J.S.I.A.M. Control
Ser. A, Vol. 2, No. 1
Printed in U.S.A., 1963

CONTROLLABILITY AND OBSERVABILITY IN MULTIVARIABLE CONTROL SYSTEMS*

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1. Introduction. The importance of linear multivariable control systems is evidenced by the large number of papers [1–12] published in recent years. Despite the extensive literature certain fundamental matters are not well understood. This is confirmed by numerous inaccurate stability analyses, erroneous statements about the existence of stable control, and overly severe constraints on compensator characteristics. The basic difficulty has been a failure to account properly for *all* dynamic modes of system response. This failure is attributable to a limitation of the transfer-function matrix—it fully describes a linear system if and only if the system is controllable and observable.

The concepts of controllability and observability were introduced by Kalman [13] and have been employed primarily in the study of optimal control.[1] In this paper, the primary objective is to determine the controllability and observability of composite systems which are formed by the interconnection of several multivariable subsystems. To avoid the limitations of the transfer-function matrix, the beginning sections deal with multivariable systems as described by a set of n first order, constant-coefficient differential equations. Later, the extension to systems described by transfer-function matrices is made. Throughout, emphasis is on the fundamental aspects of describing multivariable control systems. Detail design procedures are not treated.

2. Definitions and notation. Let a multivariable system S be represented by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ v &= Cx + Du \end{aligned} \tag{1}$$

where:

$u = u(t)$, p-dimensional input vector.
$v = v(t)$, q-dimensional output vector.
$x = x(t)$, n-dimensional state vector, n is the order of S.
$\dot{x} = \dot{x}(t)$, time derivative of state
A, constant nth order differential transition matrix.

* Received by the editors July 5, 1962 and in revised form November 1, 1962. Presented at the Symposium on Multivariable System Theory, SIAM, November 1, 1962 at Cambridge, Massachusetts.

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[1] Reference [14] gives a historical account of controllability and lists other references.

B, constant, n row, p column, input matrix.
C, constant, q row, n column, output matrix.
D, constant, q row, p column, transmission matrix.
If $n = 0$ the system is said to be *static.*

The characteristic roots λ_i, $i = 1, \cdots n$, of A are assumed to be distinct. This greatly simplifies the proof of theorems and prevents the main course of the paper from becoming obscured. Besides, there are few practical systems which cannot be satisfactorily approximated with an A which has distinct roots.[2]

Let ρ be an n-th order nonsingular matrix which diagonalizes A:[3]

$$\rho^{-1}A\rho = \Lambda = \begin{bmatrix} \lambda_1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \lambda_2 & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \lambda_n \end{bmatrix}, \tag{2}$$

Define *normal coordinates* as the components of the n-dimensional state vector y,

$$x = \rho y. \tag{3}$$

Then the *normal form* representation of S is given by

$$\begin{aligned} \dot{y} &= \Lambda y + \beta u \\ v &= \gamma y + Du, \end{aligned} \tag{4}$$

where

$$\beta = \rho^{-1}B, \quad \text{the normal form input matrix,} \tag{5}$$

$$\gamma = C\rho, \quad \text{the normal form output matrix.} \tag{6}$$

The normal coordinates are not unique. If desired, they may be made so by arranging the λ_i in order of increasing magnitude (roots with identical magnitudes may be taken in order of increasing angle) and choosing the column vectors of ρ, ρ_i, $i = 1, \cdots, n$, to have unit Euclidean length.

For the purpose considered here, the system S is *stable* if Re $\lambda_i < 0$ for all i.

The rank of the input r_u is defined as the rank of the matrix B (or equivalently, the rank of β). It is the "effective" number of inputs which can

[2] See Bellman [15, p. 198.]

[3] Familiar results of matrix theory will be used without comment. These results can be found in Bellman [15] or other standard texts.

influence the state vector. The integer $(p - r_u) \geqq 0$ is therefore the number of ineffectual inputs. It is possible with no loss of generality to reduce the number of components of u by $(p - r_u)$.

The rank of the output r_v is defined as the rank of the matrix C (or γ). It is the effective number of outputs available for observing the state of the system. The integer $(q - r_v) \geqq 0$ gives the number of outputs (components of v) which are linearly dependent if $D = 0$. It is possible without loss of generality to reduce the number of columns of C by $(q - r_v)$.[4]

3. Observability and controllability. A system S is *controllable* if β has no rows which are zero. Coordinates y_i corresponding to non-zero rows of β are called controllable; coordinates corresponding to zero rows of β are called uncontrollable. Uncontrollable coordinates can in no way be influenced by the input u. Thus a system which is not controllable has dynamic modes of behavior which depend solely on initial conditions or disturbance inputs. Disturbance inputs are not indicated in (1) and will not be treated in this study. Sometimes, they may be satisfactorily handled by means of appropriately introduced initial conditions.

A system S is *observable* if γ has no columns which are zero. Coordinates y_i corresponding to non-zero columns of γ are called observable; coordinates y_i corresponding to zero columns of γ are called unobservable. Unobservable coordinates are not detectible in the output v. Thus a system which is not observable has dynamic modes of behavior which cannot be ascertained from measurement of the available outputs.

A few general remarks are in order. First, the definition of controllability is different from Kalman's [14]: "A system is controllable if any initial state can be transferred to any desired state in a finite length of time by some control action." However, under the restrictions of the previous section the two definitions are equivalent. More recently, Kalman [16] has taken the same point of view given in this paper. For some additional remarks see the note by Ho [17].

Second, there is a striking similarity in the definitions of controllability and observability, the rows of β playing the same role as the columns of γ. This is also true of Kalman's definitions, and means that remarks similar to those of the previous paragraph can be made about observability. More importantly, for every conclusion concerning controllability, there is a corresponding one concerning observability. This will be evident in the statement and proof of theorems which follow.

Finally, the definitions become more involved when the characteristic

[4] Usually it is desirable to eliminate ineffectual inputs and superfluous columns of C. Exceptions occur when amplitude constraints are imposed on the u_i (such as $| u_i | < k_i$, $i = 1, \cdots p$) or noise is present in the measurement of the v_i.

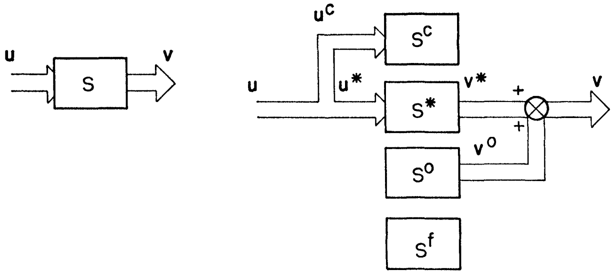

FIG. 1. System S and its partitioned representation

roots are not distinct. The diagonal matrix is replaced by a Jordan normal form and the conditions on β and γ are not so simply stated.

In order to deal more concisely with the above concepts consider:

THEOREM 1. *A system S may always be partitioned into four possible subsystems (shown in Figure 1):*

1) *a system S^* which is controllable and observable and has a transmission matrix D,*
2) *a system S^o each of whose normal coordinates are observable and uncontrollable,*
3) *a system S^c each of whose normal coordinates are controllable and unobservable,*
4) *a system S^f each of whose normal coordinates are uncontrollable and unobservable.*

All subsystems have zero transmission matrices except S^*. Also, $u^* = u^c = u$, $v = v^* + v^o$, and $n = n^* + n^o + n^c + n^f$.

The proof of Theorem 1 follows directly from equations (4) by partitioning y according to the restrictions 1) through 4). A somewhat more involved partitioning may result when the characteristic roots are not distinct.

Thus the only subsystem which has to do with the relationship of v to u is S^*. The observable system S^o only adds a disturbance v^o to the controlled part of the output v^*. Although S^o, S^c and S^f appear to have little importance in system analysis this is not necessarily so. If state variables appropriate to the description of S^o, S^c, and S^f get large, neglected nonlinear couplings may become important or physical damage of the system may result. This certainly will be the case if S^o, S^c or S^f are unstable, i.e. there are hidden instabilities.

From Theorem 1 it is clear that a necessary and sufficient condition for the absence of S^o, S^c, and S^f is that S be controllable and observable. It is possible to determine if S is controllable and observable without recourse to the normal form representation by means of the following theorem.

THEOREM 2. *Let b_i, $i = 1, \cdots, p$, be the columns of B and c_i^T, $i = 1, \cdots$*

q, be the rows of C.[5] *A system S is controllable (observable) if and only if the vectors $e_{ki} = A^k b_i$, $i = 1, \cdots, p$, $k = 0, \cdots, n-1$ ($e_{ki} = (A^T)^k(c_i)$, $i = 1, \cdots, q$, $k = 0, \cdots, n-1$) span the n-dimensional coordinate space.*

The controllability part of this theorem has been proved using the previously mentioned alternative definition of controllability [14]. By duality [13, 16] the observability part may be obtained for an alternative definition of observability [13]. The fact that the same results are obtained for the different definitions proves their equivalence.

Proof. First consider the controllability part of the theorem.

To prove necessity assume S is controllable and write

$$
\begin{aligned}
e_{ki} = A^k b_i &= (\rho\Lambda\rho^{-1})^k b_i = \rho\Lambda^k\rho^{-1} b_i \\
&= \rho\Lambda^k\beta_i\,.
\end{aligned} \tag{7}
$$

Since $\beta = [\beta_1 \cdots \beta_p]$ has no zero row it is possible to form a vector $\beta^+ = k_1\beta_1 + \cdots + k_p\beta_p$ none of whose components is zero. Clearly, the vectors $e_k^+ = \rho\Lambda^k\beta^+$, $k = 0, \cdots, n-1$ form a subspace of the space defined by the e_{ki}. But

$$
\begin{aligned}
&\det\,[e_0^+ \cdots e_{n-1}^+] \\
&= \det\left\{\rho\begin{bmatrix}\beta_1^+ & 0 & 0 & \cdots & 0 \\ 0 & \beta_2^+ & 0 & & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & & & \\ 0 & \cdots & \cdots & & \beta_n^+\end{bmatrix}\begin{bmatrix}1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1}\end{bmatrix}\right\} \\
&= (\det\rho)(\det V)(\beta_1^+\beta_2^+ \cdots \beta_n^+) \neq 0
\end{aligned} \tag{8}
$$

because the Vandermonde determinant V is nonzero for distinct λ_i, ρ is nonsingular, and the β_i^+ are all nonzero. Thus the subspace is n-dimensional. Therefore the e_{ki} must span the n-dimensional space.

To prove sufficiency assume the e_{ki} span the n-dimensional space. Then for any $r \neq 0$, say

$$
r = (\rho^T)^{-1}\begin{bmatrix}1 \\ 0 \\ \vdots \\ 0\end{bmatrix},
$$

the inner product (r, e_{ki}) cannot be zero for all k and i. But

$$
(r, e_{ki}) = (r, \rho\Lambda^k\beta_i) = (\Lambda^k\rho^T r, \beta_i) = \lambda_1^k\beta_{1i}\,. \tag{9}
$$

Assume all $\beta_{1i} = 0$ and a contradiction is obtained. Thus not all $\beta_{1i} = 0$. By

[5] The superscript T indicates the transpose of a matrix or vector.

changing r the argument also works on all other rows of β. Hence S is controllable.

To prove the observability part of the Theorem note (γ_i^T is i-th row of γ)

$$\begin{aligned} e_{ki} = (A^T)^k c_i &= \{(\rho\Lambda\rho^{-1})^T\}^k(\gamma_i^T\rho^{-1})^T \\ &= (\rho^{-1})^T\Lambda^k\rho^T(\rho^{-1})^T\gamma_i \\ &= (\rho^{-1})^T\Lambda^k\gamma_i . \end{aligned} \tag{10}$$

Since (10) is similar to (7) the remaining steps are the same as those in the controllability part.

4. Observability and controllability of composite systems. In this section the controllability and observability of composite systems are related to the controllability and observability of their subsystems. Theorems 3 and 4 treat respectively the parallel and cascade connection of two subsystems. Successive application of these theorems extends the result to composite systems which consist of many subsystems connected in parallel and cascade. Theorem 5 is the central theorem of the paper. It states conditions for the controllability and observability of a general feedback system.

THEOREM 3. *Let the parallel connection of systems S_a and S_b form a composite system S (see Figure 2). Then:*

i) $n = n_a + n_b$;

ii) $\lambda_1 , \cdots \lambda_n = \lambda_{1a} , \cdots , \lambda_{n_a a} , \lambda_{1b} , \cdots , \lambda_{n_b b}$;

iii) *a necessary and sufficient condition that S be controllable (observable) is that both S_a and S_b be controllable (observable).*

To prove Theorem 3 let S_a and S_b be represented in normal form. Then

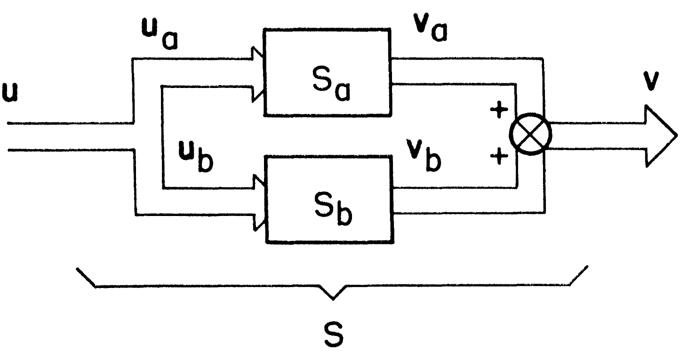

FIG. 2. Parallel connection of S_a and S_b

from the notation in Figure 2 the normal form of S can be chosen so that

$$y = \begin{bmatrix} y_a \\ y_b \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_a & 0 \\ 0 & \Lambda_b \end{bmatrix}, \tag{11}$$
$$\beta = \begin{bmatrix} \beta_a \\ \beta_b \end{bmatrix}, \quad \gamma = [\gamma_a \gamma_b], \quad D = D_a + D_b .$$

Simple inspection of (11) yields all parts of the theorem.

THEOREM 4. *Let the cascade connection of system S_a followed by S_b form a composite system S (see Figure 3). Then:*

i) $n = n_a + n_b$;

ii) $\lambda_1, \cdots, \lambda_n = \lambda_{1a}, \cdots, \lambda_{n_a a}, \lambda_{1b}, \cdots \lambda_{n_b b}$;

iii) *a necessary (but insufficient) condition for the controllability (observability) of S is that both S_a and S_b be controllable (observable);*

iv) *if S_a and S_b are both controllable (observable) any uncontrollable (unobservable) coordinates of S must originate, when designated according to characteristic root, in $S_b(S_a)$.*

Using the normal form representations of S_a and S_b yields

$$\dot{x} = \begin{bmatrix} \Lambda_a & 0 \\ \beta_b \gamma_a & \Lambda_b \end{bmatrix} x + \begin{bmatrix} \beta_a \\ \beta_b D_a \end{bmatrix} u, \quad \text{where } x = \begin{bmatrix} y_a \\ y_b \end{bmatrix}, \tag{12}$$
$$v = [D_b \gamma_a \; \gamma_b] x + D_b D_a u.$$

as the set of equations representing S.

To put these equations in normal form define

$$x = \begin{bmatrix} I & 0 \\ \phi & I \end{bmatrix} y, \quad y = \begin{bmatrix} I & 0 \\ -\phi & I \end{bmatrix} x, \tag{13}$$

where $-\phi\Lambda_a + \Lambda_b \phi = -\beta_b \gamma_a$, i.e.,

$$[\phi_{ij}] = \frac{(\beta_b \gamma_a)_{ij}}{\lambda_{ia} - \lambda_{jb}}. \tag{14}$$

$(\beta_b \gamma_a)_{ij}$ denotes the ij element of $\beta_b \gamma_a$. The assumption of distinct roots

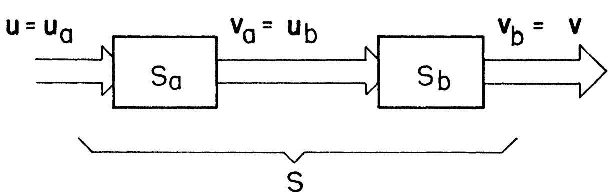

FIG. 3. Cascade connection of S_a followed by S_b

requires $\lambda_{ja} - \lambda_{ib} \neq 0$ all i and j. It is easily shown that

$$\dot{y} = \Lambda y + \beta u \qquad (15)$$
$$v = \gamma y + Du ,$$

where

$$\Lambda = \begin{bmatrix} \Lambda_a & 0 \\ 0 & \Lambda_b \end{bmatrix}, \qquad \beta = \begin{bmatrix} \beta_a \\ (-\phi\beta_a + \beta_b D_a) \end{bmatrix} \qquad (16)$$
$$\gamma = [(D_b\gamma_a + \gamma_b\phi)\gamma_b], \qquad D = D_a D_b .$$

Results i) and ii) follow immediately from inspection of (15). Consider the controllability parts of results iii) and iv). From (14) and (16) it is obvious that a null row of β_a or β_b will result in a null row of β. Thus the necessity of iii) follows. It is also clear that $-\phi\beta_a + \beta_b D_a$ may have a null row even if β_a and β_b do not. Thus iv) and the remainder of iii) hold. Corresponding reasoning applied to the columns of γ yields the observability results.

Formulas (16) can be used to determine if S is controllable or observable. Unfortunately, a fair amount of work is involved and there appears to be no way of getting simpler sufficient conditions for the controllability or observability of S.

It is helpful to consider a few simple examples where S is uncontrollable or unobservable even though S_a and S_b are controllable and observable. Let S_a and S_b be given by:

$$\dot{y}_{1a} = -y_{1a} + u_1 \qquad \dot{y}_{1b} = -2y_{1b} + u_{1b} - u_{2b} = -2y_{1b} + v_{1a} - v_{2a}$$
$$v_{1a} = y_{1a} , \qquad v_1 = y_{1b} \qquad (17)$$
$$v_{2a} = y_{1a} .$$

Then if $x_1 = y_{1a}$ and $x_2 = y_{1b}$ define the state vector of S,

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 \qquad (18)$$
$$v_1 = [0 \quad 1]x.$$

In this example S is uncontrollable and unobservable because the matrices, D_a, D_b, γ_a, β_b, which "couple" S_a and S_b are such ($D_a = D_b = 0$, $\beta_b\gamma_a = 0$) that the input u_1 never reaches the normal coordinate of S_b and the normal coordinate of S_a is not passed on to the output v_1. This particular situation cannot happen in single-input, single-output systems, since it would imply either $\gamma_a = 0$ or $\beta_b = 0$.

For the second example let

$$\begin{aligned} &\dot{y}_{1a} = -y_{1a} + u_1 \qquad && \dot{y}_{1b} = -2y_{1b} + u_{1b} = -2y_{1b} + v_{1a} \\ &v_{1a} = y_{1a} + u_1\,, && v_1 = y_{1b} - u_{1b}\,. \end{aligned} \tag{19}$$

Taking the state vector of S as $x_1 = y_{1a}$, $x_2 = y_{1b}$ gives

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_1\,, \\ v_1 &= [1 \quad -1]x - u_1\,, \end{aligned} \tag{20}$$

and for

$$\rho = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{21}$$

the normal form representation is

$$\begin{aligned} y &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 \\ v_1 &= [0 \quad -1]y - u_1\,. \end{aligned} \tag{22}$$

Equation (20) shows that $x_1 = y_{1a}$ and $x_2 = y_{1b}$ can individually be controlled and observed. Yet from equation (22), S is clearly uncontrollable and unobservable. This apparent paradox is resolved by observing that the uncontrolled (and therefore unalterable) coordinate $y_2 = -x_1 + x_2 = -y_{1a} + y_{1b}$. Therefore y_{1a} and y_{1b} cannot *independently* be controlled or observed.

A third example arises, applicable to the *parallel* connection of S_a and S_b, if the assumption in section 2 of distinct characteristic roots is waived. Then iii) of Theorem 3 becomes analogous to iii) of Theorem 4, in that the stated condition is necessary but not sufficient. Let S_a and S_b be identical first order systems

$$\dot{y}_{1c} = -y_{1c} + u_{1c}\,, \qquad c = a, b. \tag{23}$$

Then S is given by

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u. \tag{24}$$

While $y_{1a} = x_1$ and $y_{1b} = x_2$ are controllable, they are not independently controllable, since their difference is given by the solution of

$$(\dot{x}_1 - \dot{x}_2) = -(x_1 - x_2). \tag{25}$$

THEOREM 5. *Systems S_a and S_b form respectively the forward and return paths of a feedback system S (see Figure 4). Let the cascade connection of S_a*

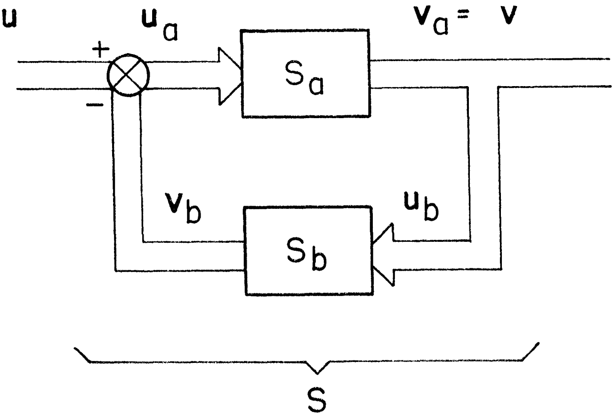

FIG. 4. Feedback system S with S_a in the forward path and S_b in the return path

followed by S_b *be* S_c *and of* S_b *followed by* S_a *be* S_o. *Assume that* $(I + D_aD_b)$ *is nonsingular. Then:*

i) $n = n_a + n_b$,

ii) *a necessary and sufficient condition that* S *be controllable* (*observable*) *is that* $S_c(S_o)$ *be controllable* (*observable*),

iii) *a necessary but not sufficient condition that* S *be controllable* (*observable*) *is that both* S_a *and* S_b *be controllable* (*observable*),

iv) *if* S_a *and* S_b *are both controllable* (*observable*) *any uncontrollable* (*unobservable*) *coordinates of* S *are uncontrollable* (*unobservable*) *coordinates of* $S_c(S_o)$ *and originate in* S_b.

Before going on with the proof, a few general observations are made. The nonsingularity of $(I + D_aD_b)$, which is equivalent to the nonsingularity of $(I + D_bD_a)$, is physically reasonable, for if it is broken the static gain $D = (I + D_aD_b)^{-1}D_a = D_a(I + D_bD_a)^{-1}$ of the closed-loop system S is undefined. Introduction of systems S_c and S_o is a natural consequence of proving separately the controllability and observability parts of the theorem. Since controllability involves only the influence of the input u on S, the system shown in Figure 5a suffices. Similarly, determination of observability leads to the system of Figure 5b. Statements analogous to ii) of Theorems 3 and 4 are not possible, since feedback alters characteristic roots.

By employing

$$\begin{aligned} u_a &= u - v_b \\ v &= v_a = u_b \end{aligned} \tag{26}$$

and the equations describing S_a and S_b, the equations describing S are obtained. Inspection of these equations shows i) is true; however, they are too complex to yield a simple proof of ii).

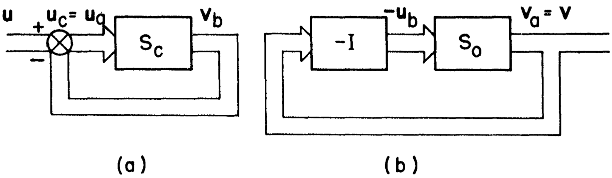

FIG. 5. Systems for determining the controllability (a) and observability (b) of a feedback system.

Consider first the controllability part of ii). From Figure 5a

$$v_b = v_c \qquad u_c = u - v_c . \tag{27}$$

Using these equations and the normal form equations for S_c gives $(x = y_c)$

$$\dot{x} = Ax + Bu \tag{28}$$

for S, where

$$A = \Lambda_c - B\gamma_c \tag{29}$$

$$B = \beta_c(I + D_bD_a)^{-1}. \tag{30}$$

It is easily shown from the nonsingularity of $(I + D_bD_a)^{-1}$ that a row of B will be zero if and only if the corresponding row of β_c is zero. Thus B has non-zero rows if and only if S_c is controllable.

The sufficiency part of ii) is proved by contradiction. Let S_c be controllable and assume that S is uncontrollable. Then from Theorem 2 the vectors e_{ki}, $k = 0, \cdots, n - 1, i = 1, \cdots, p$ cannot span the n-dimensional space. That is, a non-zero vector r exists such that

$$(r, e_{ki}) = 0, \qquad k = 0, \cdots, n - 1, i = 1, \cdots, p. \tag{31}$$

Or equivalently,

$$r^TA^kB = r^T(\Lambda_c - B\gamma_c)^kB = 0, \quad k = 0, \cdots, n - 1. \tag{32}$$

Evaluating (32) starting with $k = 0$ gives

$$\begin{aligned} r^TB &= 0 \\ r^T(\Lambda_c - B\gamma_c)B &= r^T\Lambda_cB - (r^TB)\gamma_cB = r^T\Lambda_cB = 0 \\ &\vdots \\ r^T(\Lambda_c - B\gamma_c)^{n-1}B &= r^T\Lambda_c^{n-1}B = 0. \end{aligned} \tag{33}$$

From Theorem 2 it can be seen that the columns of the matrices Λ_c^kB,

$k = 0, \cdots, n - 1$ span the n-dimensional space if and only if B has no zero row. Since by the previous paragraph B has no zero row, this and (33) imply that r is zero. Thus the contradiction is obtained.

The necessity part of ii) is obvious from the discussion at the end of section 4.

The observability part of ii) is proved by starting with Figure 5*b* and S_o. Then S is given by

$$\begin{aligned} \dot{x} &= Ax \\ v &= Cx, \end{aligned} \qquad A = \Lambda_o - \beta_o C, \qquad C = (I + D_a D_b)^{-1}\gamma_0 . \tag{34}$$

The above steps can then be applied to the columns of C with the desired results.

Theorem 4 applied to the determination of S_c and S_o gives iii).

Consider the controllability part of iv). From (28), (29), and (30) it can be seen that if the i-th row of β_c is zero, $\dot{x}_i = \lambda_i x_i$. Thus the uncontrollable coordinate $y_{ic} = x_i$ is unchanged by feedback. Moreover, by Theorem 4 this coordinate must originate in S_b. Similar arguments give the observability part of iv).

The most important result of Theorem 5 is ii). It says that closed-loop controllability and observability can be ascertained from the open-loop systems S_c and S_o. Thus one is not forced to deal with intricate closed-loop equations.

When S_b is static an even simpler situation exists. Then iv) implies that S is controllable and observable if S_a is controllable and observable.

Further information on uncontrollable and unobservable coordinates can be gleaned from Theorems 3, 4, and 5. Let S^u denote the combination of systems S^o, S^c, and S^f, that is, the part of system S which is not controllable and observable. From Theorem 1 it is clear that the coordinates of S_a^u, S_b^u, $\cdots$ are uncontrollable or unobservable in the composite system S. Thus S_a^u, S_b^u, $\cdots$ are part of S^u. To see what happens to the remaining coordinates of S_a, S_b, $\cdots$ it is sufficient to examine by means of Theorems 3, 4 and 5 the interconnection of the controllable and observable system S_a^*, S_b^*, $\cdots$. As an example take the feedback system of Theorem 5: S^u consists of S_a^u and S_b^u plus the coordinates of S_b^* which are uncontrollable in the system S_a^* followed by S_b^* and unobservable in the system S_b^* followed by S_a^*.

5. The transfer-function matrix. The traditional approach to the analysis and synthesis of multivariable systems is based on the transfer-function matrix rather than the differential equations (1). To obtain a transfer-function representation of a system S, it is assumed that the output vector v

is entirely due to input forcing u, i.e., initial conditions are zero. Let Laplace transforms be denoted by upper-case letters.
Then

$$V(s) = H(s)U(s) \tag{35}$$

where s is the Laplace-transform variable and $H(s) = [H_{ij}]$ is the q by p transfer-function matrix. The element $H_{ij}(s)$ is the scalar transfer function which relates the i-th output and the j-th input.

To obtain the transfer-function matrix from the differential-equation representation consider:

THEOREM 6. *Given a system S defined by equations* (1) *and* (4), *the transfer-function matrix is*

$$\begin{aligned} H(s) &= C(Is - A)^{-1}B + D = \gamma(Is - \Lambda)^{-1}\beta + D \\ &= \sum_{i=1}^{n^*} \frac{K_i}{s - \lambda_i^*} + D \end{aligned} \tag{36}$$

where the matrices K_i have rank one.

The first two expressions for H follow directly from the Laplace transform of (1) and (4) with $x(0) = y(0) = 0$. Since $(Is - \Lambda)^{-1}$ is diagonal, the second expression can be written out in terms of the columns of γ, γ_i, and the rows of β, β_i^T:

$$H = \sum_{i=1}^{n} \frac{\gamma_i \beta_i^T}{s - \lambda_i} + D. \tag{37}$$

For any i corresponding to an uncontrollable or unobservable coordinate, γ_i or β_i^T is zero. Thus the sum needs to be taken only over the characteristic roots associated with S^*. $K_i = \gamma_i^* \beta_i^{*T}$, being a vector outer product, is of rank one.

The important, and not surprising, conclusion of Theorem 6 is that a transfer-function matrix represents the controllable and observable part of S, S^*. It has been noted in Theorems 4 and 5 that controllability and observability of subsystems does not assure the controllability and observability of a composite system. Thus transfer-function matrices may satisfactorily represent all the dynamic modes of the subsystems but fail to represent all those of the composite system. Furthermore, the loss of hidden response modes is not easy to detect because of the complexity of the transfer-function matrices and matrix algebra. Since differential equations offer a safer basis for describing multivariable systems it is valid to ask why transfer-function matrices should be used at all. The answer is that frequency domain design procedures and the smaller size of H (it is $q \times p$ rather than $n \times n$) often make computations more manageable.

If the transfer-function matrix of a physical system is given it is generally impossible to derive the corresponding differential-equation representation. This is because the state variable choice is not unique and all information concerning systems S^c, S^o, and S^f is missing. It is possible, however, to find a set of differential equations (1) or (4) which yield the same $H(s)$ as a prescribed $H(s)$. Procedures for doing this are described below. The main result is stated here as a theorem and gives the required order of the differential equations.

THEOREM 7. *Given a rational transfer-function matrix $H(s)$ whose elements have a finite number of simple poles at $s = \lambda_i$, $i = 1, \cdots, m$ in the finite s-plane. Let the partial fraction expansion of H be*

$$H(s) = \sum_{i=1}^{m} \frac{K_i}{s - \lambda_i} + D, \tag{38}$$

where

$$K_i = \lim_{s \to \lambda_i} (s - \lambda_i) H(s), \tag{39}$$

$$D = \lim_{s \to \infty} H(s). \tag{40}$$

Let the rank of the i-th pole, r_i, be defined as the rank of K_i. Then $H(s)$ can be represented by differential equations (1) or (4) whose order is

$$n = \sum_{i=1}^{m} r_i . \tag{41}$$

The eigenvalues of A and Λ are distinct if and only if all $r_i = 1$. It is impossible to represent $H(s)$ by a differential equation whose order is less than n.

First it will be shown how $H(s)$ can be represented by a set of differential equations.

Since the matrix K_i is of rank r_i there are r_i linearly independent columns in K_i. Let e_{ji}, $j = 1, \cdots, r_i$ be such a set of columns. Then every column of K_i can be expressed as a linear combination of the e_{ji}. A compact notation is

$$K_i = E_i F_i \tag{42}$$

where E_i is a $q \times r_i$ matrix which has columns e_{ji}. To determine F_i premultiply (42) by E^T. Then

$$E_i^T K_i = E_i^T E_i F_i . \tag{43}$$

But the determinant of $E_i^T E_i$ is the Gram determinant [19] of the e_{ji}, and is nonzero because the e_{ji} are linearly independent (this is a good test for picking a linearly independent set e_{ji}). Thus

$$F_i = (E_i^T E_i)^{-1} E_i^T K_i . \tag{44}$$

Once F_i is known K_i can be expressed as

$$K_i = \sum_{j=1}^{r_i} e_{ji} f_{ij}^T \tag{45}$$

where f_{ij}^T is the j-th row of F_i. Thus

$$H(s) = \sum_{i=1}^{m} \sum_{j=1}^{r_i} \frac{e_{ji} f_{ji}^T}{s - \lambda_i} + D. \tag{46}$$

This formula is similar to (37) except that there are r_i vector outer products for each λ_i. Thus $H(s)$ can be represented by (4) where

$$\Lambda = \begin{bmatrix} \lambda_1 I_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 I_2 & & & \vdots \\ \vdots & & \ddots & & \\ \vdots & & & & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_m I_m \end{bmatrix},$$

$$\beta = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}, \qquad \gamma = [E_1 \cdots E_m] \tag{47}$$

and I_i is an identity matrix of order r_i. Thus the root λ_i is of multiplicity r_i and $n = \sum_{i=1}^{m} r_i$.

To show that a realization of lower order is not possible, the Laplace transform of (4) is taken, defining a transfer-function matrix $\bar{H}$. However, to cover all possibilities it is essential that Λ take its most general form, the Jordan normal form. For a characteristic root of multiplicity ℓ, this means that the number of Jordan blocks with this characteristic root is not fixed, only that all the blocks taken together form a matrix of order ℓ. However, $\bar{H}$ shows that all Jordan blocks must be of order one if $\bar{H}$ is to have simple poles (unless some modes are uncontrollable or unobservable, which only increases the order of (4)). Furthermore, the rank of the residue of $\bar{H}$ at $s = \lambda_i$ is no greater than the multiplicity of λ_i. Thus if $\bar{H}$ is to have the form of H in (38) the differential equations (4) must have a minimum order

$$n = \sum_{i=1}^{m} r_i .$$

If the equations (4) have order greater than n, the realization is either uncontrollable, unobservable, or both.

Theorem 7 provides a solution of the synthesis problem, since once the differential equations (1) or (4) are known, they can be realized as a physical system (example, an electronic differential analyzer). Furthermore, the synthesized system uses a minimum number of dynamic elements.[6] The assumption of simple poles can be relaxed, but at the expense of considerable additional complexity. Kalman [16] gives an alternative procedure for determining n.

Theorem 7 and a simple example illustrate how the order of a system represented by a transfer-function matrix may be underestimated. Let

$$H(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{s+1} \\ \dfrac{-1}{(s+1)(s+2)} & \dfrac{1}{s+2} \end{bmatrix} = \frac{\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}{s+2}. \tag{48}$$

At first glance it might be guessed that the system has order two, but $r_1 + r_2 = 2 + 1 = 3$, so the minimum order is three. One realization of an equivalent third order system is shown in Figure 6. It is possible that the actual order of the system may be greater than three. For example, in Figure 7 the order is five.

Underestimation of system order is the reason why most erroneous stability analyses have gone unnoticed. In a stability analysis the number of characteristic roots considered should at least be equal to the sum of the minimum orders of all the subsystems. This is easily checked by means of Theorem 7—and errors in many references gave been noted.[7]

If a transfer function matrix has any poles of rank greater than one, the assumption of distinct characteristic roots, which was made in all prior developments, is violated. If such transfer functions are encountered, an approximating system may be set up (use approximation to equations (47)) which has poles of rank one. Then all the previous results can be used.

From the above discussion it is clear that each element of $H(s)$ is an integral part of the whole description. Thus it is generally not permissible to partition a transfer-function matrix into several transfer function matrices and treat the resulting matrices as though they describe distinct systems. Yet, this has been done consistently in the representation of plants which have more inputs than outputs [9, 12]. As a consequence erroneous statements have been made concerning the existence of stable feedback systems.[8]

[6] McMillan [18] defines the degree of a square ration al matrix, which is equivalent to n, but the development is more complicated being based on the Smith normal form of a polynomial matrix. He also shows that if the m atrix is an impedance matrix, it may be synthesized by a passive network with n, and no fewer, reactive elements.

[7] See for example [10, 11].

[8] This has been noted by the author in a discussion [12].

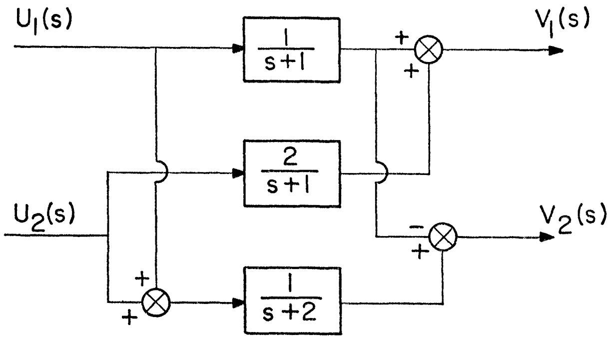

FIG. 6. Third order representation of $H(s)$.

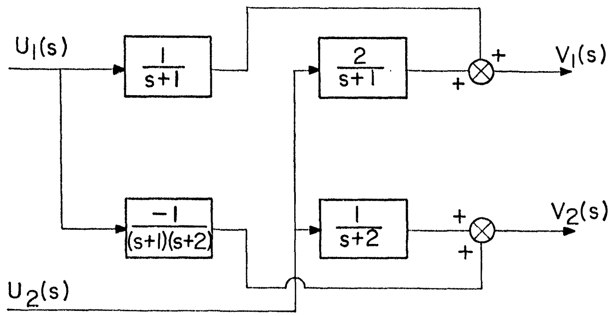

FIG. 7. Fifth order representation of $H(s)$

6. Transfer-function representation of multivariable feedback systems. Once the limitations of transfer-function matrices are recognized, it is possible to apply them successfully to the analysis and synthesis of feedback systems. In what follows it will be assumed that all transfer-function matrices have simple poles of rank one. This will keep the transfer-function representations consistent with the differential-equation representations specified earlier.

Let H_a and H_b be transfer-function matrices representing S_a and S_b in Figure 4. Then the developments,

$$
\begin{aligned}
U_a &= U - V_b = U - H_b V = U - H_b H_a U_a \\
&= (I + H_b H_a)^{-1} U,
\end{aligned} \tag{49}
$$

$$
V = H_a U_a = H_a (I + H_b H_a)^{-1} U, \tag{50}
$$

and

$$
V = H_a (U - V_b) = H_a U - H_a H_b V \tag{51}
$$

$$
= (I + H_a H_b)^{-1} H_a U, \tag{52}
$$

give alternative expressions for the transfer-function of the feedback system S,

$$H = H_a(I + H_bH_a)^{-1} = (I + H_aH_b)^{-1}H_a. \tag{53}$$

H represents the controllable and observable part of S, S^*.

The remaining part of S, S^u, was considered at the end of section 4. Systems S_a^u and S_b^u naturally are missing from the representation H because they are not represented in H_a and H_b. The coordinates of S_b^* which are not controllable and/or observable in S correspond to the poles of H_b which do not appear in H_bH_a and/or H_aH_b. In the derivation for H it is easy to see where the poles of H_b are lost: (49) gives those of H_bH_a and (51) gives those of H_aH_b. It is not so easy to see that no additional poles are lost, a difficulty which has to do with complexities in evaluating the inverse of a matrix of rational functions. This is one of the reasons that led to the more careful treatment of section 4.

Suppose that all the subsystems which make up H_a and H_b are controllable and observable. This is a reasonable assumption if transfer-function matrices are to be used. Then from the preceding it is plain that the characteristic roots of the feedback system are given by: 1) the poles of H (these roots correspond to the dynamic modes in S which are controllable and observable), 2) the poles of H_b which do not appear as poles of H_aH_b and/or H_bH_a, 3) the poles of the transfer functions representing the subsystems of S_a and S_b which do not appear respectively in H_a and H_b.

In the course of system synthesis and stability analysis *all* characteristic roots of the feedback system must be considered. Procedures for handling the characteristic roots in category 1) have been developed reasonably well in the literature. Therefore, additional effort here will be directed at 2) and 3). In particular, the problem of pole cancellation in multiplying two transfer-function matrices will be explored. This problem applies directly to 2), and often to 3), since the systems S_a and S_b are usually a cascade connection of subsystems.[9] If S_a or S_b are themselves feedback systems they must first be analyzed as feedback systems before progress can be made on the analysis of the overall feedback system.

7. Pole cancellation. Consider the cascade connection of the controllable and observable systems S_a and S_b (not the S_a and S_b of the previous section, see Figure 3). The transfer-function representation gives

$$H = H_bH_a. \tag{54}$$

If H has fewer poles than the sum of poles in H_a and H_b, pole cancellation

[9] The special case where S_b is static and S_a is the cascade connection of two subsystems has been considered in [9, 12]. The results obtained are not as general as those of the next section.

has occurred and the system S is uncontrollable or unobservable. To go further, a more detailed notation is required.

Let H_a be written as

$$H_a = \frac{\mathcal{H}_a}{h_a}, \tag{55}$$

where h_a is the characteristic polynomial for S_a,

$$h_a = k_a(s - \lambda_{1a}) \cdots (s - \lambda_{n_a a}), \qquad k_a \neq 0. \tag{56}$$

Since it has been assumed that H_a has simple poles of rank one, h_a has no repeated linear factors.[10] The elements of the matrix $\mathcal{H}_a$ are polynomials in s. Such a matrix is said to have a factor, if every element of the matrix has the same factor. Since S_a is controllable and observable $\mathcal{H}_a$ has no factors common with h_a. Similar remarks apply to H_b.

Using the notation

$$\mathcal{H} = \mathcal{H}_b \mathcal{H}_a \tag{57}$$

and

$$h = h_b h_a, \tag{58}$$

system S is controllable and observable if h and $\mathcal{H}$ do not have common factors. Any linear factor of h cancelled in H by a like factor of $\mathcal{H}$ corresponds to an uncontrollable or unobservable mode in S. Unless the elements of $\mathcal{H}_a$ and $\mathcal{H}_b$ are in some way related, the possibility that h and $\mathcal{H}$ will have common factors is remote.

The most common situation which causes $\mathcal{H}_a$ and $\mathcal{H}_b$ to be related is that of compensation where either H_a or H_b is fixed, and the other (the compensator) is chosen to make H equal a desired transfer-function matrix H_d. Clearly, if h_d does not equal $h_b h_a$ the compensated system S will be uncontrollable or unobservable. Thus certain constraints must be imposed on H_d if S is to be controllable and observable. Often it is sufficient to require that only the unstable modes of S_a and S_b be controllable and observable in S. This reduces the number of constraints.

The following treatment of constraints assumes pre-compensation (H_b fixed) and H_b square ($p_b = q_b$). The assumption that S_a is controllable and observable is reasonable because a minimum order realization of H_a must be controllable and observable. Also, it is pointless to consider an H_d which corresponds to an uncontrollable or unobservable system.

Formally, compensation requires

$$H_a = H_b^{-1} H_d. \tag{59}$$

[10] Here the term factor means a non-constant factor.

Therefore $|H_b|$ must not be identically zero. This is assured if B_b and C_b have rank p_b. Expansion of (59) yields

$$H_a = \frac{[\operatorname{adj} H_b] H_d}{|H_b|} = \frac{h_b [\operatorname{adj} \mathcal{H}_b] \mathcal{H}_d}{|\mathcal{H}_b| h_d}. \tag{60}$$

Let the greatest common divisor of the numerator and denominator be g. Then

$$\mathcal{H}_a g = h_b [\operatorname{adj} \mathcal{H}_b] \mathcal{H}_d \tag{61}$$

and

$$h_a g = h_d |\mathcal{H}_b| \tag{62}$$

because $\mathcal{H}_a$ and h_a cannot have a common factor if S_a is to be controllable and observable.

From (57), (58), (61), and (62)

$$\mathcal{H} = h_b \mathcal{H}_b [\operatorname{adj} \mathcal{H}_b] \mathcal{H}_d g^{-1} = h_b |\mathcal{H}_b| \mathcal{H}_d g^{-1} \tag{63}$$

and

$$h = h_b h_d |\mathcal{H}_b| g^{-1}. \tag{64}$$

Since $h_b |\mathcal{H}_b| g^{-1}$ is the only factor common to both h and $\mathcal{H}$ ($\mathcal{H}_d$ and h_d do not have common factors) its linear factors give all the modes which are uncontrollable and unobservable in S. Suppose all unstable modes of S are to be controllable and observable. Then all linear factors of $h_b |\mathcal{H}_b|$ which go to zero in the right-half s-plane must be included in g, or equivalently as common factors of $h_d |\mathcal{H}_b|$ and $h_b [\operatorname{adj} \mathcal{H}_b] \mathcal{H}_d$. This happens only if 1) h_d includes the right-half-plane factors of h_b, and 2) $[\operatorname{adj} \mathcal{H}_b] \mathcal{H}_d$ includes the right-half-plane factors of $|\mathcal{H}_b|$.

Very often the constraints cannot be imposed as indicated. Consider the example

$$\mathcal{H}_b = \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}, \qquad h_b = (s+1)(s-1). \tag{65}$$

Both poles have rank one. Constraint 1) requires h_d to have the factor $(s-1)$. Suppose 2) is to be satisfied with $\mathcal{H}_d$ diagonal. Then

$$[\operatorname{adj} \mathcal{H}_b] \mathcal{H}_d = \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix} \begin{bmatrix} \mathfrak{k}_{11d} & 0 \\ 0 & \mathfrak{k}_{22d} \end{bmatrix} = \begin{bmatrix} s\mathfrak{k}_{11d} & -\mathfrak{k}_{22d} \\ -\mathfrak{k}_{11d} & s\mathfrak{k}_{22d} \end{bmatrix}. \tag{66}$$

Since $|\mathcal{H}_b| = (s^2 - 1)$, each element of (66) must include the factor $(s-1)$, which in this case means both $\mathfrak{k}_{11d}$ and $\mathfrak{k}_{22d}$ have the factor $(s-1)$. But $(s-1)$ cannot be a common factor of $\mathcal{H}_d$ and h_d. The same problem also occurs if $\mathfrak{k}_{11d} = \mathfrak{k}_{22d} = 0$. As will be seen shortly, the difficulty can be resolved only by letting H_d have a pole of rank two at $s = 1$. This is the same constraint which would result from procedures described in the literature [9, 12]. It has not been noted previously that it may be relaxed if H_d is not diagonal, a fact which is of interest, since present design procedures are based on diagonalization of the open-loop transfer-function matrix.

The above analyses cannot be extended readily when H_a, H_b, or H_d have poles of rank greater than one, because then common factors in the numerator and denominator of H_a, H_b, and H_d do not necessarily imply that the systems are uncontrollable or unobservable. Theorem 7 offers a satisfactory alternative approach. System S is controllable and observable if the order of S as determined from $H = H_d$ is equal to the order of S_a plus the order of S_b. To simplify the application of this statement the following assumptions are made: i) H_a, H_b, and H_d all have simple poles, ii) H_b has poles of rank one, iii) S_a and S_b are controllable and observable, iv) H_d, is diagonal. Then S is controllable and observable if and only if

$$\operatorname{rank}\,[\lim_{s\to\lambda}(s-\lambda)H_a] + \operatorname{rank}\,[\lim_{s\to\lambda}(s-\lambda)H_b] = \operatorname{rank}\,[\lim_{s\to\lambda}(s-\lambda)H_d], \tag{67}$$
$$\lambda = \lambda_{1a}, \cdots, \lambda_{n_a a}, \lambda_{1b}, \cdots, \lambda_{n_b b}.$$

Define

$$G = H_b^{-1} = [g_1 \cdots g_p] \tag{68}$$

and let $\ell_i = 0$ if h_{iid} is analytic at $s = \lambda$, $\ell_i = 1$ if h_{iid} has a simple pole at $s = \lambda$. Using (59) and ii), (67) can be written as

$$\begin{aligned} &\operatorname{rank}\,[\lim_{s\to\lambda}\{(s-\lambda)h_{11d}g_1 \cdots (s-\lambda)h_{ppd}g_p\}] \\ &\quad = \sum_{i=1}^{p} \ell_i - 1, \qquad \lambda = \lambda_{1b}, \cdots, \lambda_{n_b b} \\ &\quad = \sum_{i=1}^{p} \ell_i, \qquad \lambda = \lambda_{1a}, \cdots, \lambda_{n_a a}, \quad \lambda \neq \lambda_{1b}, \cdots, \lambda_{n_b b}. \end{aligned} \tag{69}$$

Once the g_i are computed, constraints on the h_{iid} such that (69) is satisfied are easily found. For example, with H_b as defined by (65),

$$g_1 = \begin{bmatrix} s \\ -1 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} -1 \\ s \end{bmatrix}. \tag{70}$$

Consider $\lambda = 1$. Clearly, (69) holds only if $\ell_1 = \ell_2 = 1$. Thus both h_{11d} and h_{22d} have simple poles at $s = 1$. The same result is true at $s = -1$. Other values of $s = \lambda$ which must be considered are those where H_a (and also H_d) has poles. Since for $s \neq \pm 1$, g_1 and g_2 are linearly independent, (69) will be satisfied automatically. If g_1 or g_2 had poles they would have to be included as zeros of h_{11d} or h_{22d}. As before, it is often sufficient to impose the constraints only at λ values which have positive real parts, letting some stable modes be uncontrollable or unobservable. In this example, the only active constraint would then be that h_{11d} and h_{22d} have simple poles at $s = 1$.

Usually, the g_i are analytic at $s = \lambda_{1b}, \cdots, \lambda_{n_b b}$. In fact with ii), a necessary and sufficient condition for analyticity at $s = \lambda_{ib}$ is that K_{ib} and

$$\sum_{j=1, j\neq i}^{p} \frac{K_{jb}}{\lambda_{ib} - \lambda_{jb}} + D_b$$

have columns which span the p-dimensional coordinate space. Furthermore, if $G(s)$ is analytic at $s = \lambda_{ib}$ then it can be shown that $G(\lambda_{ib})$ is of rank $p - 1$. Thus for $\lambda = \lambda_{ib}$ equation (69) is satisfied if h_{iid}, $i = 1, \cdots, p$ have simple poles at $s = \lambda_{ib}$. Many times, a considerably less severe constraint is sufficient. For example, if $g_1(\lambda_{ib}) = 0(g_2, \cdots, g_p$ are linearly independent) only h_{11d} requires a pole at $s = \lambda_{ib}$. Or suppose $g_3(\lambda_{ib}) = k_1 g_1(\lambda_{ib}) + k_2 g_2(\lambda_{ib})$ where k_1 and k_2 are arbitrary constants; then (69) is true if $\ell_i = 1, i = 1, 2, 3$ and $\ell_i = 0, i = 4, \cdots, p$.

Finally, consider an example where $G(s)$ is not analytic at $s = \lambda_{ib}$.

$$H_b = \begin{bmatrix} \dfrac{2s}{(s-1)(s+1)} & \dfrac{-(s-3)}{2(s+1)} \\[2ex] \dfrac{-(s-1)}{(s+1)} & \dfrac{-2(s-1)}{(s+1)} \end{bmatrix} \tag{71}$$

and

$$G = \begin{bmatrix} \dfrac{4(s-1)}{(s+3)} & \dfrac{-(s-3)}{(s+3)} \\[2ex] \dfrac{-2(s-1)}{(s+3)} & \dfrac{-4s}{(s+3)(s-1)} \end{bmatrix}. \tag{72}$$

Take $\lambda = 1$. If $\ell_1 = \ell_2 = 1$ (the usual constraint [9, 12]), (69) is not satisfied (S is not controllable and observable even if a multiple pole treatment is considered); but it will be satisfied if $\ell_1 = 1$ and h_{22d} has a zero at $s = 1$. If stable modes are to be controllable and observable $\ell_1 = \ell_2 = 1$ at $\lambda = -1$ and h_{11d} and h_{22d} must have zeros at $s = -3$.

Though the above is a limited treatment, it does allow solution of many compensation problems and indicates the complexity of the situation. With obvious modifications the case of post-compensation (H_α fixed) can be handled.

8. Conclusion. From the foregoing it should be concluded that too great an emphasis on operational methods (the transfer-function matrix) is unwise. Differential equations (1) arise naturally in relating the physical properties of a system to its response characteristics, and any mathematical procedure which neglects information contained in these equations should be viewed skeptically.[11] It is surprising that physical considerations have not raised more doubts about the transfer-function representation earlier. Certainly, the errors of underestimated order would not have occurred if any effort had been made to relate the mathematical representation to the physical world—for example, by means of system simulation.

Finally, it should be noted that the synthesis of a multivariable feedback system is truly a formidable task. Unwieldy calculations, complex compensation constraints, and difficulties in evaluating the effect of disturbance inputs and parameter variations all complicate the search for satisfactory design procedures. The results developed above should at least provide a sound basis for this search.

Acknowledgment. The author wishes to thank Edward O. Gilbert and Bernard S. Morgan for helpful suggestions during the writing of the paper. The work was partially supported by the National Aeronautics and Space Administration under contract number NAS-8-1569.

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