Automatica, Vol. 18, No. 5, pp. 631–633, 1982
Printed in Great Britain.

0005–1098/82/050631–03$03.00/0
Pergamon Press Ltd.

Technical Communique

Recursive Computation of Pseudo-inverse of Matrices*

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Key Words—Matrix algebra, identification.

Abstract—Methods for recursive computation of the pseudo-inverse of a sequence of horizontal rectangular matrices in which the successive matrices differ by: (i) one column only, and (ii) one row only, are described. In both the above cases, the pseudo-inverse of the first matrix of the sequence is assumed to be known. The proposed methods avoid the direct inversion of matrices. These new methods can also be used to compute the pseudo-inverse of any arbitrary matrix iteratively. The two distinct cases mentioned above are considered in two parts in this paper. These situations arise in the fixed memory adaptive identification of discrete-data systems.

PART I

1. *Statement of the problem*

A RECTANGULAR $n \times m$ matrix, G_1, of rank n, $m > n$ and its pseudo-inverse $[G_1 G_1^T]^{-1} G_1 = W_1$ are given. G_2 is another $n \times m$ matrix of rank n which differs from G_1 by only one column as shown below

$$G_1 = [c_1 c_2 \cdots c_m]$$

$$G_2 = [c_2 c_3 \cdots c_m c_{m+1}]$$

where $c_1, c_2, \ldots, c_m, c_{m+1}$ are column vectors of dimension n. It is obvious that G_1 and G_2 contain $m-1$ identical columns. It is required to compute

$$W_2 = [G_2 G_2^T]^{-1} G_2$$

using W_1 and G_1, without using inversion subroutines. Since $m > n$, the removal or addition of one column will not change the rank of the matrix (n) in the generic case.

It is to be noted that the above stated problem is different from that described in Lee (1964). In this method, a fresh column is appended and the first column is deleted. In Lee (1964), no column is deleted but a fresh column is appended to the previous matrix. In such a case, the dimension of the matrix increases for every addition of a column.

2. *Development of the procedure*

Let

$$\underset{(n \times n)}{P_1} = [G_1 G_1^T]^{-1} \tag{1}$$

then

$$\underset{(n \times m)}{W_1} = P_1 G_1 \tag{2}$$

Let

$$\underset{(n \times n)}{P_2} = [G_2 G_2^T]^{-1} \tag{3}$$

then

$$\underset{(n \times m)}{W_2} = P_2 G_2. \tag{4}$$

If P_2 could be determined from P_1, then W_2 could be easily evaluated from (4). From (1) and (2) and by making use of the symmetry of P_1, it can be easily shown that

$$P_1 = W_1 W_1^T. \tag{5}$$

Let

$$G_1 = [a \,\vdots\, D] \tag{6}$$

$$G_2 = [D \,\vdots\, b] \tag{7}$$

where

$$a = c_1 \text{ and } b = c_{m+1}.$$

From (1)

$$P_1^{-1} = G_1 G_1^T$$
$$= aa^T + DD^T. \tag{8}$$

Similarly

$$P_2^{-1} = bb^T + DD^T \tag{9}$$
$$= bb^T + P_1^{-1} - aa^T. \tag{10}$$

Let

$$R^{-1} = P_1^{-1} + bb^T \tag{11}$$

then

$$P_2^{-1} = R^{-1} - aa^T. \tag{12}$$

Applying the matrix inversion lemma (Lee, 1964) to (11)

$$R = P_1 - \frac{P_1 bb^T P_1}{1 + b^T P_1 b}. \tag{13}$$

Thus R can be computed from P_1 and b.

A similar matrix inversion lemma can be used to recast (12)

$$P_2 + R + \frac{Raa^T R}{1 - a^T Ra}. \tag{14}$$

The proof of this lemma can be obtained by following a procedure similar to that outlined in Mendel (1973). Equations (13) and (14) are the required recurrence relationships. The algorithm can be easily implemented on a computer.

*Received September 1981; revised March 1982. This paper was recommended for publication in revised form by associate editor Y. Sunahara under the direction of editors B. D. O. Anderson and A. H. Levis.

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The pseudo-inverse of any arbitrary matrix G of dimension $n \times m$; $m > n$, can be computed, starting from

$$G_0 = [0 : I_n] \quad (15)$$

where [0] is of dimension $(n \times m - n)$ and I_n is an identity matrix of dimension $(n \times n)$.

3. *Example*

To find the pseudo-inverse of an arbitrary (3×4) matrix G, where

$$G = \begin{bmatrix} -0.0212 & 0.0023 & -0.7865 & 0.0001 \\ -0.0417 & 0.0058 & -0.9761 & -0.0032 \\ -0.0684 & -0.0029 & -0.7878 & -0.0060 \end{bmatrix}.$$

Starting matrix.

$$G_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{and } P_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 1.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & -0.0212 \\ 0 & 1 & 0 & -0.0417 \\ 0 & 0 & 1 & -0.0684 \end{bmatrix}$$

and the computed

$$P_1 = \begin{bmatrix} 0.9996 & -0.8780\text{E-}3 & -0.1440\text{E-}2 \\ -0.8780\text{E-}3 & 0.9983 & -0.2833\text{E-}2 \\ -0.1440\text{E-}2 & -0.2833\text{E-}2 & 0.9954 \end{bmatrix}.$$

Step 2.

$$G_2 = \begin{bmatrix} 0 & 0 & -0.0212 & 0.0023 \\ 1 & 0 & -0.0417 & 0.0058 \\ 0 & 1 & -0.0684 & -0.0029 \end{bmatrix}$$

and the computed

$$P_2 = \begin{bmatrix} 0.2213\text{E}4 & -0.1973\text{E}1 & -0.3174\text{E}1 \\ -0.1973\text{E}1 & 0.1000\text{E}1 & 0.1301\text{E-}4 \\ -0.3174\text{E}1 & 0.1301\text{E-}4 & 0.9999 \end{bmatrix}.$$

Step 3.

$$G_3 = \begin{bmatrix} 0 & -0.0212 & 0.0023 & -0.7865 \\ 0 & -0.0417 & 0.0058 & -0.9761 \\ 1 & -0.0684 & -0.0029 & -0.7878 \end{bmatrix}$$

and the computed

$$P_3 = \begin{bmatrix} 0.6292\text{E}4 & -0.5068\text{E}4 & 0.2589\text{E}1 \\ -0.5068\text{E}4 & 0.4085\text{E}4 & -0.2893\text{E}1 \\ 0.2589\text{E}1 & -0.2893\text{E}1 & 0.9998 \end{bmatrix}.$$

Step 4.

$$G_4 = \begin{bmatrix} -0.0212 & 0.0023 & -0.7865 & 0.0001 \\ -0.0417 & 0.0058 & -0.9761 & -0.0032 \\ -0.0684 & -0.0029 & -0.7878 & -0.0060 \end{bmatrix}$$

and the computed

$$P_4 = \begin{bmatrix} 0.3698\text{E}5 & -0.3966\text{E}5 & 0.1222\text{E}5 \\ -0.3966\text{E}5 & 0.4305\text{E}5 & -0.1375\text{E}5 \\ 0.1222\text{E}5 & -0.1375\text{E}5 & 0.4826\text{E}4 \end{bmatrix}$$

and the pseudo-inverse of $G_4 (= G)$ is

$$W = P_4 G_4 = \begin{bmatrix} 33.7378 & -180.4373 & -2.7012 & 57.2720 \\ -14.2182 & 198.3350 & 0.9557 & -59.2602 \\ -16.0666 & -65.5998 & 0.2433 & 16.2531 \end{bmatrix}$$

PART II

1. *Statement of the problem*

G_1 is a known rectangular $(m \times n)$ matrix of rank $m < n$, whose pseudo-inverse

$$W_1 = [G_1 G_1^T]^{-1} G_1 \quad (1)$$

is known. G_2 is another $(m \times n)$ matrix of rank m which differs from G_1 by only one row

$$G_1 = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_m^T \end{bmatrix}, \quad G_2 = \begin{bmatrix} c_2^T \\ c_3^T \\ \vdots \\ c_m^T \\ c_{m+1}^T \end{bmatrix} \quad (2) \,\&\, (3)$$

where $c_1^T, c_2^T, \ldots, c_m^T, c_{m+1}^T$ are row vectors of dimension n. It is obvious that G_1 and G_2 contain $(m-1)$ identical rows. It is required to compute

$$W_2 = [G_2 G_2^T]^{-1} G_2 \quad (4)$$

using W_1 and G_2, without using inversion subroutines. The removal of a $(n \times 1)$ row vector from a matrix of rank $m < n$ will reduce the rank of the resultant matrix by 1 and the addition of a $(n \times 1)$ row vector will increase the dimension by 1. This is the main difference between the present case and that covered in Part I.

2. *Development of the procedure*

Let

$$G_1 = \begin{bmatrix} a^T \\ D^T \end{bmatrix} \quad (5)$$

$$G_2 = \begin{bmatrix} D^T \\ b^T \end{bmatrix} \quad (6)$$

where $a = c_1$ and $b = c_{m+1}$. Let

$$H_1 = G_1 G_1^T = \begin{bmatrix} r^2 & x^T \\ x & R^{-1} \end{bmatrix} \quad (7)$$

where $r^2 = a^T a$; $x = D^T a$ and $R^{-1} = D^T D$. Let

$$H^{-1} = P = \begin{bmatrix} p_1 & p_2^T \\ p_2 & P_3 \end{bmatrix}. \quad (9)$$

From $H_1 H_1^{-1} = I$, we get the following set of equations

$$r^2 p_1 + x^T p_2 = 1 \quad (10)$$

$$r^2 p_2^T + x^T P_3 = 0 \quad (11)$$

$$x p_1 + R^{-1} p_2 = 0 \quad (12)$$

$$x p_2^T + R^{-1} P_3 = I \quad (13)$$

From the above equations, we get

$$R^{-1} = P_3^{-1} + \frac{x x^T}{r^2}. \quad (14)$$

Applying the matrix inversion lemma (Lee, 1964) to (14)

$$R = P_3 - \frac{P_3 x x^T P_3}{r^2 + x^T P_3 x} \quad (15)$$

which can be further simplified using (10)–(13)

$$R = P_3 - \frac{p_2 p_2^T}{p_1}. \quad (16)$$

Now let

$$H_2 = G_2 G_2^T = \begin{bmatrix} R^{-1} & y \\ y^T & s^2 \end{bmatrix} \quad (17)$$

where $y = D^T b$ and $s^2 = b^T b$. Also let

$$Q = H_2^{-1} = \begin{bmatrix} Q_1 & q_2 \\ q_2^T & q_m \end{bmatrix}. \quad (18)$$

From $H_2 H_2^{-1} = I$, we get the following set of equations

$$R^{-1} Q_1 + y q_2^T = I \quad (19)$$

$$R^{-1} q_2 + y q_m = 0 \quad (20)$$

$$y^T Q_1 + s^2 q_2^T = 0 \quad (21)$$

$$y^T q_2 + s^2 q_m = 1. \quad (22)$$

From the above equations, it can be easily shown that

$$Q_1^{-1} = R^{-1} - \frac{y y^T}{s^2}. \quad (23)$$

Applying a lemma which is similar to the one reported in Lee (1964), to the above equation, we get

$$Q_1 = R + \frac{R y y^T R}{s^2 - y^T R y} \quad (24)$$

provided $s^2 - y^T R y \neq 0$. Using (19)–(22) and (24), it can be easily shown that

$$Q_1 = R + \frac{q_2 q_2^T}{q_m} \quad (25)$$

$$q_m = \frac{1}{s^2 - y^T R y} \quad (26)$$

and

$$q_2 = -q_m R y \quad (27)$$

$$W_2 = H_2^{-1} G_2 = Q G_2. \quad (28)$$

Algorithm.

$$\begin{aligned} R &= P_3 - \frac{p_2 p_2^T}{p_1} \\ y &= D^T b \\ s^2 &= b^T b \\ q_m &= \frac{1}{s^2 - y^T R y} \\ q_2 &= -q_m R y \\ Q_1 &= R + \frac{q_2 q_2^T}{q_m}. \end{aligned} \quad (29)$$

3. *Example*

G_1 and G_2 are the given matrices, which differ by one row only.

Steps **(30)–(33).** P **is computed from** W_1**.**

Step **(34).** Q **is computed from** P**,** G_1 **and the last row of** G_2**.**

Step **(35)** W_2 **is computed using** $W_2 = QG_2$**.**

The results were verified by multiplying W_2 and G_2^T which yielded I_3.

$$G_1 = \begin{bmatrix} -0.0212 & 0.0023 & -0.7865 & 0.0001 \\ -0.0417 & 0.0058 & -0.9761 & -0.0032 \\ -0.0684 & -0.0029 & -0.7878 & -0.0060 \end{bmatrix} \quad (30)$$

$$W_1 = \begin{bmatrix} 33.7378 & -180.4373 & -2.7012 & 57.2720 \\ -14.2182 & 198.3350 & 0.9557 & -59.2602 \\ -16.0660 & -65.5998 & 0.2433 & 16.2531 \end{bmatrix} \quad (31)$$

$$G_2 = \begin{bmatrix} -0.0417 & 0.0058 & -0.9761 & -0.0032 \\ -0.0684 & -0.0029 & -0.7878 & -0.0060 \\ 0.0100 & 0.0400 & 0.0900 & 0.0160 \end{bmatrix} \quad (32)$$

$$P = \begin{bmatrix} 0.3698E05 & -0.39663E05 & 0.12225E05 \\ -0.39663E05 & 0.43052E05 & -0.13745E05 \\ 0.12225E05 & -0.13745E05 & 0.48257E04 \end{bmatrix} \quad (33)$$

$$Q = \begin{bmatrix} 0.55862E03 & -0.70821E03 & -0.16328E03 \\ -0.70821E03 & 0.90673E03 & 0.27034E03 \\ -0.163280E03 & 0.27934E03 & 0.59938E03 \end{bmatrix} \quad (34)$$

$$W_2 = \begin{bmatrix} 23.5142 & -1.2375 & -2.0359 & -0.1508 \\ -29.7844 & 4.0706 & 1.2929 & 1.1514 \\ -5.6888 & 22.2444 & 0.3474 & 8.4906 \end{bmatrix}. \quad (35)$$

4. *Area of application*

There are two distinct ways of utilizing the input–output data for identifying linear discrete-data systems.

1. The entire input–output data collected up to the present instant can be used. In this case the dimension of the data matrix keeps on expanding as new measurement vectors arrive. This is known as the expanding memory identification scheme. The recursive technique for this situation is available in the technical literature (Lee, 1964; Mendel, 1973).

2. Another option is the fixed memory identification scheme where the dimension of the data matrix is kept constant. When a new measurement vector arrives, the oldest one is deleted. For such a situation, new recursive identification schemes can be developed based on the procedures discussed in the paper. The two algorithms were developed for use in microprocessor-based measuring instruments which use identification algorithms to determine the parameters when the data is corrupted by small measurement noise. The two procedures described in Part I and Part II can be profitably used when the number of data-vectors retained at every step is (i) greater than or (ii) less than the number of parameters to be identified respectively.

COMPARISON AND USES OF THE TWO ALGORITHMS

Recursive algorithm 1.

1. Successive matrices differ by one column only.
2. The dimension of the column is n and the rank of the matrix is n.
3. A least-square fit over the latest m measurements for MIMO discrete system can be obtained using this scheme.
4. Yields the ordinary inverse when $m = n$.

Recursive algorithm 2.

1. Successive matrices differ by one row only.
2. The dimension of the row is n. The rank of the matrix is $m < n$.
3. Lyapunov stable recursive fixed memory identification scheme for MIMO discrete systems, which shows fast convergence, could be derived using this procedure.
4. Yields the ordinary inverse when $m = n$.

Conclusions

New recursive algorithms for the computation of the pseudo-inverse of a sequence of rectangular matrices in which the successive matrices differ by one column or one row only are presented. These methods can be extended to obtain the pseudo-inverse of any arbitrary rectangular matrix. A possible area of application for these algorithms is the Adaptive Identification of Multivariable discrete systems using a limited number of data vectors at every iteration.

Acknowledgement—The first author wishes to acknowledge the financial support provided by the Alexander Von Humboldt Foundation, West Germany.

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