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Estimation of Markov parameters and time-delay/interactor matrix

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Abstract

The ARMarkov-least squares method is extended to multivariable systems. This method explicitly determines the Markov parameters (impulse response coefficients) of a process using process input–output data and a standard least-squares (LS) algorithm. The parameter estimates are consistent and have tighter confidence bounds than those produced by other linear regression methods. The Interactor matrix, which defines the time delays in multivariable systems, can be directly estimated from the Markov parameters. From simulation results it is observed that the Markov parameters estimated by the ARMarkov-LS method are the closest to the actual Markov parameters irrespective of the system order and lead to a better estimate of the interactor matrix than other linear regression methods such as Correlation Analysis, ARX, FIR, etc. The identified Markov parameters and/or the time-delay/interactor matrix can be used directly in the design of model predictive controllers and control loop performance assessment.

Keywords: Identification; ARMarkov-LS; Impulse response; Markov parameter; Interactor matrix

1. Introduction

Estimation of time-delay and Markov parameters (impulse response) plays a very important role in control system design and analysis. For univariate systems, the time delay is defined by the time needed for any input to have an effect on the output of the system. This, in turn, is equal to the number of zeros or statistically insignificant impulse response coefficients. In discrete-time systems, the time delay is expressed as the number of infinite zeros (Huang, 1997). Controllers, such as model predictive controllers (MPC), use the step or impulse response coefficients of the process model to build the dynamic matrix. It is particularly important to estimate the initial (fast dynamics) part of the step/impulse response. If these Markov parameters can be determined accurately, improved performance can be achieved.

In multivariate systems, time delays are expressed in terms of a nonsingular matrix consisting of the smallest number of impulse response coefficients (Markov parameters). This matrix is called the interactor matrix. The nonsingularity of the matrix can be determined in several ways. In this paper, singular-value decomposition (SVD) is used. Huang (1997) proposed a method of determining the interactor matrix which uses Markov parameters directly rather than transfer functions. To estimate the interactor matrix directly from the Markov parameters, the first step is to determine the Markov parameters from input/output data collected from the multivariable system of interest.

The method proposed in this paper for the determination of the Markov parameters is called the ARMarkov/LS method. The ARMarkov method was first introduced by Hyland (1991) and used by Akers and Bernstein (1997) to estimate the Markov parameters. This procedure uses the standard least-squares algorithm to estimate the Markov parameters in an ARMarkov representation which relates the present output(s) to past outputs and past inputs. The ARMarkov representation has the same form as an ARX model except that the ARMarkov model explicitly contains more than one Markov parameter (Akers & Bernstein, 1997). In this paper, the ARMarkov/LS method is extended to multivariate systems to estimate the Markov parameter blocks. Once the Markov parameters are estimated, they

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are used to estimate the time delay in SISO systems and the interactor matrix in MIMO systems.

For known processes, the estimated parameters can be compared with the actual plant parameters. For unknown processes, it is necessary to use other approaches to determine how good the parameter estimates are. Statistical analysis of the estimated parameters is a common way of assessing the parameter estimates. Statistical properties such as the consistency and variance/covariance of the Markov parameters estimated by the ARMarkov method are analyzed in this paper.

The ARMarkov procedure used to determine the Markov parameters of SISO systems is outlined in Section 2 and then extended to MIMO systems in Section 3. Statistical analysis of the ARMarkov parameter estimates is discussed in Section 4. Section 5 describes the determination of the order of the interactor matrix and the estimation of the unitary interactor matrix. The results obtained in Sections 2–5 are illustrated through simulation examples in Section 6 and compared with those produced by other linear regression methods.

2. The ARMarkov method for SISO systems

Let

$$G(z) \triangleq \left[\begin{array}{c|c} A_x & B_x \\ \hline C_x & D_x \end{array}\right]$$

denote the transfer function of a discrete, linear time-invariant SISO system having the state-space form

$$x(k+1) = A_x x(k) + B_x u(k), \tag{1}$$

$$y(k) = C_x x(k) + D_x u(k), \tag{2}$$

where $A_x \in R^{n\times n}, B_x \in R^{n\times 1}, C_x \in R^{1\times n}$, and $D_x \in R^{1\times 1}$.

The Markov parameters h_j, as defined by Akers and Bernstein (1997), are

$$\begin{aligned} h_j &\triangleq D_x \quad &&\text{for } j = -1, \\ &\triangleq C_x A_x^j D_x \quad &&\text{for } j \geqslant 0. \end{aligned} \tag{3}$$

The transfer function corresponding to Eqs. (1) and (2) is

$$G(z) \triangleq C_x(zI - A_x)^{-1} B_x + D_x. \tag{4}$$

The Markov parameter representation of $G(z)$ is

$$G(z) = \sum_{j=-1}^{\infty} h_j z^{-(j+1)}. \tag{5}$$

The ARX transfer function representation of $G(z)$ is

$$G_{\mathrm{ARX}}(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n} \tag{6}$$

with the time-domain representation

$$\begin{aligned} y(k) = &-a_1 y(k-1) - \cdots - a_n y(k-n) + b_0 u(k) \\ &+ \cdots + b_n u(k-n) \quad \text{for } k \geqslant 0 \end{aligned} \tag{7}$$

The ARX coefficients in Eq. (7) and the Markov parameters in Eq. (5) satisfy the following relationship (Akers & Bernstein, 1997):

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} h_{-1} & 0 & \cdots & 0 \\ h_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_{n-1} & \cdots & h_0 & h_{-1} \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, \tag{8}$$

$$h_{n+j} = -\sum_{i=1}^{n} a_i h_{n+j-i} \quad \text{for } j \geqslant 0. \tag{9}$$

The ARMarkov representation corresponding to $G(z)$ with μ Markov parameters is given by

$$G_{\mathrm{ARMark}}(z) = \frac{h_{-1} z^{\mu+n-1} + \cdots + h_{\mu-2} z^n + \beta_{\mu,1} z^{n-1} + \cdots + \beta_{\mu,n}}{z^{\mu+n-1} + \alpha_{\mu,1} z^{n-1} + \cdots + \alpha_{\mu,n}}. \tag{10}$$

The ARMarkov time-domain representation is given by

$$\begin{aligned} y(k) = &-\sum_{j=1}^{n} \alpha_{\mu,j} y(k-\mu-j+1) + \sum_{j=1}^{\mu} h_{j-2} u(k-j+1) \\ &+ \sum_{j=1}^{n} \beta_{\mu,j} u(k-\mu-j+1) \end{aligned} \tag{11}$$

and involves only the first μ Markov parameters $h_{-1}, \ldots, h_{\mu-2}$. The parameters $\alpha_{\mu,1}, \ldots, \alpha_{\mu,n} \in R^{1\times n}$ are functions of the ARX coefficients and the Markov parameters (Akers and Bernstein 1997). Here n is the order of the ARMarkov model. The ARX time-domain representation is a special form of the ARMarkov time-domain representation with $\mu = 1$.

2.1. ARMarkov/LS Identification Algorithm

The ARMarkov regressor vector, $\boldsymbol{\phi}_\mu$ for Eq. (11) is written as

$$\begin{aligned} \boldsymbol{\phi}_\mu(k) \triangleq [&y(k-\mu) \cdots y(k-\mu-n+1), u(k) \cdots \\ &u(k-\mu-n+1)]^{\mathrm{T}}. \end{aligned} \tag{12}$$

The process output can be expressed as

$$y(k) = \mathbf{W}_\mu \boldsymbol{\phi}_\mu(k), \tag{13}$$

where the weighting matrix is given by

$$\mathbf{W}_\mu = [-\mathbf{A}_\mu \ h_{-1} \ \cdots \ h_{\mu-2} \ \mathbf{B}_\mu], \tag{14}$$

$$\mathbf{A}_\mu = [\alpha_{\mu,1}, \ldots, \alpha_{\mu,n}] \in R^{1\times n}, \tag{15}$$

$$\mathbf{B}_\mu = [\beta_{\mu,1} \ldots \beta_{\mu,n}] \in R^{1\times n}. \tag{16}$$

Let $\hat{\mathbf{W}}_\mu$ be the estimate of the weighting matrix and $\hat{y}(k)$ be the estimated output. Then

$$\hat{y}(k) = \hat{\mathbf{W}}_\mu \boldsymbol{\phi}_\mu(k). \tag{17}$$

Define the output error, $\varepsilon(k)$ as

$$\varepsilon(k) = y(k) - \hat{y}(k) \tag{18}$$

and the cost function, J, in terms of the output error as

$$J = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{2}\varepsilon^2(k) = \frac{1}{2N}\sum_{k=1}^{N} (y(k) - \hat{\mathbf{W}}_\mu \boldsymbol{\phi}_\mu(k))^2, \tag{19}$$

where N is the number of total data points. Then $\hat{\mathbf{W}}_\mu$ is a strict minimization of J iff

$$\hat{\mathbf{W}}_\mu = \left[\frac{1}{N}\sum_{k=1}^{N} y(k)\boldsymbol{\phi}_\mu^{\mathrm{T}}(k)\right]\left[\frac{1}{N}\sum_{k=1}^{N} \boldsymbol{\phi}_\mu(k)\boldsymbol{\phi}_\mu^{\mathrm{T}}(k)\right]^{-1}. \tag{20}$$

The matrix $(1/N)\sum_{k=1}^{N} \boldsymbol{\phi}_\mu(k)\boldsymbol{\phi}_\mu^{\mathrm{T}}(k)$ contains the covariance estimates of $u(k)$ and $y(k)$ and must be non-singular for the inverse to exist. When $\mu = 1$, Eq. (20) is identical to the ARX/LS identification algorithm.

3. The ARMarkov method for MIMO systems

Akers and Bernstein (1997) suggested that for MIMO systems, each individual input–output pair be considered separately to generate the corresponding Markov parameters. The estimated parameters could then be stacked in matrix form. However, as shown below, the ARMarkov/LS method can be reformulated for MIMO systems to estimate the Markov parameter blocks directly. The input–output relationship is expressed as

$$\mathbf{Y}(k) = \sum_{j=1}^{n} -\boldsymbol{\alpha}_{\mu,j}\mathbf{Y}(k-\mu-j+1) + \sum_{j=1}^{\mu} \mathbf{H}_{j-2}\mathbf{U}(k-j+1) + \sum_{j=1}^{n} \boldsymbol{\beta}_{\mu,j}\mathbf{U}(k-\mu-j+1), \tag{21}$$

where $\mathbf{Y}(k) = [y_1(k), y_2(k) \cdots y_l(k)]^{\mathrm{T}}$, $\mathbf{U}(k) = [u_1(k), u_2(k) \cdots u_m(k)]^{\mathrm{T}}$ and $\boldsymbol{\alpha}$, $\mathbf{H}_j$ and $\boldsymbol{\beta}$ are of appropriate sizes. The number of outputs and inputs are l and m, respectively. The regressor vector is expressed as

$$\boldsymbol{\Phi}_\mu(k) \triangleq [\mathbf{Y}(k-\mu)^{\mathrm{T}} \cdots \mathbf{Y}(k-\mu-n+1)^{\mathrm{T}}, \mathbf{U}(k)^{\mathrm{T}} \cdots \mathbf{U}(k-\mu-n+1)^{\mathrm{T}}]^{\mathrm{T}}.$$

Following the same least-squares procedure used for the SISO systems, the cost function becomes

$$J = \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\varepsilon}, \tag{22}$$

where $\boldsymbol{\varepsilon}$ is the output error vector and the estimate of the weighting matrix, $\hat{\mathbf{W}}_\mu$ is

$$\hat{\mathbf{W}}_\mu = \left[\frac{1}{N}\sum_{k=1}^{N} \mathbf{Y}(k)\boldsymbol{\Phi}^{\mathrm{T}}(k)\right]\left[\frac{1}{N}\sum_{k=1}^{N} \boldsymbol{\Phi}(k)\boldsymbol{\Phi}^{\mathrm{T}}(k)\right]^{-1}, \tag{23}$$

where

$$\hat{\mathbf{W}}_\mu = [-\hat{\mathbf{A}}_\mu \; \hat{\mathbf{H}}_{-1} \; \cdots \; \hat{\mathbf{H}}_{\mu-2} \; \hat{\mathbf{B}}_\mu], \tag{24}$$

$$\hat{\mathbf{A}}_\mu = [\hat{\boldsymbol{\alpha}}_{\mu,1}, \ldots, \hat{\boldsymbol{\alpha}}_{\mu,n}] \in R^{l\times nm}, \quad \hat{\mathbf{H}}_j \in R^{l\times m}, \tag{25}$$

$$\hat{\mathbf{B}}_\mu = [\hat{\boldsymbol{\beta}}_{\mu,1} \; \cdots \; \hat{\boldsymbol{\beta}}_{\mu,n}] \in R^{l\times nm}. \tag{26}$$

4. Statistical analysis of the parameter estimates

4.1. Consistency of the least-squares estimates

ARX and FIR are the simplest of the most widely used identification methods which use linear regression methods. The ARMarkov method described here also uses linear regression. Since the ARX and FIR methods are parametric and nonparametric respectively and the ARMarkov method is a blend of these two methods, it is intuitively expected that the ARMarkov method will have the properties which are in between the properties of the parametric and non-parametric formulations.

The general structure of the actual process is assumed to be

$$y = Gu + He \tag{27}$$

$$= \frac{B}{A}u + He, \tag{28}$$

and the model is written as

$$\hat{y} = \hat{G}u + \hat{H}e, \tag{29}$$

where H is noise structure, A, B are polynomials in z, $\hat{G}$ and $\hat{H}$ are different for different model structures and e is white noise with zero mean and covariance $\lambda^2 I$ where λ represents the variance of e.

In input–output form, the process can be expressed as

$$y(k) = \phi^{\mathrm{T}}(k)\theta_0 + v_0(k), \tag{30}$$

where ϕ is the regressor vector, θ_0 represents the actual process parameter vector and v_0 is the process noise with $v_0 = AHe$.

The structures of the linear regressions models are as follows:

$$\text{ARX: } \hat{y}(k) = -a_1 y(k-1) - \cdots - a_n y(k-n) + b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n), \tag{31}$$

$$\text{FIR: } \hat{y}(k) = f_0 u(k) + f_1 u(k-1) + \cdots + f_{M-1} u(k-M+1) \tag{32}$$

$$\begin{aligned}\text{ARMarkov: } \hat{y}(k) = & -\alpha_1 y(k-\mu) - \cdots \\ & -\alpha_n y(k-\mu-n+1) \\ & + h_0 u(k) + h_1 u(k-1) + \cdots \\ & + h_{\mu-1} u(k-\mu+1) \\ & + \beta_1 u(k-\mu) + \cdots \\ & + \beta_n u(k-\mu-n+1),\end{aligned} \tag{33}$$

where n is the order of the ARX model, M represents the number of FIR coefficients and μ is the number of Markov parameters in the ARMarkov model. The models can also be written in terms of regressor vectors and model parameters as

$$\text{ARX: } \hat{y}(k) = \phi^{\mathrm{T}}_{\mathrm{ARX}}(k)\theta_{\mathrm{ARX}} + v_{\mathrm{ARX}}(k), \tag{a}$$

$$\text{FIR: } \hat{y}(k) = \phi^{\mathrm{T}}_{\mathrm{FIR}}(k)\theta_{\mathrm{FIR}} + v_{\mathrm{FIR}}(k), \tag{b}$$

$$\text{ARMarkov: } \hat{y}(k) = \phi^{\mathrm{T}}_{\mathrm{ARM}}(k)\theta_{\mathrm{ARM}} + v_{\mathrm{ARM}}(k). \tag{c}$$

Now assume that *the input signal is persistently exciting and the model contains the actual process* (*no bias*). Then according to Ljung (1987), the least-squares estimate (LSE) is consistent, i.e. $\hat{\theta}$ converges to θ_0 as the number of data points, $N \to \infty$ if

(1) v_0 is white noise (not the case in most practical applications) or
(2) the input sequence is independent of the noise sequence and there is no output term in the regression vector.

In the FIR model, there is no output term in the regressor vector, so the estimated parameters converge to the actual ones as $N \to \infty$, but in the ARX method, the parameter estimate is not consistent in the presence of non-white noise (Ljung, 1987; Söderström & Stoica, 1989) because of the output terms in the regressor.

Lemma 1. *The estimated Markov parameters in the ARMarkov model are consistent, although the parameters equivalent to the ARX parameters in the ARMarkov model are not.*

Proof. The least-squares estimate,

$$\hat{\theta}^{\mathrm{LS}} = \theta_0 + \left[\frac{1}{N}\sum_{i=1}^{N} \phi(k)\phi^{\mathrm{T}}(k)\right]^{-1} \frac{1}{N}\sum_{i=1}^{N} \phi(k)v_0(k) \tag{34}$$

$$= \theta_0 + (E(\phi(k)\phi^{\mathrm{T}}(k)))^{-1}E(\phi(k)v_0(k)) \quad \text{as } N \to \infty. \tag{35}$$

Let

$$\theta_0 = \begin{bmatrix}\theta_{\mathrm{Mark}} \\ \theta_{AR}\end{bmatrix}, \quad \phi = \begin{bmatrix}\phi_{\mathrm{Mark}} \\ \phi_{AR}\end{bmatrix}, \tag{36}$$

$$\phi(k)v_0(k) = \begin{bmatrix}\phi_{\mathrm{Mark}}v_0(k) \\ \phi_{AR}v_0(k)\end{bmatrix}, \tag{37}$$

where ϕ_{Mark} corresponds to the present and past input terms. Hence $E(\phi_{\mathrm{Mark}}(k)v_0(k)) = 0$, but $E(\phi_{AR}(k)v_0(k)) \neq 0$ since ϕ_{AR} involves some output terms. Therefore, as $N \to \infty$

$$\hat{\theta}^{\mathrm{LS}} = \begin{bmatrix}\theta_{\mathrm{Mark}} \\ \theta_{AR}\end{bmatrix} + \begin{bmatrix}0 \\ (E(\phi(k)\phi^{\mathrm{T}}(k)))^{-1}E(\phi(k)v_0(k))\end{bmatrix} \tag{38}$$

$$= \begin{bmatrix}\theta_{\mathrm{Mark}} \\ \theta_{AR} + (E(\phi(k)\phi^{\mathrm{T}}(k)))^{-1}E(\phi(k)v_0(k))\end{bmatrix}, \tag{39}$$

$$\hat{\theta}^{\mathrm{LS}}_{\mathrm{Mark}} = \theta_{\mathrm{Mark}}. \quad \square \tag{40}$$

Example 4.1. Lemma 1 can be illustrated by considering the following first-order process with non-white noise:

$$y(k) = -a_1 y(k-1) + b_0 u(k-1) + c_0 e(k) + c_1 e(k-1). \tag{41}$$

Assume an ARMarkov model with two Markov parameters h_0 and h_1 so that

$$\hat{y}(k) = -\alpha_1 y(k-2) + h_0 u(k) + h_1 u(k-1) + \beta_0 u(k-2). \tag{42}$$

Here u is an independent input sequence with variance σ^2 and e is white noise with variance λ^2.

As $N \to \infty$, the least-squares parameter estimates are

$$h_0 = 0, \tag{43}$$

$$\begin{bmatrix}h_1 \\ \alpha_1 \\ \beta_0\end{bmatrix} = [R^*]^{-1}\begin{bmatrix}Ey(k)u(k-1) \\ Ey(k)y(k-2) \\ Ey(k)u(k-2)\end{bmatrix}, \tag{44}$$

$$R^* = \begin{bmatrix} Eu(k-1)^2 & -Ey(k-2)u(k-1) & Eu(k-1)u(k-2) \\ -Ey(k-2)u(k-1) & Ey(k-2)^2 & -Ey(k-2)u(k-2) \\ Eu(k-1)u(k-2) & -Ey(k-2)u(k-2) & Eu(k-2)^2 \end{bmatrix}, \tag{45}$$

$$R^* = \begin{bmatrix}\sigma^2 & 0 & 0 \\ 0 & Ey(k-2)^2 & 0 \\ 0 & 0 & \sigma^2\end{bmatrix}, \tag{46}$$

$$Ey(k-2)^2 = \frac{b_0^2\sigma^2 + (1 + c_0^2 - a_1 2c_0)\lambda^2}{1 - a_1^2},$$

$$Ey(k)u(k-1) = b_0\sigma^2, \tag{47}$$

$$Ey(k)u(k-2) = -a_1 b_0 \sigma^2, \tag{48}$$

$$Ey(k)y(k-2) = a_1^2 \frac{b_0^2\sigma^2 + (1 + c_0^2 - a_1 2c_0)\lambda^2}{1 - a_1^2} + c_0\lambda^2. \tag{49}$$

Therefore,

$$\begin{bmatrix} h_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} b_0 \\ -a_1 b_0 \end{bmatrix}. \tag{50}$$

Remark 1. The Markov parameters, h_0 and h_1, are consistent (transformed into actual parameters (Akers & Bernstein, 1997)); whereas, α_1, which is a function of the noise/disturbance parameters never converges to the actual parameter. Since the ARMarkov model is a blend of the FIR and ARX models (Eqs. (31)–(33)), it is consistent in the parameters corresponding to the FIR model but inconsistent in the parameters corresponding to the ARX model.

4.2. Covariance of the estimated parameters

4.2.1. Covariance estimates for systems with white noise

The error terms, ε in the output prediction in Eq. (18) are statistically independent having zero mean and variance λ^2 and the estimates $\hat{\theta}^{LS}$ denoted by $\hat{\mathbf{W}}_\mu$ in Eq. (20) are the unbiased estimates of the actual parameters W_μ, i.e. $E(\hat{\mathbf{W}}_\mu) = W_\mu$. The *covariance matrix* of the estimates is

$$\text{cov}(\hat{\mathbf{W}}_\mu) = \lambda^2 \left[\frac{1}{N}\sum_{i=1}^{N} \phi(k)\phi^{\mathrm{T}}(k)\right]^{-1}. \tag{51}$$

The diagonal elements of the *covariance matrix* represent the estimated variances of the estimators. The estimated squared standard error (SSE) of $\hat{\mathbf{W}}_\mu$ can be written as

$$\text{SSE} = \sum_{k=1}^{N} \varepsilon^{\mathrm{T}}(k)\varepsilon(k) \tag{52}$$

$$= \sum_{k=1}^{N} y(k)^2 - \hat{\mathbf{W}}_\mu^{\mathrm{T}}\left(\frac{1}{N}\sum_{k=1}^{N} y(k)\boldsymbol{\phi}^{\mathrm{T}}(k)\right) \tag{53}$$

and the estimate of the error variance can be written as

$$\hat{\lambda}^2 = \frac{\text{SSE}}{N-p} = \frac{\sum_{k=1}^{N} y(k)^2 - \hat{\mathbf{W}}_\mu^{\mathrm{T}}(\frac{1}{N}\sum_{k=1}^{N} y(k)\boldsymbol{\phi}^{\mathrm{T}}(k))}{N-p}, \tag{54}$$

where p is the total number of parameters to be estimated and N is the number of data points.

When the actual process is subject to white noise, the noise estimates determined by the LS methods are independently distributed with zero mean and variance $\hat{\lambda}^2$. In this case, the linear regression method is equivalent to the prediction error method (PEM). According to Ljung (1987), for PEM, the total variance of the parameter estimates (expressed in terms of the process model variance) is proportional to the ratio of the number of model parameters to the number of data points. i.e.

$$\text{var}(\hat{G}_{\text{ARX}}) \propto \frac{2n}{N}, \tag{55}$$

$$\text{var}(\hat{G}_{\text{ARMark}}) \propto \frac{\mu + 2n}{N}, \tag{56}$$

$$\text{var}(\hat{G}_{\text{FIR}}) \propto \frac{M}{N}. \tag{57}$$

The proportionality constant is the same for all the three models and equals the ratio of the input spectrum to the noise spectrum.

Lemma 2. *For the same input–output data set*

$$\text{var}(\hat{G}_{\text{FIR}}) > \text{var}(\hat{G}_{\text{ARMark}}) > \text{var}(\hat{G}_{\text{ARX}}). \tag{58}$$

Proof. Assume $M \geqslant \mu + 2n$ and $\mu > 1$. N is same for the three methods. Then the proof directly follows from the expressions of variances in Eqs. (55) through (57). □

4.2.2. Covariance estimates for systems with non-white noise

Now assume that the actual process is subjected to non-white noise, v_0 with covariance,

$$Ev_0 v_0^{\mathrm{T}} = R, \tag{59}$$

where R is a positive-definite matrix. Then the covariance of the parameter estimates (Söderström & Stoica, 1989) is

$$\text{cov}(\hat{\theta}) = \left[\frac{1}{N}\sum_{k=1}^{N} \phi(k)\phi^{\mathrm{T}}(k)\right]^{-1} [\phi(1) \ \cdots \ \phi(N)] R \begin{bmatrix} \phi^{\mathrm{T}}(1) \\ \vdots \\ \phi^{\mathrm{T}}(N) \end{bmatrix} \left[\frac{1}{N}\sum_{i=1}^{N} \phi(k)\phi^{\mathrm{T}}(k)\right]^{-1}, \tag{60}$$

which is a non-minimal, positive-definite symmetric matrix. In practice, R is always unknown and replaced by the estimate of the error covariance matrix,

$$\hat{R} = \frac{1}{N}\sum_{i=1}^{N} \varepsilon(k)\varepsilon(k)^{\mathrm{T}} \tag{61}$$

$$= \frac{1}{N}\sum_{i=1}^{N} [y(k) - \hat{y}(k)][y(k) - \hat{y}(k)]^{\mathrm{T}} \tag{62}$$

$$= \frac{1}{N}\sum_{i=1}^{N} [y(k) - \phi^{\mathrm{T}}(k)\hat{\theta}_{LS}][y(k) - \phi^{\mathrm{T}}(k)\hat{\theta}_{LS}]^{\mathrm{T}}. \tag{63}$$

Since the structure of the regressor vector and hence the parameter vector is different in different LS model structures (ARX, FIR, ARMarkov), the methods cannot be compared directly. However, since we are specifically interested in the first μ Markov parameters, one approach is to compare the total sum of the variances corresponding to the first μ Markov parameters determined by each LS method. In the parametric methods such as ARX, the Markov parameters are not determined directly. Therefore, the proposed method of comparison is not directly applicable to parametric methods. However, the model parameters can be transformed into the Markov parameters and the variances of the model parameters can be mapped to the variances of the Markov parameters. But this is an indirect measure of variance. Here the sum of the variances of the available model parameters is used as a measure. Assuming that $M > \mu + 2n$, it has been shown *through simulation* (results are shown in Section 6) that

$$\sum_{i=1}^{\mu} \mathrm{var}(f_i^{(\mathrm{FIR})}) > \sum_{i=1}^{\mu} \mathrm{var}(h_i^{(\mathrm{ARMark})}) > \sum_{i=1}^{n} (\mathrm{var}(a_i^{(\mathrm{ARX})}) + \mathrm{var}(b_i^{(\mathrm{ARX})})). \quad (64)$$

Another basis of comparison of the methods is the largest eigenvalue of the covariance matrices (the largest eigenvalue corresponds to the maxima of the covariance matrix and hence the worst-case scenario). It has also been shown *through simulation* (results are shown in Section 6) that

$$\sigma_{\max}(\hat{R}_{\mathrm{FIR}}) > \sigma_{\max}(\hat{R}_{\mathrm{ARMark}}) > \sigma_{\max}(\hat{R}_{\mathrm{ARX}}), \quad (65)$$

where $\sigma_{\max}$ corresponds to the maximum eigenvalue.

Remark 2. The ARMarkov parameter estimates have larger variances than the parametric (ARX) method estimates and smaller variances than the non-parametric (FIR) method estimates for both white and non-white noise disturbances. This is intuitively reasonable since the variances are proportional to the number of estimated parameters.

4.2.3. Confidence intervals on the parameter estimates

The confidence intervals on the parameters estimated by the ARMarkov method can be constructed, as in Multiple Linear Regression. Assume that the errors, ε in the output prediction are normally and independently distributed with zero mean and variance λ^2. Let

$$C = \left[\frac{1}{N}\sum_{i=1}^{N} \phi(k)\phi^{\mathrm{T}}(k)\right]^{-1}. \quad (66)$$

Then

$$\frac{\hat{\mathbf{W}}_j - W_j}{\sqrt{\hat{\lambda}^2 C_{jj}}}, \quad j = 1,2,\dots,p \quad (67)$$

has a *t-distribution* with $(N-p)$ degrees of freedom.

The $100(1-\alpha)$ *percent confidence interval on the regression coefficients* W_j, $j = 1,2,\dots,p$ is given by

$$\hat{\mathbf{W}}_j - t_{\alpha,N-p}\sqrt{\hat{\lambda}^2 C_{jj}} \leqslant W_j \leqslant \hat{\mathbf{W}}_j + t_{\alpha,N-p}\sqrt{\hat{\lambda}^2 C_{jj}}, \quad (68)$$

where α is the level of significance (Johnson & Wichern, 1988). The confidence bounds and hence the confidence intervals involve the parameter variance. As discussed in the previous section, the variance of the first μ Markov parameters determined by the ARMarkov method is less than the variance of the first μ Markov parameters determined by the FIR method. Therefore, the ARMarkov parameters have tighter confidence bounds than the FIR parameters. The Markov parameters are not determined directly by the ARX method and hence this analysis is not directly applicable to the ARX method. However, the relationship between confidence bounds and variance implies that the ARX method has the tightest bounds.

Remark 3. As discussed in the above sections, the *consistency* of the ARMarkov parameter estimates is in between the consistencies of the parameters estimated by the parametric (ARX) and non-parametric (FIR) linear regression methods. The *variances* and hence the *confidence bounds* have a similar relationship. The ARX method lacks an analytical measure of consistency and the FIR method has the largest variance due to the large number of parameters. However, the Markov parameters estimated using the ARMarkov method have both the properties in an acceptable form.

5. The interactor matrix

As mentioned earlier, in multivariable systems, time delays correspond to the smallest number of Markov block matrices whose linear combination is nonsingular (Huang & Shah, 1999; Huang, 1997). The resulting block matrix is referred to as the interactor matrix (Tsiligiannis & Svoronos, 1988; 1989). Knowledge of the interactor matrix is important in high performance control strategies such as minimum variance control and in the analysis of closed-loop control performance (Peng & Kinnaert, 1992; Tsiligiannis & Svoronos, 1988; Huang & Shah, 1999). The procedure introduced in Huang and Shah (1999) to estimate the interactor matrix without complete knowledge of the process transfer function is discussed in the following sections.

5.1. Open-loop systems

Consider the multivariable system,

$$Y = GU + Ha_t,$$

where G and H are the system and disturbance transfer function matrices, respectively. The minimum variance controller is designed to make the variance of the interactor-filtered output ($\tilde{Y} = q^{-d}DY$) minimal where D is the interactor matrix and the positive integer d is the order of the interactor matrix which makes $q^{-d}D$ proper (Huang & Shah, 1999). There are different forms of the interactor matrix such as diagonal, triangular, etc. When $D^{\mathrm{T}}D = I$, the interactor matrix is called the unitary interactor matrix which has advantageous properties over the other forms as described in Peng and Kinnaert (1992) and Huang and Shah (1999).

5.1.1. *Determination of the order of the interactor matrix*

The interactor matrix is written as

$$D = D_0 q^d + D_1 q^{d-1} + \cdots + D_{d-1} q. \tag{69}$$

As described in Huang and Shah (1999), the interactor matrix can be factored out from the process Markov parameters as

$$[D_{d-1}, \ldots, D_0] \begin{bmatrix} G_0 & 0 & 0 & \ldots & 0 \\ G_1 & G_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ G_{d-2} & G_{d-1} & \cdots & \ddots & 0 \\ G_{d-1} & G_{d-2} & \cdots & \cdots & G_0 \end{bmatrix}$$

$$= [K, 0, \ldots, 0] \tag{70}$$

or simply

$$\mathbf{DG} = \mathbf{K}, \tag{71}$$

where $\mathbf{G}$ is a block-Toeplitz matrix consisting of the Markov parameters, $\mathbf{D}$ is the matrix form of the interactor and $\mathbf{K}$ is a full rank matrix. Existence of the solution of Eq. (71) to determine $\mathbf{D}$ depends on the order of the interactor matrix.

The order of the unitary interactor matrix can be determined following the procedure described in Huang and Shah (1999) using the process Markov block parameters obtained from the ARMarkov/LS method and singular-value decomposition (SVD) of $\mathbf{G}$.

5.1.2. *Estimation of the interactor matrix*

A block matrix consisting of the first d Markov parameters can be written as

$$\Lambda = [G_0^{\mathrm{T}}, G_1^{\mathrm{T}}, \ldots, G_d^{\mathrm{T}}]^{\mathrm{T}}. \tag{72}$$

Once the block matrix, Λ is formed, the unitary interactor matrix, $D(q)$ can be factored out from Λ. The details of the estimation procedure are not repeated here but are available in Huang and Shah (1999).

5.2. *Closed-loop systems*

In industrial applications, closed-loop data is much easier to obtain and/or more readily available than open-loop data. Therefore, it is preferable to estimate the interactor matrix using closed-loop data. Huang and Shah (1999) showed that the unitary interactor matrix is "feedback invariant", i.e. the linear combination of the Markov parameters of the process under open-loop and closed-loop conditions yields the same interactor matrix and the interactor matrix of the *open-loop* system can be estimated directly from *closed-loop* data. In closed-loop systems, identification using persistently exciting dither signals gives good estimates of the Markov parameters and in turn leads to a good estimate of the interactor matrix. Markov parameters obtained by the ARMarkov method using closed-loop input–output data can be used to estimate the interactor matrix as illustrated in Section 6.

6. Simulation examples

This section describes simulations carried out to estimate the Markov parameters from process input–output data. Statistical properties of the parameters estimated using the ARMarkov method are compared with the properties of the same parameters estimated by other least-squares identification methods. The time delays/interactor matrices estimated using the Markov parameters determined by different linear regression methods are also compared. Closed-loop input–output data as well as open-loop data are used to estimate the interactor matrix to confirm the "feedback invariance" of the interactor matrix.

6.1. *Open-loop SISO systems*

Example 6.1. Consider the third-order SISO system with the transfer function,

$$G(z) = \frac{0.0077z^{-1} + 0.0212z^{-2} + 0.0036z^{-3}}{1 - 1.9031z^{-1} + 1.1514z^{-2} - 0.2158z^{-3}} \tag{73}$$

subjected to colored noise generated by passing white noise through the disturbance transfer function

$$H(z) = \frac{1}{1 - 0.95z^{-1}}. \tag{74}$$

Since the ARMarkov method involves simple linear regression, the estimated Markov parameters are compared with the Markov parameters determined by simple correlation analysis (or FIR) and the ARX method which also uses linear regression. The ARMarkov method gives the best estimate of the process impulse reponse (Markov

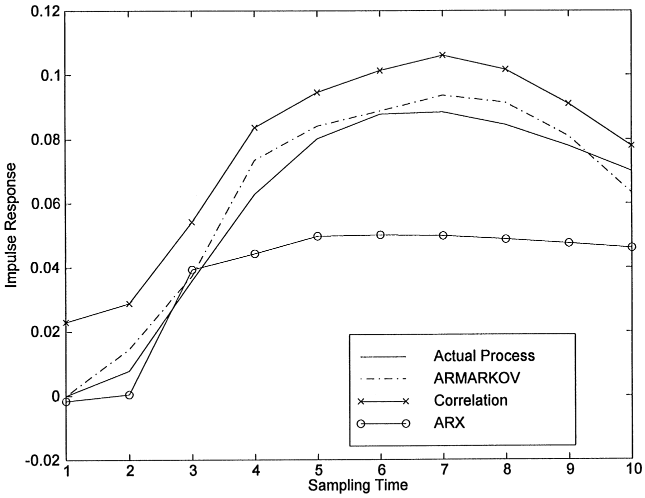

Fig. 1. Impulse response of a third-order system.

parameters) as shown in Fig. 1. The Markov parameters determined by the other methods have much larger errors because these methods try to fit the process over the whole range. The ARMarkov method gives better estimates of the first few Markov parameters which are needed to determine the process delay/interactor matrix.

$$G_p(z) = \begin{bmatrix} \frac{0.1091z^{-1}+0.07z^{-2}}{1-1.0844z^{-1}+0.2636z^{-2}} & \frac{0.0406z^{-1}-0.0113z^{-2}+0.0009z^{-3}}{1-0.6495z^{-1}+0.0482z^{-2}} \\ \frac{0.0406z^{-1}-0.0299z^{-2}-0.0047z^{-3}}{1-0.7849z^{-1}+0.7902z^{-2}} & \frac{0.0077z^{-2}+0.0212z^{-3}+0.0036z^{-4}}{1-1.9031z^{-1}+1.1514z^{-2}-0.2158z^{-3}} \end{bmatrix} \quad (75)$$

6.1.1. *Variance*

The same process described in (73) is considered.

White noise disturbance:

	ARMarkov ($\mu = 10, n = 3$)	FIR ($M = 25$)
total variance	2.51	3.34
$\sigma_{\max}$	3.16	4.45

Non-white disturbance (same colored noise described in Eq. (74)):

	ARMarkov ($\mu = 10, n = 3$)	FIR ($M = 25$)
total variance	6.67	12.77
$\sigma_{\max}$	5.31	9.01

6.1.2. *Confidence bounds*

Consider the process in Eq. (73) subjected to white noise. The confidence interval of the first μ Markov parameters estimated by the ARMarkov method and by the FIR method were determined by following the procedure described in Section 4.

The results (upper and lower confidence bounds) shown in Fig. 2 clearly show that the ARMarkov estimates have a tighter confidence bounds than the FIR estimates. Since the Markov parameters are not directly estimated in the ARX method, the confidence bounds for the ARX method were not compared here.

6.2. *Open-loop MIMO systems*

Example 6.2. This example is concerned with the Markov parameter estimation of a process which has different delays and different orders in the input–output pairs of the transfer function matrix. The elements of the diagonal disturbance transfer function matrix approximate random step-type disturbances which are common in chemical processes.

Consider a 2×2 multivariable system with the transfer function matrix

and the disturbance transfer function matrix

$$H = \begin{bmatrix} \frac{1}{1-0.95z^{-1}} & 0 \\ 0 & \frac{1}{1-0.99z^{-1}} \end{bmatrix} \quad (76)$$

From the impulse response plot in Fig. 3, it is obvious that the Markov parameters determined by the ARMarkov method are very close to the Markov parameters of the actual plant and better than the Markov parameters determined by the ARX and the Correlation methods (FIR). The first and second Markov parameter blocks determined by different methods are given in Table 1.

Example 6.3. This example, used by Huang (1997), illustrates the interactor matrix estimation of a simple 2×2 first-order system with single delay in all the input–output pairs in the transfer function matrix.

The plant transfer function matrix is

$$G_p(z) = \begin{bmatrix} \frac{z^{-1}}{1-0.1z^{-1}} & \frac{z^{-1}}{1-0.1z^{-1}} \\ \frac{2z^{-1}}{1-0.3z^{-1}} & \frac{2z^{-1}}{1-0.4z^{-1}} \end{bmatrix} \quad (77)$$

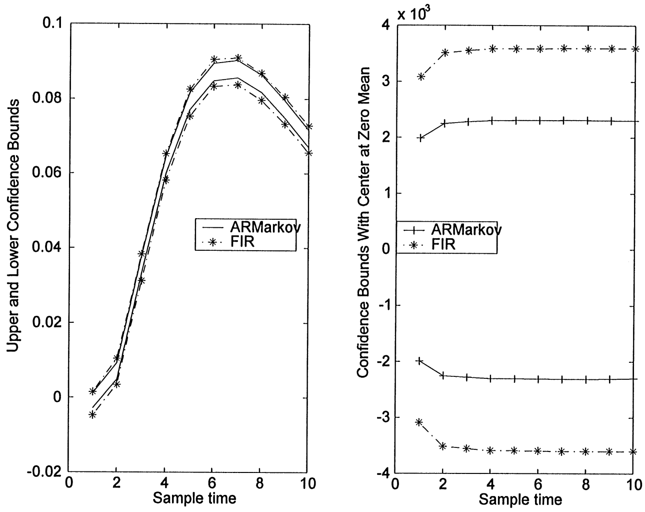

Fig. 2. Confidence bounds on Markov parameters.

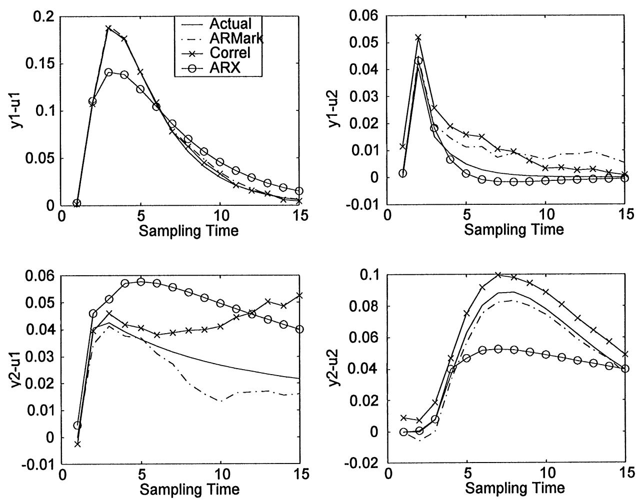

Fig. 3. Impulse response of a 2×2 MIMO system.

and the disturbance transfer function matrix is

$$H=\begin{bmatrix}\frac{1}{1-0.5z^{-1}} & \frac{-0.6}{1-0.5z^{-1}}\\ \frac{0.5}{1-0.5z^{-1}} & \frac{1}{1-0.5z^{-1}}\end{bmatrix}. \tag{78}$$

The unitary interactor matrices estimated by using the Markov parameters determined by different methods are shown in Table 2. The unitary interactor matrix estimated by using the Markov parameters determined by the ARMarkov method is exactly the same as in the actual process and the best estimate among all the methods used.

The above example considered first order transfer functions and a unit delay in all the input-output pairs. For this reason, the interactor matrices estimated by using the Markov parameters determined by the different methods are close to the actual interactor matrix and close to one another. Processes with higher order transfer functions and higher order delays (also different delays in different input–output pairs) result in greater differences in the estimated parameters. The following example is concerned with such a process.

Table 1
Markov parameter blocks of a 2×2 MIMO system

Method	*go*	*g*1
Actual	$\begin{bmatrix}0.1091 & 0.0406\\ 0.0406 & 0\end{bmatrix}$	$\begin{bmatrix}0.1883 & 0.0151\\ 0.0426 & 0.0077\end{bmatrix}$
ARMARKOV	$\begin{bmatrix}0.1063 & 0.0429\\ 0.0410 & 0.0060\end{bmatrix}$	$\begin{bmatrix}0.1844 & 0.0143\\ 0.0376 & 0.0035\end{bmatrix}$
FIR	$\begin{bmatrix}0.1098 & 0.0476\\ 0.0246 & -0.0100\end{bmatrix}$	$\begin{bmatrix}0.1866 & 0.0194\\ 0.0201 & -0.0099\end{bmatrix}$
ARX	$\begin{bmatrix}0.1451 & 0.0437\\ 0.0447 & 0.0032\end{bmatrix}$	$\begin{bmatrix}0.1557 & 0.0166\\ 0.0385 & -0.0036\end{bmatrix}$

Table 2
Interactor matrix for a 2×2 first-order system

Method	Unitary interactor matrix
Actual process	$\begin{bmatrix}-0.4472*q & -0.8944*q\\ 0.8944*q^2 & -0.4472*q^2\end{bmatrix}$
ARMARKOV	$\begin{bmatrix}-0.4472*q & -0.8944*q\\ 0.8944*q^2 & -0.4472*q^2\end{bmatrix}$
Correlation analysis	$\begin{bmatrix}-0.4437*q & -0.8962*q\\ 0.8962*q^2 & -0.4437*q^2\end{bmatrix}$
ARX	$\begin{bmatrix}-0.4469*q & -0.8974*q\\ 0.8974*q^2 & -0.4469*q^2\end{bmatrix}$

Example 6.4. Consider a 2×2 MIMO system with transfer function matrices

$$G_p(z)=\begin{bmatrix}\frac{0.3z^{-6}}{1-0.95z^{-1}} & \frac{0.0077z^{-1}+0.0212z^{-2}+0.0036z^{-3}}{1-1.9031z^{-1}+1.1514z^{-1}+0.2158z^{-3}}\\ \frac{z^{-6}}{1-0.7z^{-1}} & \frac{0.1091z^{-1}+0.07z^{-2}}{1-1.8044z^{-1}+0.2636z^{-2}}\end{bmatrix}, \tag{79}$$

$$H=\begin{bmatrix}\frac{1}{1-0.95z^{-1}} & 0\\ 0 & \frac{1}{1-0.99z^{-1}}\end{bmatrix}. \tag{80}$$

The unitary interactor matrices estimated by different methods are shown in Table 3.

Table 3
Coefficients in the interactor matrix

$$\text{Interactor structure: } \begin{bmatrix} m1*q^1 + \cdots + m5*q^5 & p5*q^1 + \cdots + p1*q^5 \\ p1*q^2 + \cdots + p5*q^6 & m5*q^2 + \cdots + m1*q^6 \end{bmatrix}$$

	$m1$	$m2$	$m3$	$m4$	$m5$	$p1$	$p2$	$p3$	$p4$	$p5$
Actual	0.0680	0.1957	0.0897	0.1087	0.0609	0.0043	0.0203	0.0353	0.0517	0.9639
ARMark	0.0722	0.1939	0.0896	0.1146	0.0539	0.0040	0.0196	0.0360	0.0524	0.9637
Correl	0.0876	0.1914	0.0550	0.0704	0.1268	0.0115	0.0321	0.0281	0.0443	0.9632
ARX	0.0596	0.1987	0.1127	0.0036	0.1109	0.0068	0.0229	0.0202	0.0361	0.9642

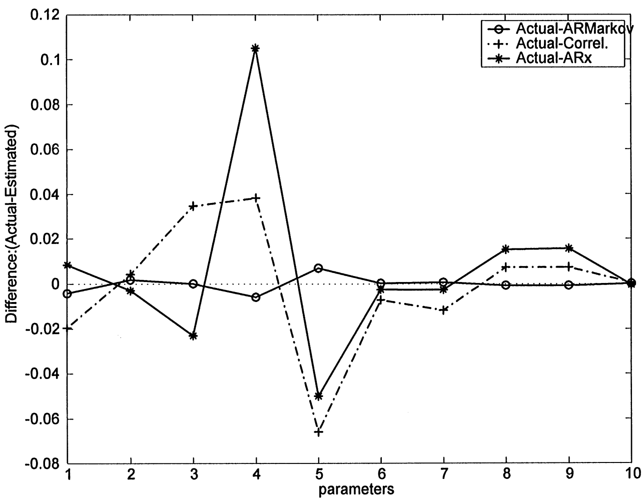

Fig. 4. Errors in the estimated coefficients of the interactor.

The differences between each coefficient in the actual unitary interactor matrix and the unitary matrix estimated by different methods are plotted in Fig. 4 so that the accuracy of the estimated parameters can be compared more easily.

It can be clearly seen that the coefficients estimated by the ARMarkov method have the smallest error.

6.3. Closed-loop systems

Since closed-loop data are more readily available and/or easier to collect in practical/industrial applications, the ARMarkov method was also applied to estimate the Markov parameters from closed-loop input-output data. A simple closed-loop system is shown in Fig. 5. The purpose here was not to design a controller but to show that the open-loop and closed-loop unitary interactor matrices are the same as proved theoretically by Huang (1997). Therefore, a simple proportional controller was used. The next example uses the same

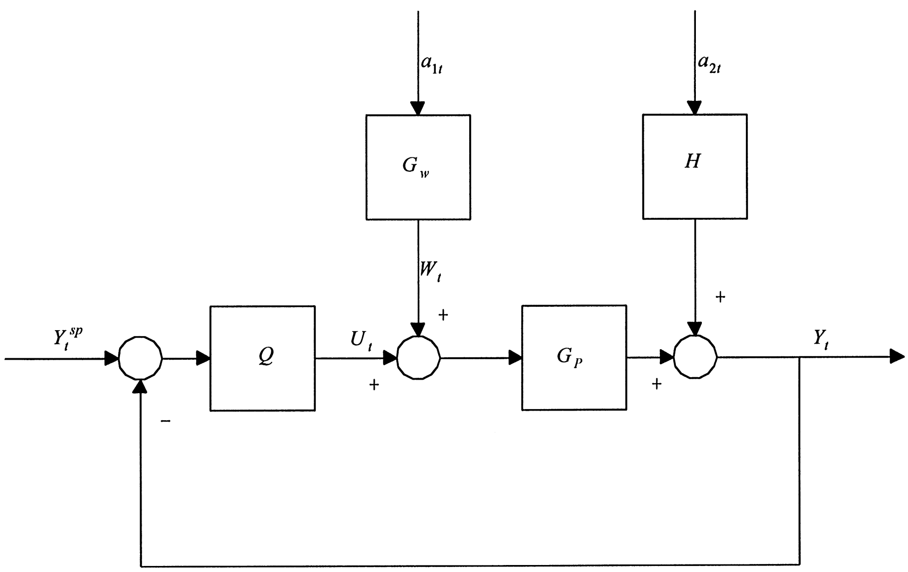

Fig. 5. Closed-loop system.

Table 4
Interactor matrix from closed-loop data

Method	Unitary interactor matrix
Actual process	$\begin{bmatrix} -0.4472*q & -0.8944*q \\ 0.8944*q^2 & -0.4472*q^2 \end{bmatrix}$
ARMarkov method	$\begin{bmatrix} -0.4469*q & -0.8945*q \\ 0.8945*q^2 & -0.4469*q^2 \end{bmatrix}$
Correlation analysis	$\begin{bmatrix} -0.4494*q & -0.8933*q \\ 0.8933*q^2 & -0.4494*q^2 \end{bmatrix}$
ARX	$\begin{bmatrix} -0.4534*q & -0.8913*q \\ 0.8913*q^2 & -0.4534*q^2 \end{bmatrix}$

first-order plant and disturbance transfer functions as in Example 6.3 for the open-loop case.

Example 6.5. The transfer function matrices used are

$$G_p(z) = \begin{bmatrix} \frac{z^{-1}}{1-0.1z^{-1}} & \frac{z^{-1}}{1-0.1z^{-1}} \\ \frac{2z^{-1}}{1-0.3z^{-1}} & \frac{2z^{-1}}{1-0.4z^{-1}} \end{bmatrix},$$

$$H = \begin{bmatrix} \frac{1}{1-0.5z^{-1}} & \frac{-0.6}{1-0.5z^{-1}} \\ \frac{0.5}{1-0.5z^{-1}} & \frac{1}{1-0.5z^{-1}} \end{bmatrix}, \tag{81}$$

$$Q = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad G_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y_t^{sp} = 0. \tag{82}$$

The unitary interactor matrices estimated by the different methods using closed-loop input–output data are given in Table 4. In this example also, the ARMarkov method gives the best match with the actual process and the interactor matrix is almost the same as in the open-loop case shown in Table 2. Other simulations (not shown here) show that the disturbance dynamics have less effect on the ARMarkov method than on the other methods.

7. Conclusions

- The ARMarkov/LS identification method is extended to MIMO systems.
- The Markov parameters estimated by the ARMarkov/LS method have better statistical properties (e.g. consistency, variance, confidence bounds) than the parameters estimated by the other linear regression methods such as ARX, FIR, etc.
- The estimated Markov parameters and the time delay/interactor matrix are closer to the actual process response than the results obtained by the other linear regression methods described above.
- The effect of disturbances on the ARMarkov/LS parameter estimates appears to be smaller than on the parameters estimated by the other linear regression methods such as ARX and FIR.
- The Interactor matrix is theoretically feedback invariant. Simulation examples show that the interactor matrix estimated from closed-loop data is almost identical to the one determined from open-loop data.

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