Subspace Identification (Tijl De Bie '05)

TTT 2009

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07/14/09

Dynamical Systems

Dynamical systems describe **change** (through a continuous phase space) according to a **fixed update rule**—often with external inputs.

- Physical processes (convection, chemical reactions)
- Economic processes (price of corn, oil)
- A vehicle
- Changes in a robot's sensor values... :)

Pop culture significance:

- Chaos
- The "butterfly effect"
- Strange attractors

Some Restrictions...

This talk is about a restricted subset:

- Linear (no hidden triggers)
- Time invariant
- Causal

Linear Model

We feed inputs (u) into the model, and observe outputs (y). Unobserved state (x) evolves behind the scenes:

$$x_{t+1} = Ax_t + Bu_t + v_x$$
$$y_t = Cx_t + Du_t + v_y$$

- u_t are inputs at time t,
- y_t are outputs at time t,
- x_t are (hidden) state variables at time t,
- v_* are noise terms.

Future depends only on past outputs through current state (**Markov property**).

Linear Model

This subset still captures some important phenomena:

- Projectiles and springs
- Inductors, capacitors
- Wave propagation

Moreover many systems are linear around working points (e.g., aircraft dynamics).

$$x_{t+1} = Ax_t + Bu_t + v_x$$
$$y_t = Cx_t + Du_t + v_y$$

System Identification of a Linear System

System identification: to parametrize a model of how state changes. Linear case:

- Given an input sequence ($[u_0 \ldots u_n]$),
- and an output sequence ($[y_0 \ldots y_n]$),
- tell me A, B, C, D.

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + v_x \\ y_t &= Cx_t + Du_t + v_y \end{aligned}$$

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$$y_t = Cx_t + Du_t + v_y$$

- The game's afoot!

Subspace Methods

Subspace methods entail:

- Determine a state sequence X_i
- Fit (A,B,C,D) to X_i via regression

Also,

- No worries about convergence
- Readily available solution tools
- The state-space representation is nice
- Conceptually clear

Subspace Methods

Key observation (1):

- We know $[u_i \dots u_{i+j+1}]$ and $[y_0 \dots y_{i+j+1}]$.
- If we knew the state sequence, $X_i = [x_i \dots x_{i+j}]$, we could use simple linear regression to determine A,B,C,D:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_i & x_{i+1} & \dots & x_{i+j-1} \\ u_i & u_{i+1} & \dots & u_{i+j+1} \end{bmatrix} \approx \begin{bmatrix} x_{i+1} & x_{i+2} & \dots & x_{i+j} \\ y_i & y_{i+1} & \dots & y_{i+j+1} \end{bmatrix}$$

- ...but we don't know X_i.

Subspace Methods

Key observation (2):

- State x_t is a summary of i steps of the past (where $u_p = [u_{t-i} \dots u_t]'$ and $y_p = [y_{t-i} \dots y_t]'$):

$$w_X^T x_t \approx w_{U_p}^T u_p + w_{Y_p}^T y_p$$

Subspace Methods

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- State x_t is also all we need to predict future outputs (where $u_f = [u_{t+1} \dots u_{t+i}]'$ and $y_f = [y_{t+1} \dots y_{t+i}]'$):

$$w_{Y_f}^T y_f \approx w_{U_f}^T u_f + w_X^T x_t$$

Subspace Methods

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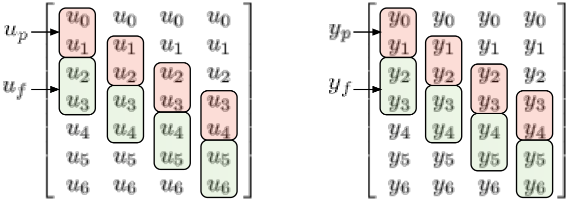
$$w_{Y_f}^T y_f \approx w_{U_f}^T u_f + w_X^T x_t$$

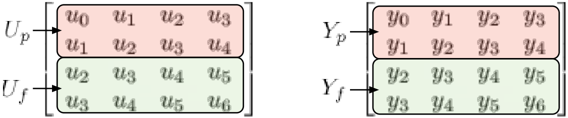
- Combined algebraically (and absorbing the signs):

$$w_{Y_f}^T y_f + w_{U_f}^T u_f + w_{U_p}^T u_p + w_{Y_p}^T y_p \approx 0$$

Invariance



$$w_{Y_f}^T y_f + w_{U_f}^T u_f + w_{U_p}^T u_p + w_{Y_p}^T y_p \approx 0$$



$$w_{Y_f}^T Y_f + w_{U_f}^T U_f + w_{U_p}^T U_p + w_{Y_p}^T Y_p \approx 0$$

Optimization

If this is true:

$$w_{Y_f}^T Y_f + w_{U_f}^T U_f + w_{U_p}^T U_p + w_{Y_p}^T Y_p \approx 0$$

Our problem becomes:

$$\min_{w_{Y_f}, w_{U_f}, w_{U_p}, w_{Y_p}} ||w_{Y_f}^T Y_f + w_{U_f}^T U_f + w_{U_p}^T U_p + w_{Y_p}^T Y_p||^2$$

...subject to constraints so that no w_* shrink to 0:

$$1 = ||w_{Y_f}^T Y_f||^2 = ||w_{U_f}^T U_f||^2 = ||w_{Y_p}^T Y_p||^2 = ||w_{U_p}^T U_p||^2$$

Optimization

Taking the derivative with a single Lagrange multiplier λ gives system of 4 linear equations:

$$\begin{bmatrix} 0 & Y_f U_f^T & Y_f U_p^T & Y_f Y_p^T \\ U_f Y_f^T & 0 & U_f U_p^T & U_f Y_p^T \\ U_p Y_f^T & U_p U_f^T & 0 & U_p Y_p^T \\ Y_p Y_f^T & Y_p U_f^T & Y_p U_p^T & 0 \end{bmatrix} \begin{bmatrix} w_{Y_f} \\ w_{U_f} \\ w_{U_p} \\ w_{Y_p} \end{bmatrix}$$

$$= \lambda \begin{bmatrix} Y_f Y_f^T & 0 & 0 & 0 \\ 0 & U_f U_f^T & 0 & 0 \\ 0 & 0 & U_p U_p^T & 0 \\ 0 & 0 & 0 & Y_p Y_p^T \end{bmatrix} \begin{bmatrix} w_{Y_f} \\ w_{U_f} \\ w_{U_p} \\ w_{Y_p} \end{bmatrix}$$

$$\Rightarrow J\mathbf{w} = \lambda K \mathbf{w}$$

$$\Rightarrow \boxed{\mathbf{w} = \mathrm{eig}(K^{-1}J)}$$

State Sequence

A single eigenvector provides 1-D basis for state:

$$w_X^T X_i \approx w_{U_p}^T U_p + w_{Y_p}^T Y_p,$$

Multiple eigenvectors provide more elaborate bases for state:

$$\boxed{W_x X_i \approx W_{U_p}^T U_p + W_{Y_p}^T Y_p}$$

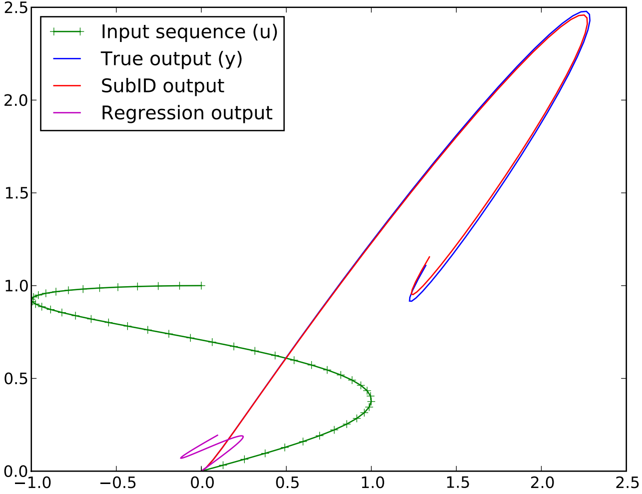
...onto some examples!

Simple Example

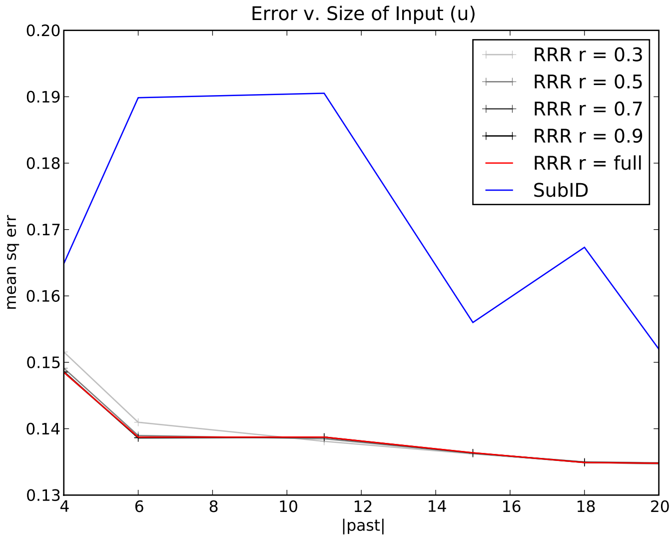
Output generated explicitly with known A, B, C, D.

- Two-dimensional inputs.
- Two-dimensional outputs.
- Low noise levels: N(0,0.05).

Simple Example



Critterbot Light Sensors Example

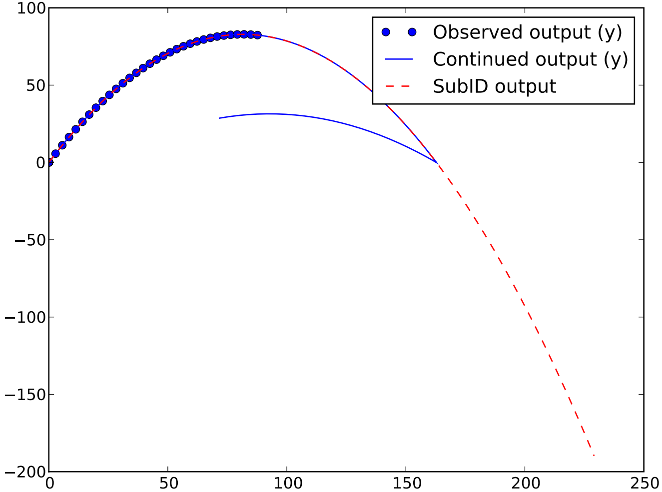


Oddball Example

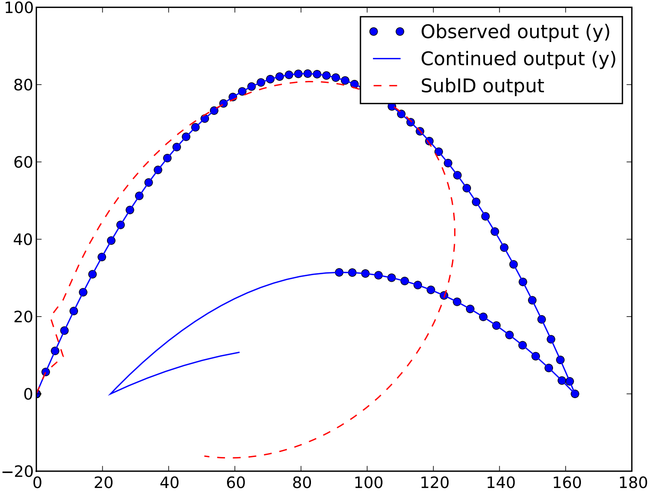
A ball at rest. It gets kicked and flies through the air.

- Two-dimensional input (horizontal/vertical force).
- Two-dimensional outputs (position).
- No noise.
- Ball bounces—oddly.

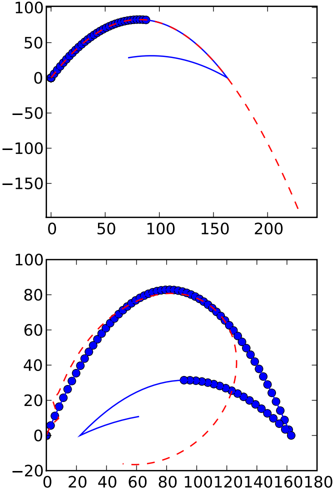
Oddball Example



Oddball Example 2



Oddball State Manifold



Extensions

- Change optimization constraints (CVA):

$$1 = ||w_{Y_f}^T Y_f||^2 = ||w_{U_f}^T U_f||^2 = ||w_{Y_p}^T Y_p||^2 = ||w_{U_p}^T U_p||^2$$

- Regularize objective function (Tijl De Bie '05):

$$\min_{w_{Y_f}, w_{U_f}, w_{U_p}, w_{Y_p}} ||w_{Y_f}^T Y_f + w_{U_f}^T U_f + w_{U_p}^T U_p + w_{Y_p}^T Y_p||^2 + \gamma(||w_{U_p}||_{\ell 2}^2 + ||w_{U_f}||_{\ell 2}^2)$$

- Restrict the rank of:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Further Reading

Articles on the subject include:

- Overschee and De Moor's "Subspace Identification for Linear Systems" (or any papers they've written together).
- Katayama's "Subspace Methods for System Identification" (reading group).

The derivation in this talk is (only) available online:

- http://videolectures.net/slsfs05_bie_slasi
- Presented at SLSFS '05 in Bohinj by Tijl De Bie, who introduced regularization and kernelization of SubID methods.
- No publication—yet.

Why?

Why should you care?

- Methods are very strong **when the problem is right.**
- Potential for integration with machine learning.

Why should you be skeptical?

- These methods can be **bad** when the problem is wrong.
- Linearity is a strong assumption.

Thanks!