence, the goal of any identification methodology should be the determination of the matrix $\mathbf{H}(s)$. For this purpose, the well-known property that $\mathbf{H}(s)$ is a symmetric matrix will be of great help. Again, for ease of presentation, we consider only displacement measurements but analogous formulations can be derived for velocity and acceleration measurements, as shown before.

Let us first consider the case where, in the identification process, we have N outputs and N inputs available ($m=N$ and $r=N$). This will correspond to the case of N co-located pairs of sensors and actuators. In the notation of Section 3, this case corresponds to having $\mathcal{C}_p\equiv\mathcal{C}_p^E$ and $\mathcal{B}\equiv\mathcal{B}^E$ and the matrix $\mathbf{H}(s)$ is directly determined.

If the system has been identified using N outputs and 1 input ($m=N$ and $r=1$) with the ith output co-located with the input, only the ith column of the transfer function matrix $\mathbf{H}(s)$ can be *directly* identified. This will be equivalent to knowing the matrix ψ, since in Eqs. (22) the matrix $\mathcal{C}_p$ is the identity matrix, and, consequently, the entire transfer function matrix can be obtained. In this case (N outputs and 1 input), it is well known that the physical parameters of the second-order system of Eqs. (1) can be retrieved from the identified state-space model, as discussed previously by many authors (see, e.g., the works of Alvin and Park [21] or Tseng et al. [25,26]).

On the other hand, if the identified system has N inputs and 1 output ($m=1$ and $r=N$) with the ith input co-located with the ith output, only the ith row of the transfer function matrix $\mathbf{H}(s)$ can be *directly* identified. In this case, the matrix $\mathcal{B}$ in Eqs. (22) is the identity matrix, and analogous to the previous case, it is possible to completely determine the matrix $\mathbf{H}(s)$. A solution for this case was presented by Tseng et al. [25,26].

In system identification literature, these two previous cases are considered as the two limit cases. In fact, there is no methodology available that allows us to combine information coming from m outputs and r inputs, and the possibility of combining these two types of information is one of the innovations of the proposed approach. To present this generalization, let us identify an N degree-of-freedom system with m outputs and r inputs (with $m<N$ and $r<N$ and $m+r=N+1$), with one co-located sensor–actuator pair on the ith degree-of-freedom. At this point it is useful to remind the importance of having at least one pair of co-located sensor and actuator for the determination of the transformation matrix $\mathcal{T}$, which leads to the presence of $+1$ in the $m+r=N+1$ condition. What is noteworthy in this case is the fact that neither $\mathcal{C}_p$ nor $\mathcal{B}$ are square (identity) matrices and this implies that neither a column nor a row of $\mathbf{H}(s)$ is fully identified.

Due to the co-located sensor–actuator pair at the ith degree-of-freedom the entry at the ith row and ith column of $\mathbf{H}(s)$ ($H_{ii}(s)$) is identified. Now, if we consider an input on the lth degree-of-freedom and an output on the kth degree-of-freedom, we are capable of determining $H_{kl}(s)$, which represents the component of $\mathbf{H}(s)$ on the kth row and lth column. The main innovation in this study is that the formulations developed herein allow us to use the

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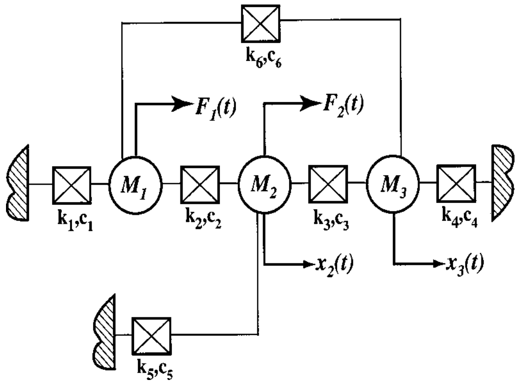

Fig. 1 Three-degree-of-freedom system considered for the application of the proposed approach

property that $\mathbf{H}(s)$ is symmetric, and hence even though we have not identified the component $H_{lk}(s)$, we can use $H_{kl}(s)$ instead. Therefore, if all the degrees-of-freedom have either an actuator or a sensor, the entire ith row and/or ith column of $\mathbf{H}(s)$ can be determined directly. This implies that it is possible to transform the general case of m sensors and r actuators to an equivalent case of a full set of sensors or of a full set of actuators. This has been possible because of the concept of ''input–output equivalence,'' so that for this methodology, it is indifferent to have either an input or an output at each degree-of-freedom.

This concept of input–output equivalence is possible because of the particular eigenvector basis discussed, i.e., the eigenvectors for the symmetric eigenvalue problem of the system in Eqs. (2). On the other hand, if we were to use the eigenvectors of the nonsymmetric problem, the transpose of the eigenvector matrix in Eqs. (9a) would be replaced with the inverse of the matrix $\boldsymbol{\varphi}$ (dimension $2N\times 2N$), and hence, we would be limited to the case of either a full set of sensors (Alvin and Park [21] or Tseng et al. [25,26]) or a full set of actuators (Tseng et al. [25,26]).

5 Numerical Examples

To show the validity of the proposed approach, first a simple but general numerical example is presented. The system, shown in Fig. 1, has been previously studied by Agbabian et al. [1] and Koh and See [30]; the values for the mass and stiffness matrices used in this study are given in Table 1.

To consider the effects of the modal coupling on the structure of the eigenvectors, we consider two different damping matrices, as shown in Table 2. The first one leads to the more classical case of modal damping. The second matrix instead induces coupling of the second-order vibrational modes, and therefore, more conventional methods that employ the modal damping assumption are not applicable. Furthermore, we assume that the system is excited by only two actuators, located at the first and the second degrees-of-freedom and that accelerations are also measured only at two degrees-of-freedom (second and third degrees-of-freedom). With this particular setup, methodologies that require either a full set of sensors, or a full set of actuators, are also not applicable.

The state-space model is identified using the simulated pulse response data of the system (with a sampling time of $\Delta T = 0.05$ sec.), and by employing the ERA/DC algorithm (Juang et al. [13]). Using the identified state-space models for both the coupled and the uncoupled cases, the scaling factors in $\mathcal{T}$, the eigenvectors $\boldsymbol{\psi}$, and the mass, damping, and stiffness matrices of the second-order model ($\mathcal{M}$, $\mathcal{L}$, and $\mathcal{K}$, respectively) are retrieved using the methodology presented in this work.

Table 1 Mass and stiffness matrices used for the system of Fig. 1

Mass			Stiffness		
0.8	0.0	0.0	4.0	-1.0	-1.0
0.0	2.0	0.0	-1.0	4.0	-1.0
0.0	0.0	1.2	-1.0	-1.0	4.0

Table 2 Damping matrices leading to uncoupled and coupled second-order vibrational modes for the system of Fig. 1

uncoupled			coupled		
0.4	-0.1	-0.1	0.5	-0.1	-0.2
-0.1	0.4	-0.1	-0.1	0.7	-0.3
-0.1	-0.1	0.4	-0.2	-0.3	0.6

Table 3 Identified discrete time matrices of the state-space model for the uncoupled damping case

$\boldsymbol{\Phi}\times 10$

9.9520	0.0956	-0.6987	0.2243	0.0072	0.0178
-0.0761	9.8881	-0.5836	-0.6557	0.0264	0.0032
0.6886	0.5640	9.9450	-0.0075	-0.4241	-0.0190
-0.2833	0.7991	0.0135	9.8893	0.1779	-0.0022
0.1586	-0.0056	0.4669	-0.1255	9.6269	0.8757
0.0158	-0.0335	-0.0190	0.0679	-1.0219	9.9322

$\mathbf{C}\times 10$

1.6132	-1.2168	0.4631	-1.1516	0.1377	-0.0220
1.4748	0.6048	1.8841	0.2602	0.2798	-0.0377

$\boldsymbol{\Gamma}^T\times 10$

0.4917	0.0904	0.2763	0.0800	-2.5024	-0.4362
-0.4112	1.2005	0.2179	-1.2017	0.1888	0.3338

5.1 Uncoupled Second-Order Modes. For this case, the identified system matrices for the discrete time state space model are presented in Table 3. Once these matrices have been obtained, they are converted to their continuous time counterparts, and the equations are written in the modal coordinates, as in Eqs. (8). At this point, it is possible to calculate the diagonal transformation matrix $\mathcal{T}$ using the information at the co-located sensor–actuator pair, leading to: $\mathrm{diag}(\mathcal{T})=(2.966\mp j2.322,\ 8.996\mp j8.164,\ 6.449\mp j4.789)$, where $\mathrm{diag}(\mathcal{T})$ refers to the components on the main diagonal of the transformation matrix $\mathcal{T}$ (with all off-diagonal terms equal to zero). As expected, they appear in complex conjugate pairs.

Once these scaling factors have been evaluated, the eigenvector matrix $\boldsymbol{\psi}$ can be identified, as discussed in Section 3. The eigenvector matrix has the form $\boldsymbol{\psi}=[\boldsymbol{\psi}_1\boldsymbol{\psi}_1^*\boldsymbol{\psi}_2\boldsymbol{\psi}_2^*\boldsymbol{\psi}_3\boldsymbol{\psi}_3^*]$ and for this case the identified complex eigenvectors $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$, and $\boldsymbol{\psi}_3$ are

$$\boldsymbol{\psi}_1=\begin{bmatrix}-0.159-j\ 0.159\\-0.276-j\ 0.276\\-0.185-j\ 0.185\end{bmatrix};\quad \boldsymbol{\psi}_2=\begin{bmatrix}0.109+j\ 0.109\\-0.135-j\ 0.135\\0.274+j\ 0.274\end{bmatrix};$$

$$\boldsymbol{\psi}_3=\begin{bmatrix}0.334+j\ 0.334\\-0.031-j\ 0.031\\-0.114-j\ 0.114\end{bmatrix}.$$

As discussed in Section 2, for a proportionally damped system, the particular scaling choice employed in the proposed methodology leads to complex eigenvectors whose components have real and imaginary parts of equal magnitude. Once these eigenvectors

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Table 4 Identified discrete time matrices of the state-space model for the coupled damping case

$\Phi \times 10$

9.9461	0.3794	-0.5714	0.1773	0.0032	0.0157
-0.3310	9.8771	-0.5938	-0.6294	0.1211	0.0084
0.5782	0.4676	9.8825	-0.2419	-0.3919	-0.0220
-0.2784	0.8119	0.3483	9.8572	0.0833	-0.0785
0.1925	-0.1088	0.5346	0.0501	9.5282	0.9199
0.0117	-0.0764	-0.0880	0.1883	-0.9465	9.9287

$\mathbf{C} \times 10$

1.5236	-1.2545	0.2148	-0.8687	0.0960	-0.0348
1.3362	0.0325	1.5306	0.2360	0.2277	-0.0230

$\Gamma^T \times 10$

0.5234	0.0030	0.6595	0.3176	-2.8212	0.0959
-0.5469	1.2525	1.0654	-1.3654	0.2666	0.6835

have been obtained, the mass, stiffness, and damping matrices can be evaluated using the expressions presented in Eq. (18)

$$\mathcal{M}=\begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 2.0 & 0 \\ 0 & 0 & 1.2 \end{bmatrix};\ \mathcal{K}=\begin{bmatrix} 4.0 & -1.0 & -1.0 \\ -1.0 & 4.0 & -1.0 \\ -1.0 & -1.0 & 4.0 \end{bmatrix};$$

Table 5 Mean values of the identified samples for the mass, damping, and stiffness coefficients. The estimates for the coefficients are obtained at 5% RMS noise level.

Mass			Damping			Stiffness		
0.797	0.000	0.000	0.501	-0.099	-0.201	3.984	-0.998	-0.995
0.000	2.002	0.000	-0.099	0.702	-0.301	-0.998	4.003	-1.004
0.000	0.000	1.203	-0.201	-0.301	0.600	-0.995	-1.004	4.006

Table 6 Absolute values of the percentage errors in the mean values of the identified samples for the mass, damping, and stiffness coefficients. The estimates for the coefficients are obtained at 5% RMS noise level. The "-" entries in the tables correspond to coefficients for which the true values are 0.

Mass			Damping			Stiffness		
0.36	–	–	0.15	1.03	0.48	0.39	0.18	0.50
–	0.08	–	1.03	0.29	0.29	0.18	0.06	0.36
–	–	0.30	0.48	0.29	0.04	0.50	0.36	0.15

Table 7 Coefficients of variation (%) of the identified samples for the mass, damping, and stiffness coefficients. The estimates for the coefficients are identified at 5% RMS noise level, and the "-" entries in the tables correspond to coefficients for which the true values are 0.

Mass			Damping			Stiffness		
6.11	–	–	23.05	8.97	5.29	5.67	6.26	4.57
–	0.65	–	8.97	2.96	3.90	6.26	0.63	4.92
–	–	3.29	5.29	3.90	14.36	4.57	4.92	2.54

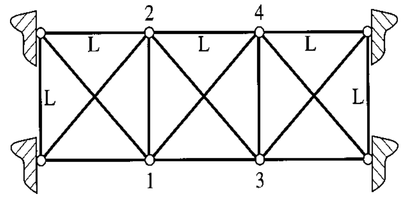

Fig. 2 Truss structure with eight unrestrained degrees-of-freedom (one horizontal and one vertical for each of the nodes denoted by 1, 2, 3, and 4)

$$\mathcal{L}=\begin{bmatrix} 0.4 & -0.1 & -0.1 \\ -0.1 & 0.4 & -0.1 \\ -0.1 & -0.1 & 0.4 \end{bmatrix}$$

which are exactly the system matrices we used to obtain the dynamic data. These matrices automatically come out as real, i.e., the imaginary components are of the order of 10^{-15} and therefore are numerical zeros for all purposes.

5.2 Coupled Second-Order Modes. The procedure for coupled systems are exactly the same as for uncoupled systems, only now the matrices we obtain at each step will look different than the ones obtained in the uncoupled case. In this case, the identified discrete time system matrices are presented in Table 4 while the diagonal entries of the matrix $\mathcal{T}$ are diag$(\mathcal{T})=(0.256 \pm j4.218,\ 0.479 \mp j16.492,\ 9.980 \mp j0.754)$. The complex eigen-

Table 8 Mass, damping, and stiffness matrices for the truss system of Fig. 2. Only the unrestrained degrees-of-freedom are included in these matrices, and the order of the degrees-of-freedom are chosen as u_1, v_1, u_2 v_2, u_3, v_3, u_4, v_4.

Mass							
100	0	0	0	0	0	0	0
0	100	0.	0	0	0	0	0
0	0	100	0	0	0	0	0
0	0	0	100	0	0	0	0
0	0	0	0	100	0	0	0
0	0	0	0	0	100	0	0
0	0	0	0	0	0	100	0
0	0	0	0	0	0	0	100
Damping							
136.4	0.0	0.0	0.0	-50.0	0.0	-17.7	-17.7
0.0	86.4	0.0	-50.0	0.0	0.0	-17.7	-17.7
0.0	0.0	136.4	0.0	-17.7	17.7	-50.0	0.0
0.0	-50.0	0.0	86.4	17.7	-17.7	0.0	0.0
-50.0	0.0	-17.7	17.7	136.4	0.0	0.0	0.0
0.0	0.0	17.7	-17.7	0.0	86.4	0.0	-50.0
-17.7	-17.7	-50.0	0.0	0.0	0.0	136.4	0.0
-17.7	-17.7	0.0	0.0	0.0	-50.0	0.0	86.4
Stiffness							
27071.1	0.0	0.0	0.0	-10000.0	0.0	-3535.5	-3535.5
0.0	17071.1	0.0	-10000.0	0.0	0.0	-3535.5	-3535.5
0.0	0.0	27071.1	0.0	-3535.5	3535.5	-10000.0	0.0
0.0	-10000.0	0.0	17071.1	3535.5	-3535.5	0.0	0.0
-10000.0	0.0	-3535.5	3535.5	27071.1	0.0	0.0	0.0
0.0	0.0	3535.5	-3535.5	0.0	17071.1	0.0	-10000.0
-3535.5	-3535.5	-10000.0	0.0	0.0	0.0	27071.1	0.0
-3535.5	-3535.5	0.0	0.0	0.0	-10000.0	0.0	17071.1

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Table 9 Properly scaled complex mode shapes (amplified by a factor of 100 for presentation) for the truss system of Fig. 2 identified with five sensors and four actuators via the proposed approach. Note that all the eigenvectors appear in complex conjugate pairs.

ψ_1, ψ_1^*	ψ_2, ψ_2^*	ψ_3, ψ_3^*	ψ_4, ψ_4^*
$-0.122 \mp -0.122j$	$-0.243 \mp -0.243j$	$-0.714 \mp -0.714j$	$0.621 \pm 0.621j$
$-1.050 \mp -1.050j$	$0.764 \pm 0.764j$	$-0.227 \mp -0.227j$	$0.197 \pm 0.197j$
$0.122 \pm 0.122j$	$0.243 \pm 0.243j$	$-0.714 \mp -0.714j$	$-0.621 \mp -0.621j$
$-1.050 \mp -1.050j$	$0.764 \pm 0.764j$	$0.227 \pm 0.227j$	$0.197 \pm 0.197j$
$0.122 \pm 0.122j$	$-0.243 \mp -0.243j$	$-0.714 \mp -0.714j$	$0.621 \pm 0.621j$
$-1.050 \mp -1.050j$	$-0.764 \mp -0.764j$	$0.227 \pm 0.227j$	$-0.197 \mp -0.197j$
$-0.122 \mp -0.122j$	$0.243 \pm 0.243j$	$-0.714 \mp -0.714j$	$-0.621 \mp -0.621j$
$-1.050 \mp -1.050j$	$-0.764 \mp -0.764j$	$-0.227 \mp -0.227j$	$-0.197 \mp -0.197j$

ψ_5, ψ_5^*	ψ_6, ψ_6^*	ψ_7, ψ_7^*	ψ_8, ψ_8^*
$0.191 \pm 0.191j$	$0.183 \pm 0.183j$	$-0.579 \mp -0.579j$	$-0.527 \mp -0.527j$
$-0.601 \mp -0.601j$	$-0.575 \mp -0.575j$	$0.067 \pm 0.067j$	$-0.168 \mp -0.168j$
$0.191 \pm 0.191j$	$0.183 \pm 0.183j$	$0.579 \pm 0.579j$	$-0.527 \mp -0.527j$
$0.601 \pm 0.601j$	$0.575 \pm 0.575j$	$0.067 \pm 0.067j$	$0.168 \pm 0.168j$
$0.191 \pm 0.191j$	$-0.183 \mp -0.183j$	$0.579 \pm 0.579j$	$0.527 \pm 0.527j$
$0.601 \pm 0.601j$	$-0.575 \mp -0.575j$	$0.067 \pm 0.067j$	$-0.168 \mp -0.168j$
$0.191 \pm 0.191j$	$-0.183 \mp -0.183j$	$-0.579 \mp -0.579j$	$0.527 \pm 0.527j$
$-0.601 \mp -0.601j$	$0.575 \pm 0.575j$	$0.067 \pm 0.067j$	$0.168 \pm 0.168j$

vector matrix $\boldsymbol{\psi}$ still has the same structure as in the previous case but now the identified complex eigenvectors $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$, and $\boldsymbol{\psi}_3$ are

$$\boldsymbol{\psi}_1=\begin{bmatrix} 0.166+j\ \ 0.154 \\ 0.266+j\ \ 0.284 \\ 0.207+j\ \ 0.171 \end{bmatrix};\quad \boldsymbol{\psi}_2=\begin{bmatrix} 0.127+j\ \ 0.088 \\ -0.161-j\ \ 0.120 \\ 0.251+j\ \ 0.296 \end{bmatrix};$$

$$\boldsymbol{\psi}_3=\begin{bmatrix} -0.327-j\ \ 0.345 \\ 0.018+j\ \ 0.045 \\ 0.139+j\ \ 0.0093 \end{bmatrix}.$$

It is important to see that, since the system is not proportionally damped, the relation between the real and imaginary parts (that they are equal in magnitude in a proportionally damped system) is not valid anymore. However, this makes no difference on the rest of the procedure, and the identified physical parameters are

$$\mathcal{M}=\begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 2.0 & 0 \\ 0 & 0 & 1.2 \end{bmatrix};\quad \mathcal{K}=\begin{bmatrix} 4.0 & -1.0 & -1.0 \\ -1.0 & 4.0 & -1.0 \\ -1.0 & -1.0 & 4.0 \end{bmatrix};$$

$$\mathcal{L}=\begin{bmatrix} 0.5 & -0.1 & -0.2 \\ -0.1 & 0.7 & -0.3 \\ -0.2 & -0.3 & 0.6 \end{bmatrix}$$

which are identical to the initial second-order matrices.

5.3 Effects of Noise on Identified Parameters. In order to discuss, in a statistically meaningful framework, the effects of noise perturbations on the proposed approach, we perform Monte Carlo type simulations on the 3-degree-of-freedom system with nonproportional damping. Here we assume that a long duration pulse response data in the form of acceleration measurements is available at the second and the third masses, and that the response of the structure is due to unit pulses applied at degrees-of-freedom 1 and 2 only. The output data is then polluted with Gaussian, zero-mean, white noise sequences, whose root-mean-squared (RMS) values are adjusted to be 5% of the unpolluted time histories. We consider 200 different noise patterns, and each of the polluted time histories are used to identify a discrete time state-space model with ERA.

Tables 5, 6, and 7 concisely summarize the results of this study. It can be seen in Table 5 that the mean values of the identified samples are very close to the exact values; indeed Table 6 reveals that the maximum relative error (in the absolute value sense) in the identified mean values is about 1%. In addition, the coefficients of variation presented in Table 7 show that the scatters around the mean values for the mass and stiffness estimates are quite acceptable, especially for the degree-of-freedom with the co-located sensor−actuator pair (degree-of-freedom 2). The coefficients for the damping matrix, however, are generally larger than those of the mass and stiffness matrices. This could partially be attributed to the high sensitivity of the damping to the phase relations between the mode shape components which generically makes the identification of the damping matrix a harder task than the identification of the mass and stiffness matrices. Overall the results show that the proposed methodology provides extremely satisfactory results even in the presence of noise perturbations.

5.4 Identification of a Truss Structure. In order to present the applicability of the proposed methodology to a more complex case, we now consider a two-dimensional truss structure with limited number of sensors and actuators. This system, shown in Fig. 2, has a total number of eight nodes of which four are fully restrained, and hence the total number of active degrees-of-freedom

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Table 10 Mass, damping, and stiffness matrices for the truss system of Fig. 2 identified with five sensors and four actuators via the proposed approach

			Mass				
100	0	0	0	0	0	0	0
0	100	0	0	0	0	0	0
0	0	100	0	0	0	0	0
0	0	0	100	0	0	0	0
0	0	0	0	100	0	0	0
0	0	0	0	0	100	0	0
0	0	0	0	0	0	100	0
0	0	0	0	0	0	0	100
			Damping				
136.4	0.0	0.0	0.0	-50.0	0.0	-17.7	-17.7
0.0	86.4	0.0	-50.0	0.0	0.0	-17.7	-17.7
0.0	0.0	136.4	0.0	-17.7	17.7	-50.0	0.0
0.0	-50.0	0.0	86.4	17.7	-17.7	0.0	0.0
-50.0	0.0	-17.7	17.7	136.4	0.0	0.0	0.0
0.0	0.0	17.7	-17.7	0.0	86.4	0.0	-50.0
-17.7	-17.7	-50.0	0.0	0.0	0.0	136.4	0.0
-17.7	-17.7	0.0	0.0	0.0	-50.0	0.0	86.4
			Stiffness				
27071.1	0.0	0.0	0.0	-10000.0	0.0	-3535.5	-3535.5
0.0	17071.1	0.0	-10000.0	0.0	0.0	-3535.5	-3535.5
0.0	0.0	27071.1	0.0	-3535.5	3535.5	-10000.0	0.0
0.0	-10000.0	0.0	17071.1	3535.5	-3535.5	0.0	0.0
-10000.0	0.0	-3535.5	3535.5	27071.1	0.0	0.0	0.0
0.0	0.0	3535.5	-3535.5	0.0	17071.1	0.0	-10000.0
-3535.5	-3535.5	-10000.0	0.0	0.0	0.0	27071.1	0.0
-3535.5	-3535.5	0.0	0.0	0.0	-10000.0	0.0	17071.1

is 8 (one horizontal and one vertical per each node). The horizontal degrees-of-freedom are denoted by u_i and the vertical degrees-of-freedom are denoted by v_i, with the subscript referring to the node number (i.e., $i=1,2,3,4$). The mass, damping, and stiffness matrices for this system are presented in Table 8. Note that these second-order matrices contain the coefficients for only the unrestrained degrees-of-freedom and that these degrees-of-freedom are ordered such that the displacement vector can be written as $\mathbf{q}(t) = [u_1(t)v_1(t)\ \ldots\ u_4(t)v_4(t)]^T$.

The instrument scheme we consider is such that there are five output sensors and four actuators: u_1, v_1, v_2, v_3, and v_4 are instrumented with output sensors, the forces $f_2^u(t)$ and $f_3^u(t)$, are applied horizontally at degrees-of-freedom 2 and 3, whereas the other two, denoted by $f_1^v(t)$ and $f_4^v(t)$ are applied vertically at degrees-of-freedom 1 and 4, such that the force vector may be defined as $\mathbf{u}(t)=[f_1^v(t)f_2^u(t)f_3^u(t)f_4^v(t)]^T$. In this case the initial discrete time state-space model is identified from unpolluted general input/output data using the OKID/ERA approach.

The co-location requirement for this case can be written as $\mathcal{C}_p^E(2,:)\boldsymbol{\psi}=(\boldsymbol{\psi}^T\mathcal{B}^E(:,2))^T$, or equivalently

$$\mathbf{C}_C^E(2,:)\boldsymbol{\varphi}=(\boldsymbol{\varphi}^{-1}\mathbf{B}_C^E(:,2))^T\boldsymbol{\mathcal{T}}^2. \tag{23}$$

Once the transformation matrix is evaluated from Eq. (23), the rows of the eigenvector matrix $\boldsymbol{\psi}$ can be identified either from

$$\boldsymbol{\psi}(i,:)=\mathbf{C}_C^E(i,:)\boldsymbol{\varphi}\boldsymbol{\mathcal{T}}^{-1} \quad \text{for} \quad i=1,2,4,6,7 \tag{24}$$

for the rows corresponding to the degrees-of-freedom with output sensors, or from

$$\boldsymbol{\psi}(i,:)=(\boldsymbol{\mathcal{T}}\boldsymbol{\varphi}^{-1}\mathbf{B}_C^E(:,i))^T \quad \text{for} \quad i=2,3,5,8 \tag{25}$$

for the rows corresponding to the degrees-of-freedom with actuators (note that the row corresponding to v_1 can be identified from either (24) or (25) due to the co-location). Since all degrees-of-freedom of this structure are instrumented with either a sensor or an actuator, all the rows of the matrix $\boldsymbol{\psi}$ can be identified, and these eigenvectors are presented in Table 9. Analogous to the case of the 3-degrees-of-freedom system with proportional damping, also in this case the real and imaginary parts of the eigenvectors are equal to each other in magnitude since the damping matrix of the truss structure was constructed so as to lead to a classical damping case.

Using the identified complex eigenvector matrix ψ, the mass, damping, and stiffness matrices can once again be constructed via Eqs. (18), and these are presented in Table 10. All the identified quantities are exactly equal to those reported in Table 8 and so the proposed methodology has once again provided an exact solution.

6 Conclusions

In this study, a new methodology for the identification of second-order structural parameters from identified state-space representations was presented. It was shown that, with the formulation developed herein, it is possible to formulate the inverse problem as a problem of transforming the identified complex eigenvectors to a certain basis. The requirements for a successful transformation are that there should be a co-located sensor–actuator pair, and that all the degrees-of-freedom should contain *either* a sensor *or* an actuator. The numerical results included in this study emphasize the efficiency and generality of the proposed approach.

The main innovation in this study is that, with the proposed methodology, it is possible to utilize mixed types of information, thereby enabling one to treat the information from a sensor or an actuator in an analogous fashion. This conceptual ''input–output equivalence'' helps relaxing the necessity of having either a full set of sensors or a full set of actuators, allowing a more general sensor–actuator setup than those required in previously discussed approaches.

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