Construction of Minimal Linear State-Variable Models from Finite Input–Output Data

ANTHONY J. TETHER

***Abstract*—An algorithm for constructing minimal linear finite-dimensional realizations (a minimal partial realization) of an unknown (possibly infinite-dimensional) system from an external description as given by its Markov parameters is presented. It is shown that the resulting realization in essence models the transient response of the unknown system. If the unknown system is linear, this technique can be used to find a smaller dimensional linear system having the same transient characteristics. If the unknown system is nonlinear, the technique can be used either 1) to determine a useful nonlinear model, or 2) to determine a linear model, both of which approximate the transient response of the nonlinear system.**

I. Introduction

When one designs a new system, the performance requirements of the system, as given by an input–output description, must be specified. The design is completed when the specification is met by connecting some physical components, resulting in a state description. Furthermore, if one has a system and wishes to derive a mathematical model (state description), the externally observed data input–output description must also be used.

The translation of one description into another is an important problem which, while noted and discussed in the late 1950's, first became clear when Kalman [5] proved a theorem which showed that the input–output description reveals only the controllable and observable part of a dynamical system, and that this part is a dynamical system with the smallest state-space dimension among systems having the same input–output relations. Henceforth, the determination of an internal description from an input–output description became known as the minimal realization problem.

The minimal realization problem was first solved by Gilbert [2] who, with the restriction that each element of the transfer-function matrix has distinct poles, gave an algorithm for computing the map, i.e., transfer-function matrix to state-variable differential equations. At the same time, Kalman [6] gave an algorithm for this same problem which reduced the state space of a nonminimal realization until it became minimal.

Ho and Kalman [4] approached this problem from a new viewpoint. They showed that the minimal realization problem is equivalent to a representation problem involving an infinite sequence of real matrices Y_i, $i = 1, \cdots$, known as the Markov parameters. They then gave an algorithm for computing the map

$$\{Y_i\}_{1 \leq i \leq \infty} \rightarrow \text{state-variable differential equations.}$$

However, this algorithm has three implicit assumptions.

1) The transfer function is known exactly.
2) There are an infinite number of Markov parameters available.
3) The underlying system is finite dimensional.

Since a transfer function can be represented as a power series in the Markov parameters [4], e.g., in the continuous case $Z(s) = \sum_{i=1}^{\infty} s^{-i} Y^i$, where s is the complex number, assumptions 1) and 2) are equivalent. Also, if 3) is true, then 2) can be removed, since a finite number of Markov parameters is sufficient to determine a realization for a finite-dimensional system.

In this paper it will be assumed that only partial information about the system is available in the form of a finite sequence of Markov parameters $\{Y_1, \cdots, Y_{N_0}\}$, $Y_i = HF^{i-1}G$. This will be called the partial realization problem, i.e., the realization of all systems whose first N_0 Markov parameters are equal to the given sequence. The determination of realizations with the smallest state-space dimension whose first N_0 Markov parameters are equal to the given sequence will be called the minimal partial realization problem.

The solution of the minimal partial realization problem will be discussed in Section VIII. The following claims can be made for this solution.

1) From the computational standpoint, the solution is attractive, since only simple numerical operations are needed.

2) Given experimental input–output data, there is no reason to assume the underlying system is finite-dimensional. If the dimension is finite, the solution will give an exact realization with the correct dimension as more information is obtained.

Manuscript received November 17, 1969. Paper recommended by P. Dorato, Chairman of the IEEE G-AC Optimal Systems Committee. This paper is based on a dissertation submitted in partial fulfillment of the requirements for the Ph.D. degree at Stanford University, Stanford, Calif., and was presented at the 1970 JACC, Atlanta, Ga. This work was supported in part by the U. S. Air Force under Contract AF-49(638)-1440.
The author is with Systems Control, Inc., Palo Alto, Calif.

II. General Background Material

It is well known [8] that a real finite-dimensional linear constant dynamical system has an internal description given by the state-variable equations

$$x(t+1) = Fx(t) + Gu(t), \quad y(t) = Hx(t) \tag{1}$$

for the discrete-time case and

$$dx/dt = Fx + Gu, \quad y = Hx \tag{2}$$

for the continuous-time case, where t is the time, $x(n \times 1)$, $u(p \times 1)$, and $y(m \times 1)$ are real vectors, and F, G, and H are matrices with real elements with sizes $n \times n$, $n \times p$, and $m \times n$, respectively. The dimension of the system is equal to the size of F.

The external description of a linear system may be stated in either the time domain or in the transform domain. For system (1), the time-domain description is given by the pulse-response function

$$t \rightarrow Y_t = HF^{t-1}G, \quad t = 1,2,\cdots \tag{3}$$

while the transform-domain description is given by the transfer function

$$z \rightarrow T(z) = H(zI - F)^{-1}G, \quad z = \text{complex number}. \tag{4}$$

For system (2), the impulse–response function is

$$t \rightarrow W(t) = He^{Ft}G, \quad t = \text{real number} \geq 0 \tag{5}$$

and the transfer function is

$$s \rightarrow Z(s) = H(sI - F)^{-1}G. \tag{6}$$

Definition 1: The systems described by (1) and (2) will be denoted by the triple

$$\Sigma \sim \{F,G,H\}.$$

For later results, it will be useful to define the notion of equivalence. Two systems are normally said to be equivalent if and only if they have the same output response for all inputs. For systems described by (1) and (2) this can be formalized by the following definition.

Definition 2: Two systems Σ_1 and Σ_2 are equivalent if and only if they have the same 1) impulse-response function for the continuous case, and 2) pulse-response function for the discrete-time case, i.e.,

1) $H_1 e^{F_1 t} G_1 = H_2 e^{F_2 t} G_2, \quad \text{for all } t \geq 0$

2) $H_1 F_1^{t-1} G_1 = H_2 F_2^{t-1} G_2, \quad t = 1,2,\cdots.$

However, it can be shown that there exists a commonality by which both continuous- and discrete-time systems can be described, namely, the infinite sequence of matrices $\{Y_i\} = \{HF^{i-1}G\}$, $i = 1,2,\cdots$. The matrices $\{Y_k\}$ are called the Markov parameters of the system due to their similarity to certain classical results (1).

Proposition 1: A real finite-dimensional system Σ (either continuous- or discrete-time) is completely determined by the Markov parameters of the system $Y_i = HF^{i-1}G$, $i = 1,2,\cdots$.

Proof: See [4].

Therefore, if we are given a sequence of Markov parameters $\{Y_i\}$, the relationships (3)–(6) are completely determined. In other words, the external description of a real finite-dimensional linear constant dynamical system, either continuous- or discrete-time, is given by a sequence of Markov parameters.

Hence a new definition of equivalent system, which will be used for the remainder of this paper, follows from these results.

Definition 3: Two systems Σ_1 and Σ_2 are equivalent if and only if they have the same Markov parameters.

III. The Realization Problem

As shown in the previous section, the input–output description of both continuous- and discrete-time systems are completely specified by the Markov parameters (Definition 1), which are in turn explicitly defined by the matrices $\{F,G,H\}$, which completely describe the internal description of both continuous- and discrete-time systems (1), (2). Therefore, the minimal realization problem can be stated (for either continuous- or discrete-time systems) in terms of the Markov parameters.

Minimal Realization Problem

Given a sequence of $m \times p$ constant matrices $\{Y_i\} 1 \leq i \leq \infty$, find a triple $\{F,G,H\}$ of constant matrices such that

$$Y_i = HF^{i-1}G, \quad i = 1,2,\cdots$$

where the size of F is as small as possible. From Proposition 1, we see that this is equivalent to finding a minimal system Σ given input–output data in the form of the Markov parameters.

Implicit in the realization problem is the question of determining when the sequence $\{Y_i\}$ (or Σ) has a finite-dimensional realization and, if so, the corresponding minimal (smallest) dimension. The following results give necessary and sufficient conditions for $\{Y_i\}$ (or Σ) to have a finite-dimensional realization and a method for determining the minimal dimension. The proofs will not be given here but can be found elsewhere [8]. First, it is necessary to state some definitions.

Definition 4: Given a sequence of real $m \times p$ matrices $\{Y_i\}$, $i = 1,2,\cdots$, the generalized Hankel matrix $H_{l',l}$ is defined as

$$H_{l',l} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_l \\ Y_2 & Y_3 & \cdots & Y_{l+1} \\ \vdots & \vdots & & \vdots \\ Y_{l'} & Y_{l'+1} & \cdots & Y_{l+l'-1} \end{bmatrix} (ml' \times pl).$$

If the Y_i are 1×1 matrices, then $H_{l',l}$ is the classical Hankel matrix, which is symmetric. In the general case, the matrix $H_{l',l}$ is symmetric if and only if all Y_i are symmetric $m \times m$ matrices.

Definition 5: Let σ^K denote the shift operator so that if $H_{l',l}$ is defined as in Definition 4, then

$$\sigma^K H_{l',l} = \begin{bmatrix} Y_{1+K} & Y_{2+K} & \cdots & Y_{l+K} \\ Y_{2+K} & Y_{3+K} & \cdots & Y_{l+K+1} \\ \vdots & \vdots & & \vdots \\ Y_{l'+K} & Y_{l'+K+1} & \cdots & Y_{l'+l+K-1} \end{bmatrix} (ml' \times pl).$$

Proposition 2: The sequence of $m \times p$ Markov parameters $\{Y_i\}$, $1 \leq i \leq \infty$, has a finite-dimensional realization if and only if there exist integers N',N such that

$$\rho H_{N',N} = \rho H_{N'+i,N+j} = n_0, \quad \rho \triangleq \text{rank of}$$

for all $i = 0,1,2,\cdots$, $j = 0,1,2,\cdots$. If so, the integers N' and N are each separately less than or equal to n_0.

Proposition 3: Assume the sequence $\{Y_i\}$, $1 \leq i \leq \infty$, has a finite-dimensional realization and let N',N be integers satisfying Proposition 2. Then the minimal dimension of a realization of $\{Y_i\}$, $1 \leq i \leq \infty$, is equal to

$$n_0 = \rho H_{N',N}.$$

Assuming that 1) there is an infinite number of Markov parameters available and 2) the unknown system is finite-dimensional, we can determine a minimal realization $\{F,G,H\}$. This can be accomplished using an algorithm which has become known as Ho's algorithm. This will be stated here but not proved. The proof can be found elsewhere [4].

Ho's Algorithm

1) Form $H_{N',N}$, where N' and N satisfy Proposition 2.
2) Find nonsingular matrices P,Q such that

$$PH_{N',N}Q = \begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \end{bmatrix} = U_{n_0}\tilde{U}_{n_0} = J$$

where I_{n_0} is an $n_0 \times n_0$ unit matrix, $n_0 = \rho H_{N',N}$, and

$$U_{n_0} = \begin{bmatrix} I_{n_0} \\ 0 \end{bmatrix} (N'm \times n_0), \quad \tilde{U}_{n_0} = [I_{n_0} \quad 0] \; (n_0 \times N_p).$$

3) Let E_m be the block matrix

$$[I_m \quad 0_m \quad \cdots \quad 0_m](m \times mN')$$

and E_p be

$$[I_p \quad 0_p \quad \cdots \quad 0_p] \; (p \times N_p).$$

Then

$$\begin{aligned} F &= \tilde{U}_{n_0}[JP(\sigma H_{N',N})QJ]U_{n_0} \\ G &= \tilde{U}_{n_0}[JPH_{N',N}E_p'] \\ H &= E_p[H_{N',N}QJ]U_{n_0}. \end{aligned} \tag{7}$$

Finally, there is one property of minimal realization which will be used later and is, in itself, one of the most interesting and important results discovered in realization theory.

Proposition 4: Two minimal systems $\Sigma_1 \sim \{F_1,G_1,H_1\}$ and $\Sigma_2\{F_2,G_2,H_2\}$ are equivalent if and only if there exists T, $\det T \neq 0$, such that

$$\begin{aligned} F_1 &= TF_2T^{-1} \\ G_1 &= TG_2 \\ H_1 &= H_2T^{-1}. \end{aligned}$$

IV. The Concept of Partial Realizations

Ho's algorithm, specifically, and the other results of Sections II and III depend heavily on the assumption that the input–output map has a finite-dimensional realization and/or there are an infinite number of Markov parameters available.

However, if we have an infinite sequence of $m \times p$ Markov parameters $Y_1,Y_2,\cdots$, which does not have a finite-dimensional realization, i.e., Proposition 2 is not satisfied for any N',N, is it possible to approximate this infinite-dimensional system with a finite-dimensional system in the following sense; the first N_0 Markov parameters of a finite-dimensional approximation are equal to the first N_0 Markov parameters of the infinite-dimensional system? Alternately, if we have only a finite sequence of Markov parameters available initially, is it possible to find a finite-dimensional linear constant dynamical system whose first N_0 Markov parameters are equal to the given finite sequence?

Anticipating the partial realization description of systems, it is necessary to introduce some terminology.

Definition 6: $\Sigma \sim \{F,G,H\}$ is said to be a partial realization of order N_0 of an input–output map if and only if $\{Y_i\} = \{HF^{i-1}G\}$ holds for $i = 1,\cdots,N_0$.

Definition 7: $\Sigma \sim \{F,G,H\}$ is said to be a minimal partial realization of order N_0 if and only if the size of F in Σ is minimal among all other $\Sigma' \sim \{F',G',H'\}$ satisfying Definition 6.

The determination of a realization, if there exists any, which satisfies Definition 6 will be called the partial realization problem. The determination of a realization satisfying Definition 7 will be called the minimal partial realization problem.

V. The Partial Realization Concept for Discrete Systems

Let Σ be a discrete-time system (1) of dimension m (m may be infinite). Assuming zero initial conditions, the output of Σ, $y(i)$, is related to the inputs $u(i)$ via the Markov parameters by the formulas

$$\begin{aligned} y(1) &= Y_1u(0) \\ y(2) &= Y_2u(0) + Y_1u(1) \\ &\vdots \\ y(N_0) &= Y_{N_0}u(0) + \cdots + Y_1u(0). \end{aligned} \tag{8}$$

Let Σ^* be a partial realization of order N_0 of Σ. Then we can state the following result.

Proposition 5: The output of Σ^*, $y^*(i)$, is equal to the output of Σ, $y(i)$, for the first N_0 time values for any input $u(i)$, $i = 0,\cdots,N_0 - 1$.

Proof: The result is obvious from (8).

This proposition shows that if we have a discrete-time system and if only the output is desired (the states of the system are truly abstract quantities) for a finite number of time values, a partial realization, if there exists one, models the original system exactly for the first N_0 time values.

If we are given the transform-domain description $T(z)$ (4) of a discrete-time system, we can consider $T(z)$ as a power series expansion (see Proposition 1)

$$z \to T(z) = \sum_{i=1}^{\infty} Y_i z^{-i}.$$

Here a partial realization of order N_0 is equivalent to matching the first N_0 terms of $T(z)$

$$Y_1 z^{-1} + Y_2 z^{-2} + \cdots + Y_{N_0} z^{-N_0}.$$

VI. The Partial Realization Concept for Continuous Systems

If we are given the transfer function $Z(s)$ (6), or the impulse-response function $W(t)$ (5) of a continuous-time system, we can consider these as power series expansions (Proposition 1)

$$s \to Z(s) = \sum_{i=1}^{\infty} Y_i s^{-i}$$

$$t \to W(t) = \sum_{i=1}^{\infty} Y_i t^{i-1}/(i-1)!$$

respectively. Therefore, the partial realization concept of continuous-time systems is clearly equivalent to matching the first N_0 terms of $Z(s)$ or $W(t)$

$$Y_1 s^{-1} + \cdots + Y_{N_0} s^{-N_0}$$

or

$$Y_1 + Y_2 t + \cdots + Y_{N_0} t^{N_0 - 1}/(N_0 - 1)!.$$

VII. Existence of Partial Realizations

That there always exists a partial realization of order N_0 which satisfies Definition 6 is shown by the following lemma.

Lemma 1

Every finite sequence of N_0 $m \times p$ matrices $\{Y_i,\cdots,Y_{N_0}\}$ admits an extension sequence $\{Y_{N_0+1},\cdots\}$ for which a completely controllable and observable partial realization $\Sigma \sim \{F,G,H\}$ of order N_0 exists via Ho's algorithm.

Proof: See [8].

This result proves that there always exists a partial realization. However, it has not been shown if there always exists a minimal partial realization (Definition 7). Intuitively, at least, the existence of minimal partial realizations is obvious. Since there always exist partial realizations (Lemma 1) there exists at least one which has a smaller dimension than the others. This can be seen by noting that according to Lemma 1 there always exists a partial realization of finite dimension n'. Since the set of dimensions less than or equal to n' is finite, the minimum will be attained.

However, what are the properties of minimal partial realizations? Are they unique in the sense of Proposition 4? Can we calculate the minimal dimension in a similar manner as in Proposition 3? Can we construct a minimal partial realization directly from the given data? These are the questions which have yet to be answered. This is the problem we intend to solve and whose complete solution will be found in the following sections.

VIII. The Minimal Partial Realization Problem

In Section III, the minimal realization problem for real finite-dimensional linear constant dynamical systems was discussed and a solution (Ho's algorithm) was given. In Sections IV–VII, the partial realization problem (Definition 6) was stated and solved (Lemma 1). However, we wish to solve the minimal partial realization problem in Definition 7 in the sense of Sections V and VI for discrete- and continuous-time dynamical systems. From Sections V and VI, we see that the minimal partial realization problem is the same for both discrete and continuous time in the following sense: given a sequence of N_0 Markov parameters, find a triple $\{F,G,H\}$ such that Definition 7 holds. Therefore, we do not have to make a distinction between continuous- and discrete-time systems. In fact, we do not even have to consider the N_0 sequence of matrices as Markov parameters, but only as matrices. Hence we can state the minimal partial realization problem in a purely mathematical framework.

Minimal Partial Realization Problem

Given a finite sequence $\{Y_1,\cdots,Y_{N_0}\}$ of $m \times p$ matrices with real elements, find a triple $\{F,G,H\}$ such that

1) $Y_i = HF^{i-1}G, \quad i = 1,\cdots,N_0$

2) size F = minimum.

IX. Uniqueness of Extensions of a Partial Sequence

In this paper, we wish to find a minimal partial realization using Ho's algorithm. Assume we have found a minimal partial realization $\tilde{\Sigma} \sim \{\tilde{F},\tilde{G},\tilde{H}\}$. $\tilde{\Sigma}$ is unique (in the sense of Proposition 4) if and only if the extension sequence defined by $Y_i = HF^{i-1}G$, $i = N_0 + 1,\cdots$, is unique. In other words, if there exist two extension sequences $Y_i,\hat{Y}_i$, $i = N_0 + 1,\cdots$, which give us two minimal realizations $(\Sigma,\hat{\Sigma})$ by Ho's algorithm (both of which satisfy Definition 7), then Σ and $\hat{\Sigma}$ are not unique in the sense of Proposition 4.

Therefore, our first problem is determining if and/or when minimal partial realizations can be nonunique. First, it is necessary to prove the following results. (The proofs can be found in the Appendix.)

Lemma 2

Given the matrices $A(p \times p')$, $B(p \times r)$, and $C(s \times p')$ such that

$$\rho(A) = \rho(A \mid B) = \rho\begin{bmatrix} A \\ \hline C \end{bmatrix} \tag{9a}$$

there exists, at most, one matrix $D(s \times r)$ for which

$$\rho\begin{bmatrix} A & \vdots & B \\ \hline C & \vdots & D \end{bmatrix} = \rho(A \mid B) = \rho\begin{bmatrix} A \\ \hline C \end{bmatrix} = \rho(A). \tag{9b}$$

This result will be used later in determining the uniqueness of elements in the extension sequence $\{Y_{N_0+1},\cdots\}$ of a partial sequence $\{Y_i,\cdots,Y_{N_0}\}$. The following is an obvious corollary of this result.

Corollary 1

Given a finite sequence of $m \times p$ matrices $\{Y_1,\cdots,Y_{N_0}\}$ satisfying

$$\rho H_{N',N} = \rho H_{N'+1,N} = \rho H_{N',N+1}$$

for some N,N' such that $N' + N = N_0$, the extension of the sequence $\{Y_1,\cdots,Y_{N_0}\}$ to $\{Y_1,\cdots,Y_{N_0},Y_{N_0+1},\cdots, Y_{N_0+K},\cdots\}$, $1 \leq K \leq \infty$, for which

$$\rho H_{m',m} = \rho H_{N',N}$$

where $m' + m = N_0 + k$, is unique.

X. Realizability Criterion

The following result gives conditions for the existence of a unique minimal partial realization of a partial sequence $\{Y_1,\cdots,Y_{N_0}\}$.

Let $\{Y_1,\cdots,Y_{N_0}\}$ be an arbitrary finite sequence of $m \times p$ real matrices and let $H_{i,j}$, $i + j \leq N_0$, be a corresponding Hankel matrix. Then a minimal partial realization Σ given by Ho's algorithm is unique (modulo a change in basis) and realizes the sequence up to and including the term N_0 if and only if there exist positive integers N',N such that

1) $$N' + N = N_0$$

2) $$\rho H_{N',N} = \rho H_{N'+1,N} = \rho H_{N',N+1}.$$

Remark

This result is a consequence of Ho's algorithm for the following reasons. If 1) and 2) are satisfied for a finite sequence $\{Y_1,\cdots,Y_{N_0}\}$, then Ho's algorithm can be employed with

$$n_0 = \rho H_{N',N}$$

$$H_{N',N} = \begin{bmatrix} Y_1 & \cdots & Y_N \\ \vdots & & \vdots \\ Y_{N'} & \cdots & Y_{N_0-1} \end{bmatrix}$$

$$\sigma H_{N',N} = \begin{bmatrix} Y_2 & \cdots & Y_{N+1} \\ \vdots & & \vdots \\ Y_{N'+1} & \cdots & Y_{N_0} \end{bmatrix}.$$

The resulting minimal partial realization $\Sigma \sim \{F,G,H\}$ clearly satisfies Definition 7. The realization is unique (in the sense of Proposition 4) since the extension sequence $Y_i = HF^{i-1}G$, $i = N_0 + 1,\cdots$, generated by it will satisfy

$$\rho H_{N'+i,N+j} = \rho H_{N',N}$$

for all $i,j \geq 0$. By Corollary 1 the extension sequence must be unique. Therefore, by Definition 3 and Proposition 4, the realization is unique.

If 1) and 2) are not satisfied, then a minimal partial realization, if one exists, may not be unique for the following reason. In order to use Ho's algorithm in this case, new matrices $\{Y_{N_0+1},\cdots,Y_{P_0}\}$ must be specified until $\rho H_{M',M} = \rho H_{M'+1,M} = \rho H_{M',M+1}$, where $M' + M = P_0$. However, these matrices may be completely or partially arbitrary. Since F, G, and H are functions of $Y_{N_0+1},\cdots,Y_{P_0}$ (Ho's algorithm), they may not be unique.

XI. Determination of the Minimal Dimension

The first step in computing a minimal partial realization is the computation of the minimal dimension. If the sequence satisfies the realizability criterion, its minimal dimension is clearly

$$n_0 = \rho H_{N',N}.$$

Therefore, assume that the realizability criterion is not satisfied and consider the incomplete Hankel array associated with a given partial sequence $\{Y_1,\cdots,Y_{N_0}\}$

$$\begin{bmatrix} Y_1 & Y_2 & Y_3 & \cdots & Y_{N_0-2} & Y_{N_0-1} & Y_{N_0} \\ Y_2 & Y_3 & Y_4 & \cdots & Y_{N_0-1} & Y_{N_0} & * \\ Y_3 & Y_4 & Y_5 & \cdots & Y_{N_0} & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ Y_{N_0-2} & Y_{N_0-1} & Y_{N_0} & \cdots & * & * & * \\ Y_{N_0-1} & Y_{N_0} & * & \cdots & * & * & * \\ Y_{N_0} & * & * & \cdots & * & * & * \end{bmatrix} \tag{10}$$

where the positions indicated by the asterisks are left blank, since no data is available. Obviously, a lower bound for the dimension of any extension can be obtained by counting the number of linearly independent rows (or columns) already in (10), since their number cannot decrease when the asterisks are filled in. Since the row rank of a matrix is equal to its column rank, it is sufficient to count the linearly independent rows.

Lemma 3

Let $\{Y_1,\cdots,Y_{N_0}\}$ be a finite sequence of $m \times p$ matrices with real elements and let Σ be a partial realization whose first N_0 Markov parameters are equal to the given sequence. Then the dimension of a minimal partial realization min dim Σ satisfies the following inequality:

$$\min \dim \Sigma \geq n(N_0) = \rho H_{1,N_0} + (\rho H_{2,N_0-1} - \rho H_{1,N_0-1}) + \cdots + (\rho H_{N_0,1} - \rho H_{N_0-1,1}) = \sum_{j=1}^{N_0} \rho H_{j,N_0+1-j} - \sum_{j=1}^{N_0} \rho H_{j,N_0-j}.$$

Proof: See the Appendix.

Therefore, $n(N_0)$ is the lower bound of the dimension of the minimal realization for any extension of the partial sequence $\{Y_1,\cdots,Y_{N_0}\}$.

However, as of yet, we have not proven the existence of extensions for which $n(N_0)$ is the dimension of the resulting realization. It does turn out that this lower bound can be achieved for suitably chosen extensions.

Definition 8: Let $N'(N_0)$ equal the first integer such that every row of the block row $[Y_{N'(N_0)+1},\cdots,Y_{N_0}]$ is linearly dependent on the rows of the Hankel matrix $H_{N'(N_0),N_0-N'(N_0)}$.

Let $N(N_0)$ equal the first integer such that every column of the block column

$$\begin{bmatrix} Y_{N(N_0)+1} \\ \vdots \\ Y_{N_0} \end{bmatrix}$$

is linearly dependent on the columns of $H_{N_0-N(N_0),N(N_0)}$.

Finally, with the help of the following lemma, the main result can be stated and proved.

Lemma 4

Let $n(N_0),N'(N_0),N(N_0)$ be as defined in Lemma 3 and Definition 8. Then any extension $\{Y_{N_0+1},\cdots\}$ of $\{Y_1,\cdots, Y_{N_0}\}$ whose realization achieves the minimal lower bound $n(N_0)$ for its dimension also satisfies

$$\rho H_{N'(N_0),N(N_0)} = \rho H_{N'(N_0)+1,N(N_0)} = \rho H_{N'(N_0),N(N_0)+1}$$

for that extension.

Proof: See the Appendix.

The preceding lemma also shows that for any extension $Y_{N_0+1},\cdots$, if there exists any, whose realization achieves the lower bound $n(N_0)$, $N'(N_0)$ and $N(N_0)$ are true invariants. Note that $N'(N_0)$ and $N(N_0)$ are related to the controllability and observability properties of the realization. Therefore, a realization satisfying Lemma 4 has the property that

$$\rho[G,FG,\cdots,F^{N-1}] = n(N_0)$$

$$\rho \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{N'-1} \end{bmatrix} = n(N_0).$$

For discrete-time systems, this means that all states in the state space can be reached in N time values for some control input $\{u_0,\cdots,u_{N-1}\}$ and all states can be determined (observed) after N' observations of the output are made [9].

Note that the ranks of G and H are clearly related to N and N'. Since N' and N are constant over all partial realizations with dimension $n(N_0)$ (if there exist any), the ranks of G and H should also remain constant and therefore be invariants.

XII. The Minimal Partial Realization Theorem

Theorem 1[1]

Let $\{Y_1,\cdots,Y_{N_0}\}$ be a fixed partial sequence of $m \times p$ matrices with real coefficients and let $n(N_0)$, $N(N_0)$, and $N'(N_0)$ be the integers defined previously. Then

1) $n(N_0)$ is the dimension of the minimal realization Σ;
2) $N(N_0)$ and $N'(N_0)$ are (separately) the smallest integers such that the realizability criterion holds simultaneously for all minimal extensions;
3) there is a minimal extension of order $P(N_0) = N(N_0) + N'(N_0)$ for which $n(N_0)$ is the dimension of the realization computed by Ho's algorithm, but which is in general not unique; and
4) every extension which is fixed up to $P(N_0)$ is uniquely determined thereafter.

Proof: If $P(N_0) \leq N_0$, the sequence $\{Y_1,\cdots,Y_{N_0}\}$ satisfies the realizability criterion and 1)–4) are clearly true. So we will assume that $P(N_0) > N_0$. Also, the argument N_0 is dropped from $n(\cdot)$, $N'(\cdot)$, and $N(\cdot)$ for this proof.

If there does exist an extension 3) whose realization has dimension n, then 1), 2), and 4) are true by Lemmas 3 and 4 and Corollary 1, respectively. Therefore, all we have to do is show the existence of such an extension. This will be done by a method by which the extension may actually be constructed. Consider the Hankel array

$$\begin{bmatrix} Y_1 & \cdots & Y_{N_0-N'} & \cdots & Y_{N-1} & Y_N & Y_{N+1} \\ \vdots & & \vdots & & \vdots & \vdots & \vdots \\ Y_{N_0-N} & \cdots & Y_{2N_0-P-1} & \cdots & Y_{N_0-2} & Y_{N_0-1} & Y_{N_0} \\ Y_{N_0-N+1} & \cdots & Y_{2N_0-P} & \cdots & Y_{N_0-1} & Y_{N_0} & Y_{N_0+1} \\ Y_{N_0-N+2} & \cdots & Y_{2N_0-P+1} & \cdots & Y_{N_0} & Y_{N_0+1} & Y_{N_0+2} \\ \vdots & & \vdots & & \vdots & \vdots & \vdots \\ Y_{N'} & \cdots & Y_{N_0-1} & \cdots & & & \\ Y_{N'+1} & \cdots & Y_{N_0} & \cdots & & & \end{bmatrix} \tag{11}$$

where $\{Y_{N_0+1},Y_{N_0+2},\cdots,Y_p\}$ are to be chosen so that $\rho H_{N',N} = n$.

Procedure

If a row in the last block row of $H_{N_0-N+1,N}$ is linearly dependent, we fill in the corresponding row of Y_{N_0+1} in $H_{N_0-N+1,N+1}$ so as to maintain linear dependence. By Lemma 2, this determines the elements of that row uniquely. This process is continued until all linearly dependent rows of the last block row of $H_{N_0-N+1,N}$ are exhausted.

[1] A similar result was discovered independently and simultaneously by Kalman [7].

Now we pass to the last block row of $H_{N_0-N+2,N}$. The parts of the linearly dependent rows of the last block row of $H_{N_0-N+1,N}$ in the last block row of $H_{N_0-N+2,N}$ will still be linearly dependent with the previously computed elements for Y_{N_0+1} *because of the special nature of the Hankel pattern*. We now consider the linearly dependent rows in the last block row of $H_{N_0-N+2,N-1}$ which were linearly independent in the last block row of $H_{N_0-N+1,N}$. The corresponding rows of Y_{N_0+1} in $H_{N_0-N+2,N}$ are filled in to preserve linear dependence as before. Here these rows of Y_{N_0+1} could be nonunique since Lemma 2 may not be valid.

If there are still rows in the last block row of $H_{N_0-N+2,N-1}$ which are independent, we go to the last block row of $H_{N_0-N+3,N-2}$ and so on until linear dependence of all rows is obtained. This always happens since the last block row of $H_{N'+1,N_0-N'}$ is defined to be linearly dependent on H_{N',N_0-N}. Choosing Y_{N_0+1} in this way does not increase the minimum rank of $H_{N',N}$. In other words, if we used Lemma 3 on the sequence $\{Y_1,\cdots,Y_{N_0},Y_{N_0+1}\}$ the value of $n(N_0+1)$ would equal $n(N_0)$.

The process now continues by induction until all of the matrices $\{Y_{N_0+1},\cdots,Y_p\}$ are determined. By virtue of the way $\{Y_{N_0+1},\cdots,Y_p\}$ are determined, $n(p) = n(p-1) = \cdots = n(N_0)$ and, therefore, $\rho H_{N',N} = n(N_0) = \rho H_{N'+1,N}$, where the second equality is true by construction.

By definition of N

$$\rho H_{N_0-N,N} = \rho H_{N_0-N,N+1}.$$

The matrix Y_{N_0+1} was evaluated such that the number of independent rows in $H_{N_0-N+1,N}$ are equal to the number of independent rows in $H_{N_0-N+1,N+1}$. Therefore, the increase in rank of the matrix $H_{N_0-N+1,N}$ over the rank of $H_{N_0-N,N}(\rho H_{N_0-N+1,N} - \rho H_{N_0-N,N})$ is equal to the increase in rank of $H_{N_0-N+1,N+1}$ over $H_{N_0-N+1,N+1}(\rho H_{N_0-N+1,N+1} - \rho H_{N_0-N,N+1})$. Hence

$$\rho H_{N_0-N+1,N+1} - \rho H_{N_0-N,N+1} = \rho H_{N_0-N+1,N} - \rho H_{N_0-N,N}$$

which implies that

$$\rho H_{N_0-N+1,N+1} = \rho H_{N_0-N+1,N}.$$

Continuing by induction, we obtain

$$\rho H_{N',N} = \rho H_{N',N+1}.$$

Hence we can find a realization Σ by Ho's algorithm which has dimension $n(N_0)$. It is not unique since some elements of $\{Y_{N_0+1},\cdots,Y_p\}$ may not be unique.

This proves the existence of extensions whose realization achieves the dimension $n(N_0)$. Q.E.D.

The following example illustrates the use of Theorem 1.

XIII. Example

Let

$$Y_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}$$

$$Y_3 = \begin{bmatrix} 10 & 7 \\ 1 & 1 \end{bmatrix}, \quad Y_4 = \begin{bmatrix} 22 & 15 \\ 3 & 3 \end{bmatrix}.$$

Then

$$\rho \begin{bmatrix} 1 & 1 & 4 & 3 & 10 & 7 & 22 & 15 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 & 3 \end{bmatrix} = \rho H_{1,4} = 2$$

$$\rho \begin{bmatrix} 1 & 1 & 4 & 3 & 10 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \rho H_{1,3} = 2$$

$$\rho \begin{bmatrix} 1 & 1 & 4 & 3 & 10 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 4 & 3 & 10 & 7 & 22 & 15 \\ 0 & 0 & 1 & 1 & 3 & 3 \end{bmatrix} = \rho H_{2,3} = 4$$

$$\rho \begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 4 & 3 & 10 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \rho H_{2,2} = 3$$

$$\rho \begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 4 & 3 & 10 & 7 \\ 0 & 0 & 1 & 1 \\ 10 & 7 & 22 & 15 \\ 1 & 1 & 3 & 3 \end{bmatrix} = \rho H_{3,2} = 4$$

$$\rho \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 4 & 3 \\ 0 & 0 \\ 10 & 7 \\ 1 & 1 \end{bmatrix} = \rho H_{3,1} = \rho H_{4,1} = \rho \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 4 & 3 \\ 0 & 0 \\ 10 & 7 \\ 1 & 1 \\ 22 & 15 \\ 3 & 3 \end{bmatrix} = 2.$$

Therefore,

$$\begin{aligned} n(4) &= \rho H_{1,4} + (\rho H_{2,3} - \rho H_{1,3}) + (\rho H_{3,2} - \rho H_{2,2}) \\ &\quad + (\rho H_{4,1} - \rho H_{3,1}) \\ &= 2 + (4-2) + (3-2) + (2-2) \\ &= 5. \end{aligned}$$

Hence the minimal dimension is equal to 5. Since $\rho H_{1,4} = \rho H_{1,3}$ and $\rho H_{4,1} = \rho H_{3,1}$, there exists a Y_5 and Y_6 (Theorem 1) such that $\rho H_{3,3} = \rho H_{4,3} = \rho H_{3,4}$. After determining the dependent rows and columns, we choose Y_5 and Y_6 as in Theorem 1

$$Y_5 = \begin{bmatrix} 46 & 31 \\ x_1 & x_1 \end{bmatrix}, \quad Y_6 = \begin{bmatrix} 94 & 63 \\ x_2 & x_2 \end{bmatrix}.$$

Notice that there are only two arbitrary elements in Y_5 and Y_6. The system from which the four $\{Y_1,\cdots,Y_4\}$ Markov parameters were computed is

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 3 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and the resulting system is

$$F = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ 6 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & x_1 - 13 & 4 & -6x_1 + x_2 + 27 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with the characteristic polynomial equal to

$$(s^2 - 3s + 2)((s+1)(s-4)s + s(13 - x_1) + 6x_1 - x_2 - 27).$$

XIV. Conclusion

The realization problem of multiple input–output linear constant systems when only partial information is available has been solved in the preceding sections of this paper.

The problem as formulated and solved matches exactly the first N_0 Markov parameters of a linear constant system, which is possibly infinite dimensional, to that of a finite-dimensional linear constant system.

For continuous-time systems, this corresponds to determining a transfer-function matrix (6) or impulse-response function (5) whose first N_0 terms in their respective power series expansion (Proposition 1) are equal to the given finite sequence $\{Y_1,\cdots,Y_{N_0}\}$. This is clearly, at best, a matching of the high-frequency response, if the transfer function is considered, or the transient response, if the impulse-response function is considered, of the system. However, techniques of this sort have been used in the past [10, ch. 13] with mixed success. For example, the well-known time-delay transfer function e^{-as} has been approximated in essentially this manner in the past using the Padé theory [11].

For discrete-time systems, the problem corresponds to determining a transfer-function matrix (4) or impulse-response function (3) whose first N_0 Markov parameters are equal to the given finite sequence. However, because of Proposition 5, we can say that the resulting realization is an excellent approximation of the transient-response for the first N_0 time values. Hence the theory developed in this paper can be used directly in discrete-time problems where the input–output relationships are of concern for a finite amount of time.

Also, the unknown system need not be linear for this theory to apply. If the system under consideration (known or unknown) is truly linear and only the transient (initial) response is under consideration, the minimal partial realization can be used to reduce the size of the original system.

If the system is nonlinear, the minimal partial realization concept can be used to obtain a linear model of the system which approximates the initial or transient response of the system for a given input. If minimal partial realizations are then obtained for various inputs, it may be possible to

1) determine a useful nonlinear model of the system by noting how the individual elements of F, G, and H vary with the different inputs, or
2) determine a linear model which approximates the transient response of the unknown system for inputs of interest.

Some development has been accomplished in this area in modeling the time behavior of complex nonlinear systems. For example, this technique has been illustrated on a simple example and appears to be a potentially very powerful tool for developing specialized problem-oriented identification procedures for approximating the transient responses of boilers and turbines [3].

Another research area which is likely to benefit from the present method is the building of adaptive systems by real-time identification of Σ, in which the estimates of F, G, and H are continuously updated as new data are received. If the realizability criterion is satisfied, we can determine what the system was over the previous time interval. Furthermore, we have conditions for determining if the system is changing, i.e., until we receive information for which the realizability criterion no longer holds, the realization remains constant. If the realizability criterion is not satisfied, or no longer holds, Theorem 1 gives us a method for determining what class of systems we are now considering.

Appendix

Proof of Lemma 2

Proof: From (9a), it is obvious that there exist two equivalence classes $M = \{\alpha_i\}$, with α_i equal to a $p' \times r$ matrix with real elements, and $N = \{\beta_i\}$, with β_i equal to an $s \times p$ matrix with real elements, such that

$$A\alpha_i = B, \quad \beta_i A = C$$

respectively.

Let D^+ be an $s \times r$ matrix which satisfies (9b). Then there exist matrices $\alpha_1(p' \times r)$, $\beta_1(s \times p)$ (not unique) such that

$$\left[\begin{array}{c} A \\ \hline C \end{array}\right] \alpha_1 = \left[\begin{array}{c} B \\ \hline D^+ \end{array}\right]$$

$$\beta_1[A \mid B] = [C \mid D^+]$$

which separate into

$$A\alpha_1 = B, \qquad \beta_1 A = C$$

$$C\alpha_1 = D^+, \quad \beta_1 A = D^+.$$

This implies that for any D^+ which satisfies (9b), the corresponding α_i and β_i are in the classes M and N, respectively. Assume that there exist $\alpha_2 \in M$ and $\beta_2 \in N$ such that

$$C\alpha_2 = \hat{D}, \quad \beta_2 B = \hat{D}$$

where $\hat{D} \neq D^+$. But

$$D^+ = C\alpha_1 = \beta_2 A \alpha_1 = \beta_2 B = \hat{D}.$$

Hence, if there exists a D which satisfies (9b), it is unique.

Q.E.D.

Proof of Corollary 1

Proof: The proof is by induction.

1) $k = 1$, let

$$A = H_{N',N}, \quad B = \begin{bmatrix} Y_{N+1} \\ \vdots \\ Y_{N_0} \end{bmatrix}$$

$$C = [Y_{N'+1}, \cdots, Y_{N_0}], \quad D = Y_{N_0+1}$$

and apply Lemma 2

2) assume true for k and assume there exists Y_{N_0+K+1} such that

$$\rho \left[\begin{array}{c|c} H_{N'+K,N} & \begin{matrix} Y_{N+1} \\ \vdots \\ Y_{N_0+K} \end{matrix} \\ \hline Y_{N'+K+1} \cdots Y_{N_0+K} & Y_{N_0+K+1} \end{array}\right] = \rho \left[\begin{array}{c|c} H_{N'+K,N} & \begin{matrix} Y_{N+1} \\ \vdots \\ Y_{N_0+K} \end{matrix} \end{array}\right]$$

$$= \rho \left[\begin{array}{c} H_{N'+K,N} \\ \hline Y_{N'+K+1} \cdots Y_{N_0+K} \end{array}\right].$$

Then apply Lemma 2 with

$$A = H_{N'+K,N}, \quad B = \begin{bmatrix} Y_{N+1} \\ \vdots \\ Y_{N_0+K} \end{bmatrix}$$

$$C = [Y_{N'+K+1}, \cdots, Y_{N_0+K}], \quad D = Y_{N_0+K+1}.$$

Q.E.D.

Proof of Lemma 3

Proof: For any extension of the sequence $\{Y_1, \cdots, Y_{N_0}\}$,

$$\begin{aligned} \rho H_{N_0,N_0} \equiv \rho H_{1,N_0} &+ (\rho H_{2,N_0} - \rho H_{1,N_0}) \\ &+ \cdots + (\rho H_{j+1,N_0} - \rho H_{j,N_0}) \\ &+ \cdots + (\rho H_{N_0,N_0} - \rho H_{N_0-1,N_0}). \end{aligned}$$

But $(\rho H_{j+1,N_0} - \rho H_{j,N_0})$ means the increase in rank of H_{j,N_0} after a block row is adjoined. This implies

$$(\rho H_{j+1,N_0} - \rho H_{j,N_0}) \geq 0.$$

Therefore,

$$\begin{aligned} \min \dim \Sigma \geq \min \rho H_{1,N_0} &+ \cdots + \min (\rho H_{j+1,N_0} - \rho H_{j,N_0}) \\ &+ \cdots + \min (\rho H_{N_0,N_0} - \rho H_{N_0,N_0-1}). \end{aligned}$$

Obviously, $\min \rho H_{1,N_0} = \rho H_{1,N_0}$ since H_{1,N_0} consists only of the given sequence. Now consider

$$\min (\rho H_{2,N_0} - \rho H_{1,N_0}).$$

Since Y_{N_0+1} is unknown, the smallest increase occurs when the dependent rows of the second block row of H_{2,N_0} remain dependent after the block column

$$\begin{bmatrix} Y_{N_0} \\ \\ Y_{N_0+1} \end{bmatrix}$$

is adjoined (9). Clearly, this increase is equal to

$$(\rho H_{2,N_0-1} - \rho H_{1,N_0-1}).$$

Now consider

$$\min (\rho H_{j+1,N_0} - \rho H_{j,N_0}).$$

Again, since in this case $Y_{N_0+1}, \cdots, Y_{N_0+j}$ are unknown, the smallest increase occurs when the dependent rows of the $(j+1)$th block row of H_{j+1,N_0-j} remain dependent after the matrix

$$\begin{bmatrix} Y_{N_0-j+1} & \cdots & Y_{N_0} \\ Y_{N_0-j+2} & \cdots & Y_{N_0+1} \\ \vdots & & \vdots \\ Y_{N_0+1} & & Y_{N_0+j} \end{bmatrix}$$

is adjoined. Notationally, this increase is equal to

$$(\rho H_{j+1,N_0-j} - \rho H_{j,N_0-j}).$$

Note that H_{j+1,N_0-j} and H_{j,N_0-j} consist only of the given sequence of matrices.

Hence

$$\min \dim \Sigma \geq n(N_0) = \rho H_{1,N_0} + \sum_{j=1}^{N_0-1} (\rho H_{j+1,N_0-j} - \rho H_{j,N_0-j})$$

$$= \sum_{j=1}^{N_0} \rho H_{j,N_0+1-j} - \sum_{j=1}^{N_0-1} \rho H_{j,N_0-j}.$$

Q.E.D.

Proof of Lemma 4

Proof: From the definition of $N'(N_0),N(N_0)$, we see that the minimum number of linearly independent rows (or columns) in (10) can be counted by only considering the matrix

$$\begin{bmatrix} Y_1 & \cdots & Y_{N_0-N'(N_0)} & \cdots & Y_{N(N_0)-1} & Y_{N(N_0)} \\ \cdot & & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & & \cdot & \cdot \\ Y_{N_0-N(N_0)+2} & \cdots & Y_{2N_0-N(N_0)+N'(N_0)+1} & \cdots & Y_{N_0} & * \\ \cdot & & \cdot & & * & * \\ \cdot & & \cdot & & * & * \\ \cdot & & \cdot & & * & * \\ Y_{N'(N_0)-1} & \cdots & Y_{N_0} & *** & * & * \\ Y_{N'(N_0)} & \cdots & * & *** & * & * \end{bmatrix}$$

where the asterisks, as before, are locations of the unknown parameters $\{Y_{N_0+1},\cdots\}$. Hence any realization which achieves the lower bound $n(N_0)$ must also have the property that

$$n(N_0) = \rho H_{N'(N_0),N(N_0)}.$$

Now assume that there exists a minimal partial realization of order N_0 with dimension $n(N_0)$. Assume that for this extension $\rho H_{N'(N_0)+1,N(N_0)} > \rho H_{N'(N_0),N(N_0)}$. But, by Propositions 2 and 3, this implies that the dimension of the realization is greater than $n(N_0)$. A similar argument holds for $\rho H_{N'(N_0),N(N_0)+1} > \rho H_{N'(N_0),N(N_0)}$. Hence

$$\rho H_{N'(N_0),N(N_0)} = \rho H_{N'(N_0)+1,N(N_0)} = \rho H_{N'(N_0),N(N_0)+1}$$

for any extension whose realization has dimension $n(N_0)$.

Q.E.D.

References

[1] F. R. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1960.
[2] E. G. Gilbert, "Controllability and observability in multivariable control systems," *SIAM J. Control*, vol. 2, no. 1, pp. 128–151, 1963.
[3] L. P. Hajdu, Ed., "New developments in system state estimation and anomaly detection," Systems Control, Inc., Palo Alto, Calif., Contract 14-03-79968, Final Rept., October 1969.
[4] B. L. Ho and R. E. Kalman, "Effective construction of linear state-variable models from input-output functions," *Proc. 3rd Ann. Allerton Conf. Circuit and System Theory*, 1965, pp. 449–459; also *Regelungstech.*, vol. 14, pp. 545–548, 1966.
[5] R. E. Kalman, "Canonical structure of linear dynamical systems," *Proc. Natl. Acad. Sci. U. S.*, vol. 48, pp. 596–600, 1962.
[6] ——, "Mathematical description of linear dynamical systems," *SIAM J. Control*, vol. 1, no. 2, pp. 152–192, 1963.
[7] ——, *On Partial Realizations of a Linear Input/Output Map*. New York: Holt, Rinehart and Winston, 1970.
[8] R. E. Kalman, P. L. Falb, and M. A. Arbib, *Topics in Mathematical System Theory*. New York: McGraw-Hill, 1957.
[9] R. C. K. Lee, *Optimal Estimation, Identification, and Control*, Res. Mono. 28. Cambridge, Mass.: M.I.T. Press, 1964.
[10] J. E. Storer, *Passive Network Synthesis*. New York: McGraw-Hill, 1957.
[11] H. S. Wall, *Analytical Theory of Continued Fractions*. Princeton, N. J.: Van Nostrand, 1948.

Anthony J. Tether was born in Middletown, N. Y., on November 28, 1941. He received the B.S. degree from Rensselaer Polytechnic Institute, Troy, N. Y., in 1964, and the M.S. and Ph.D. degrees from Stanford University, Stanford, Calif., in 1965 and 1969, respectively, all in electrical engineering.

He was a NASA Fellow and Research Assistant at Stanford University from 1964 to 1969. He has been with Systems Control, Inc., Palo Alto, Calif., since 1969, applying estimation and control theory for the real-time control of stochastic nonlinear systems.

Dr. Tether is a member of Tau Beta Pi, Sigma Xi, and Eta Kappa Nu.