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Subspace model identification
Part 1. The output-error state-space model identification class of algorithms

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In this paper, we present two novel algorithms to realize a finite dimensional, linear time-invariant state-space model from input–output data. The algorithms have a number of common features. They are classified as one of the subspace model identification schemes, in that a major part of the identification problem consists of calculating specially structured subspaces of spaces defined by the input–output data. This structure is then exploited in the calculation of a realization. Another common feature is their algorithmic organization: an RQ factorization followed by a singular value decomposition and the solution of an overdetermined set (or sets) of equations. The schemes assume that the underlying system has an output-error structure and that a measurable input sequence is available. The latter characteristic indicates that both schemes are versions of the MIMO Output-Error State Space model identification (MOESP) approach. The first algorithm is denoted in particular as the 'elementary MOESP scheme'. The subspace approximation step requires, in addition to input–output data, knowledge of a restricted set of Markov parameters. The second algorithm, referred to as the 'ordinary MOESP scheme', solely relies on input–output data. A compact implementation is presented of both schemes. Although we restrict our presentation here to error-free input–output data, a framework is set up in an identification context. The identification aspects of the presented realization schemes are treated in the forthcoming Parts 2 and 3.

1. Introduction

The identification of multiple-input multiple-output (MIMO) linear time-invariant state space models from input–output measurements is a problem of central importance in system analysis, design and control. In general terms it can be viewed as the problem of finding a mapping between the available input–output data sequences and unknown parameters in a user-defined class of models, e.g. state-space models.

A particular class of solutions, discussed in Ljung (1987) or Söderström and Stoica (1989), tackles versions of this general problem in a direct way using iterative optimization schemes. The major drawbacks of this direct approach are the difficulty of model class selection and the overparametrization of the model.

An alternative class of solutions are the subspace model identification (SMI) schemes. The latter terminology, introduced in Verhaegen and Deprettere (1991), covers the family of identification schemes that formulate and solve a major part of the identification problem on a signal level. On this level, the main characteristic of these schemes is the approximation of a subspace, defined by the span of the column or row space of matrices determined by the

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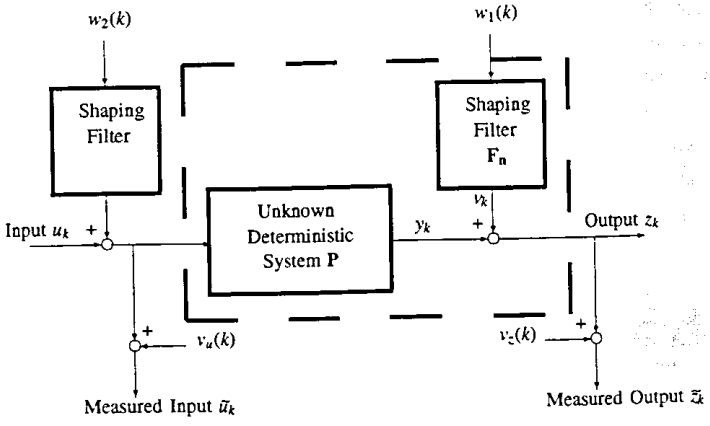
input–output data. The parametric time-invariant model, in this case, is calculated from these spans by exploiting their special structure, such as the shift-invariance property defined in Definition 2.

The SMI approach has the potential either to overcome a number of drawbacks from the direct approach or to complement them. The latter is, for example, highlighted by Ljung (1991). Of the drawbacks we mention: first, the task of model selection might become much simpler. On a signal level, this becomes a signal separation problem, where the recorded perturbed sequences have to be split in a 'signal' component and a 'noise' component. Secondly, the treatment of the MIMO case might become a trivial extension of the scalar case. Thirdly, the problem formulation in this framework is much more appealing to an application engineer. The potential advantage of the latter is demonstrated by the following prototype problem formulation, which makes use of the Figure.

A collective identification problem: GIVEN a perturbed finite length sequence of measurements of the input quantity u_k, denoted in the Figure by $\tilde{u}_k$, and the output quantity z_k, denoted by $\tilde{z}_k$, and ASSUME that the various additive perturbations $v_u(k)$, $v_z(k)$ and those due to the white noise sources $w_1(k)$ and $w_2(k)$ are (statistically) independent of the error-free input u_k, then APPROXIMATE a shift-invariant subspace of spaces defined by the input–output data which characterizes a state-space model. The latter model represents the input–output behaviour of the deterministic (or ideal) part of the system to be identified. This part is indicated in the Figure by block **P**. The perturbations considered can be divided into two classes: (1) internal error-sources, due to parasitic effects or unmeasurable system inputs, represented in the Figure by the error term v_k lumped to the error-free output y_k; and (2) external error-sources, due to the measurement equipment or transducers, represented by the error signals $v_u(k)$, $v_z(k)$ and another due to $w_2(k)$. All internal and external errors are assumed to be zero-mean arbitrary stochastic processes.

The appealing nature of such a problem formulation is again due to its formulation on a signal level. It presents the identification problem in terms of the recorded signals and their relationships, which often are familiar in engineering problems. The overall mathematical model, representing the input–output behaviour denoted by the dashed box in the Figure, is not specified in terms of



Block schematic view of a general system identification set-up.

a particular model class, such as the ARMAX model. Furthermore, the above prototype problem formulation is unifying in the sense that it covers a wide variety of existing identification problems. For example it includes problem formulations where *a priori* model structures, such as ARMAX (Ljung 1987), are assumed for the combination of the deterministic part **P** and the stochastic part $\mathbf{F}_n$. However, no numerical schemes exist that tackle this collective problem in a reliable way. Only subdivisions of the problem have been successfully solved thus far. Examples are the following.

(*a*) The stochastic realization problem. Referring to the Figure, this is the characterization of the filter $\mathbf{F}_n$ by a state-space model making use of the covariance information of the observations $\tilde{z}_k$ only. In this case, only the transfer from the zero-mean white noise sequence $w_1(k)$ to $\tilde{z}_k$ is considered. Pioneering work in this area has been done by Akaike (1975). For a survey of solutions to this problem we refer to the book by Caines (1988).

(*b*) The deterministic realization problem. With the Figure, this problem is defined as the realization of a linear state-space representation of the plant **P** neglecting the presence of all perturbations in the Figure. Original results are due to Ho and Kalman (1966). Later on, different proposals were made from a numerical robustness point of view (Kung 1978, Zeiger and McEwen 1974). Recently, in Moonen *et al.* (1989), Moonen and Vandewalle (1990), a numerically robust solution to this problem was proposed. The scheme presented in the latter papers allows handling additional errors on the input and output signals, namely $v_u(k)$ and $v_z(k)$, which are zero-mean white noise (Moonen *et al.* 1989) or zero-mean with a known colouring (Moonen and Vandewalle 1990). With the schematic outline of the Figure, we observe that in both cases a restricted part of the collective identification problem is treated. Hence, its practical usefulness will be limited.

Various attempts have been undertaken in the literature to adapt existing realization schemes to the collective identification problem. A classical approach is to apply the Ho and Kalman method and to make use of estimated Markov parameters. Based on these parameters only, the shift-invariant subspace, which in this case is the extended reachability or observability matrix, is approximated. Only a very limited number of publications have appeared on this subject in a pure identification context, such as in Juang and Pappa (1985).

In the first part of this series of papers, we describe two novel algorithms of the SMI class. In these algorithms, the output-error structure of the model in the dashed box of the Figure is essential to these solutions, as will be demonstrated in the second part (Verhaegen and Dewilde 1992) but see also Verhaegen (1991 a). Therefore, we refer to these algorithms as versions of the MOESP (standing for MIMO output-error state space model realization) approach. The first version, indicated by the elementary MOESP algorithm, uses all available input–output data in combination with a restricted set of estimated Markov parameters to determine a shift-invariant or structured subspace. The second version, indicated by the ordinary MOESP scheme, only relies on input–output data. The basic algorithm steps of these algorithms is, however, the same—namely an RQ factorization followed by a SVD. The algorithms are

presented as solutions to the deterministic realization problem. An analysis of the elementary MOESP algorithm is presented in the second part of this series of papers. Here, we only treat a simple identification problem, indicated by the model in the dashed box with v_k equal to zero-mean white noise. The analysis of the ordinary MOESP scheme for a realistic version of the collective identification problem is presented in the third part of this series of papers.

The outline of the paper is as follows. We describe in § 2 some basic notions and the model and data representation used throughout this series of papers. In § 3, we deal with a qualitative description of using the RQ factorization in the MOESP approach, the influence of the input sequence choice on the algorithmic organization of the elementary versions and evaluating the effects of the persistency of input excitation conditions. A compact implementation is presented in § 4 of the elementary MOESP scheme, i.e. MOESP1; a compact implementation of the ordinary MOESP algorithm is presented in § 5. Finally, § 6 summarizes the contributions of this paper. In order to preserve the continuity of the material presented in this paper, we have summarized the proofs of the different theorems and lemma in an appendix at the end of the paper.

2. Model description and notational preliminaries

2.1. *Some basic notation*

In this section, we define some frequently used notions in this series of papers.

2.1.1. *Matrix partitioning.* A first way of indicating the partitioning of a matrix or a vector is illustrated by the following example.

Example 1: Let $A \in \mathbb{R}^{(m+l)\times N}$, $m \leq N$, $l \geq 0$, then the following representation of A:

$$A = \begin{array}{c} \\ m \\ l \end{array} \begin{array}{c} \begin{array}{cc} m & N-m \end{array} \\ \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \end{array}$$

indicates the partitioning of A in respectively $A_{11} \in \mathbb{R}^{m\times m}$, $A_{12} \in \mathbb{R}^{m\times(N-m)}$, $A_{21} \in \mathbb{R}^{l\times m}$ and $A_{22} \in \mathbb{R}^{l\times(N-m)}$.
A second way is conformable to the notation used in the MATLAB package (Moler *et al.* 1987). □

Again, we illustrate this by example.

Example 2: Let $A \in \mathbb{R}^{m\times n}$ and let $k < n$, then a partitioning of A is represented by:

$$A = [A(:, 1{:}k)\ A(:, k+1{:}n)]$$

where $A(:, 1{:}k)$ denotes the first k columns of A and $A(:, k+1{:}n)$ denotes the last $n-k$ columns. □

2.1.2. *The rank of a matrix.* The rank of a matrix A, defined for example in Golub and Van Loan (1989), is denoted by $\rho(A)$.

2.1.3. *The RQ factorization.* The RQ factorization of a matrix $A \in \mathbb{R}^{m\times N}$ is a

factorization of this matrix into a lower triangular matrix $R \in \mathbb{R}^{m \times N}$ and a square orthogonal matrix $Q \in \mathbb{R}^{N \times N}$, such that:

$$A = RQ$$

2.1.4. *The symbol* =: *or* :=. This symbol is used for implicit definition; for example

$$\begin{bmatrix} A + B \\ CD \end{bmatrix} =: \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

defines P_1 to be $A + B$ and P_2 to be CD.

2.1.5. *The quadruple* $[A_T, B_T, C_T, D]$. Defines a quadruple of system matrices that are equal to a similarity transformation T to the quadruple of system matrices $[A, B, C, D]$, i.e.

$$[A_T, B_T, C_T, D] = [TAT^{-1}, TB, CT^{-1}, D]$$

2.1.6. *The symbol* $\subset$. Let A, B be matrices in $\mathbb{R}^{N \times m}$ with $m < N$, then $A \subset B$ means that there exists a square m by m matrix T such that:

$$A = BT$$

2.2. *The output-error (OE) state-space model*

Let the deterministic linear time-invariant system **P** in the Figure be represented by the following linear time-invariant state-space model:

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$

$$y_k = Cx_k + Du_k \tag{2}$$

where $u_k \in \mathbb{R}^m$ (the input vector), $y_k \in \mathbb{R}^l$ (the undisturbed output vector) and $x_k \in \mathbb{R}^n$ (the finite-dimensional state vector). The unknown system matrices A, B, C, D have appropriate dimensions. Furthermore, it is assumed that the model representation is minimal (Kailath 1980) that is, the system is completely reachable and observable.

As represented in the dashed box of the Figure, the output of the deterministic system is perturbed by the noise term v_k. Hence, the output equation reads:

$$z_k = y_k + v_k \tag{3}$$

When the stochastic process v_k is assumed to be an arbitrary zero-mean stochastic process (hence $\mathbf{F}_n$ in the Figure can be an arbitrary linear dynamical system), the model representation (1–3) encompasses a wide variety of existing linear, time-invariant parametric and non-parametric models, such as ARMAX, ARX and OE models referred to in Ljung (1987), as well as stochastic state-space models with a so-called process noise term as an additional unmeasurable input to our system (Kailath 1980).

In our presentation of the algorithms in the first part, we will assume, apart from Remark 1 in § 3.3, that all perturbations defined in the collective identification problem are zero. The above defined output-error model structure and the presence of a measurable input, is crucial to the identification problems considered in the MOESP framework.

2.3. *Data representation, a definition of persistency of excitation, a basic lemma and the shift-invariance property*

The signals occurring in the identification problem considered in this paper will be assumed to be (finite segments) of realizations of ergodic stochastic processes. That is, for $N \to \infty$ there exists ergodic stochastic processes $\mathbf{u}_j \in \mathbb{R}^m$ and $\mathbf{v}_k \in \mathbb{R}^l$ such that:

$$[u_j\ u_{j+1} \ldots u_{N+j-1}] \quad \text{and} \quad [v_k\ v_{k+1} \ldots v_{N+k-1}] \tag{4}$$

are realizations of $\mathbf{u}_j$ and $\mathbf{v}_k$ respectively, and the following (or similar) expression(s) holds:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} u_{j+i-1} v_{k+i-1}^{\mathrm{T}} = E[\mathbf{u}_j \mathbf{v}_k^{\mathrm{T}}] \tag{5}$$

We adopt the notation, in bold, to represent the stochastic process and use the symbol $E[\cdot]$ to represent the mathematical expectation operator. An alternative way of expressing the above limit is:

$$\frac{1}{N} \sum_{i=1}^{N} u_{j+i-1} v_{k+i-1}^{\mathrm{T}} + \varepsilon_N E_N = E[\mathbf{u}_j \mathbf{v}_k^{\mathrm{T}}] \tag{6}$$

where ε_N is a sequence of real numbers such that $\lim_{N\to\infty} \varepsilon_N = 0$ and E_N is a matrix of appropriate dimensions with $\|E_N\| \leqslant 1$.

A representation that proves to be particularly useful in studying properties of a state-space realization based on derived input and output sequences, see e.g. Moonen *et al.* (1989), Kailath (1980), Gopinath (1969), Liu and Suen (1977), De Moor (1988), is given by:

$$\begin{bmatrix} y_j & y_{j+1} & \cdots & y_{j+N-1} \\ y_{j+1} & y_{j+2} & \cdots & y_{j+N} \\ \vdots & & & \\ y_{j+i-1} & y_{j+i} & \cdots & y_{j+N+i-2} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix} \begin{bmatrix} x_j & x_{j+1} & \cdots & x_{j+N-1} \end{bmatrix}$$

$$+ \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ CAB & CB & \ddots & \vdots \\ \vdots & & & D \\ CA^{i-2}B & & \cdots & \end{bmatrix} \begin{bmatrix} u_j & u_{j+1} & \cdots & u_{j+N-1} \\ u_{j+1} & u_{j+2} & \cdots & u_{j+N} \\ \vdots & & & \\ u_{j+i-1} & u_{j+i} & \cdots & u_{j+N+i-2} \end{bmatrix} \tag{7}$$

Let Γ_i and H_i define the matrices

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ CAB & CB & \ddots & \vdots \\ \vdots & & & D \\ CA^{i-2}B & & \cdots & \end{bmatrix}$$

respectively, then we can denote the data equation (7) more compactly as

$$Y_{j,i,N} = \Gamma_i X_{j,N} + H_i U_{j,i,N} \tag{8}$$

Similarly, the output z_k can be expressed as:

$$Z_{j,i,N} = \Gamma_i X_{j,N} + H_i U_{j,i,N} + V_{j,i,N} \tag{9}$$

where $Z_{j,i,N}$ and $V_{j,i,N}$ are Hankel matrices constructed from z_k respectively v_k in a similar way as $Y_{j,i,N}$ was constructed from y_k.

An important property of the input sequence in system identification is its *persistency of excitation*. This property of the input signal is defined in Ljung (1987) p. 363, as follows.

Definition 1: An ergodic sequence $u_k \in \mathbb{R}^m$ is *persistently exciting of order i from time instant j on,* if and only if:

$$\rho\left(\lim_{N\to\infty} \frac{1}{N} U_{j,i,N} U_{j,i,N}^{\mathrm{T}}\right) = mi \tag{10}$$

□

The condition in (10) can be denoted as:

$$\rho\left(\frac{1}{N} U_{j,i,N} U_{j,i,N}^{\mathrm{T}} + \varepsilon_N^1 E_N^1\right) = mi \tag{11}$$

Therefore, when N is sufficiently large, so that the smallest singular value of $1/N(U_{j,i,N}U_{j,i,N}^{\mathrm{T}})$ is larger than $|\varepsilon_N^1|$, the definition of the persistency of excitation of the sequence u_k can be interpreted for the finite data length case as a full row rank condition

$$\rho(U_{j,i,N}) = mi \tag{12}$$

When the signal u_k is a zero-mean discrete white noise sequence it has the statistical property:

$$E\left(\begin{bmatrix} \mathbf{u}_j \\ \mathbf{u}_{j+1} \\ \vdots \\ \mathbf{u}_{j+i-1} \end{bmatrix} \begin{bmatrix} \mathbf{u}_j^{\mathrm{T}} & \mathbf{u}_{j+1}^{\mathrm{T}} & \dots & \mathbf{u}_{j+i-1}^{\mathrm{T}} \end{bmatrix}\right) = \sigma_u^2 \begin{bmatrix} I_m & 0 & \dots & 0 \\ 0 & I_m & \ddots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & I_m \end{bmatrix} \tag{13}$$

Under the assumption of ergodicity this gives:

$$\lim_{N\to\infty} \frac{1}{N} U_{j,i,N} U_{j,i,N}^{\mathrm{T}} = \sigma_u^2 I_{m.i}$$

or

$$\frac{1}{N} U_{j,i,N} U_{j,i,N}^{\mathrm{T}} = \sigma_u^2 I_{m.i} + \varepsilon_N^2 E_N^2 \tag{14}$$

From this definition of white noise we see that it is persistently exciting of arbitrary order. As outlined in the paragraph following the definition of persistency of excitation (above) this holds for the finite data length case by choosing N properly.

Throughout this paper, we often have to analyse the rank of the product of matrices. For that purpose, we will make use of the following basic lemma.

Lemma 1: Sylvester's inequality (Kailath 1980, p. 655): *Let* $M_1 \in \mathbb{R}^{m\times n}$ *and* $M_2 \in \mathbb{R}^{n\times p}$ *then*:

$$\rho(M_1) + \rho(M_2) - n \leq \rho(M_1 M_2) \leq \min\{\rho(M_1), \rho(M_2)\} \tag{15}$$

Finally, we define the shift-invariance property in the finite dimensional context of this paper as follows.

Definition 2: Let Γ be an $l.i \times n$ matrix with $i > n$ and with $\Gamma^{(1)}$ and $\Gamma^{(2)}$ denoting the submatrices composed of the first respectively last $(i-1).l$ rows of Γ, then the matrix Γ is called *shift-invariant* if and only if it satisfies:

$$\Gamma^{(2)} \subset \Gamma^{(1)} \qquad \square$$

3. The RQ factorization in approximating shift invariant or structured subspaces

3.1. *Qualitative description of the various approaches*

A basic feature that characterizes the MOESP approach is the use of the RQ factorization not just to compress the data matrices as a pre-processing step prior to other algorithmic operations (such as a SVD), but to reveal matrices with specially structured column or row spaces. This step characterizes a broad and alternative family of realization algorithms, as will be demonstrated in this paper and as is demonstrated in related papers (Verhaegen and Deprettere 1991, Verhaegen 1991 a, Verhaegen 1990 and Parts 2 and 3).

In this first part, we will present two schemes that fit into this alternative framework in detail. In the first scheme, called the elementary version of the MOESP approach, the RQ factorization of the data matrix pair

$$\left(\frac{U_{1,i,N}}{Y_{1,i,N}}\right)$$

is used to retrieve explicitly the matrix product $\Gamma_i X_{1,N}$ in (8). Under certain mild conditions, the column space and row space of this matrix product is equal to that of Γ_i and $X_{1,N}$ respectively. In the second scheme, called the ordinary MOESP scheme, we use the RQ factorization of the above data matrix pair to retrieve a matrix with column space equal to the column space of Γ_i, again under similar mild conditions. Having calculated a span of the column space of Γ_i, we exploit its shift-invariance property to compute the matrix A_T. To see this, let the columns of the matrix U_n span the column space of Γ_i. Then there exists a square, non-singular matrix T_1 such that:

$$U_n = \Gamma_i T_1 \tag{16}$$

Since

$$\Gamma_i^{(1)} A := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-2} \end{bmatrix} \quad A = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{i-1} \end{bmatrix} =: \Gamma_i^{(2)} \tag{17}$$

the matrix Γ_i is shift-invariant (see Definition 2). Next we combine (16) and (17) as follows,

$$(U_n^{(1)})(A_{T_1}) := (\Gamma_i^{(1)} T_1)(T_1^{-1} A T_1) = \Gamma_i^{(2)} T_1 =: U_n^{(2)} \tag{18}$$

to show that the matrix U_n remains shift-invariant and that when the matrix $U_n^{(1)}$ has full (column) rank, we can solve (18) for A_{T_1}. In addition (16) shows that

U_n directly displays the matrix C_{T_1}. The calculation of the matrices B_{T_1} and D then proceeds via the solution of an overdetermined set of equations, see § 5.1 for more details. It is important to note that knowledge of a span of the column space of Γ_i plays a key role in constructing this overdetermined set of equations.

In the case where we calculate a span of the row space of $X_{1,N}$, we cannot use the shift-invariance property in the strict sense of Definition 2 to compute the system matrices. However, we can exploit the special structure of this span to compute a quadruple of system matrices all at once. To see this, let the rows of the matrix $\chi_{1,N}$ span the row space of $X_{1,N}$. Then there exists a square, non-singular matrix T_2 such that:

$$\chi_{1,N} = T_2 X_{1,N} \tag{19}$$

The matrix $X_{1,N}$ is not shift-invariant. However, for $k = 1{:}N$ in (1–2) we can combine the state, input and output trajectory as follows:

$$\begin{bmatrix} x_2 & x_3 & \dots & x_N \\ y_1 & y_2 & \dots & y_{N-1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 & x_1 & \dots & x_{N-1} \\ u_1 & u_2 & \dots & u_{N-1} \end{bmatrix}$$

denoted more compactly as

$$\begin{bmatrix} X_{2,N-1} \\ Y_{1,1,N-1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_{1,N-1} \\ U_{1,1,N-1} \end{bmatrix} \tag{20}$$

The compatibility of this set of equations can be interpreted in the following less restrictive 'shift-invariance' sense,

$$\begin{bmatrix} X_{2,N-1} \\ Y_{1,1,N-1} \end{bmatrix}^{\mathrm{T}} \subset \begin{bmatrix} X_{1,N-1} \\ U_{1,1,N-1} \end{bmatrix}^{\mathrm{T}}$$

Substitution of (19) into (20) yields,

$$\begin{bmatrix} \chi_{2,N-1} \\ Y_{1,1,N-1} \end{bmatrix} = \begin{bmatrix} T_2 A T_2^{-1} & T_2 B \\ C T_2^{-1} & D \end{bmatrix} \begin{bmatrix} \chi_{1,N-1} \\ U_{1,1,N-1} \end{bmatrix}$$

Therefore, we conclude that when the matrix

$$\begin{bmatrix} \chi_{1,N-1} \\ U_{1,1,N-1} \end{bmatrix}$$

has full (row) rank, we can solve for the quadruple of system matrices $(A_{T_2}, B_{T_2}, C_{T_2}, D)$. In the second part of this series of papers, we demonstrate that the latter approach only gives unbiased estimates for a very restrictive identification problem.

In the following two subsections, we subsequently demonstrate the use of the RQ factorization for the case in which the input u_k is zero-mean white noise and for the case in which the input u_k is arbitrary.

3.2. *The white noise input case*

First, we state the following lemma.

Lemma 2: *Let the input u_k to the system (1)–(2) be discrete zero-mean white noise, having the statistical properties given by* (13) *and being independent of the initial state* $\mathbf{x}_0$. *Then*:

$$E[\mathbf{x}_k \mathbf{u}_l^T] = 0 \quad \text{for} \quad l \geqslant k \geqslant 0 \tag{21}$$

Proof: For the proof, see § A.1 of the Appendix. □

Based on this Lemma, we have the following theorem.

Theorem 1: *Let the input u_k be an ergodic zero-mean white noise sequence with the property stated in* (14), *assume the following RQ factorization:*

$$\begin{bmatrix} U_{1,i,N} \\ \hline Y_{1,i,N} \end{bmatrix} = \begin{matrix} mi \\ li \end{matrix} \left[\begin{array}{c|c} \overset{mi}{R_{11}^N} & \overset{li}{0} \\ \hline R_{21}^N & R_{22}^N \end{array}\right] \begin{bmatrix} Q_1^N \\ \hline Q_2^N \end{bmatrix} \tag{22}$$

and let,

$$\lim_{N\to\infty} \frac{1}{N} X_{1,N} X_{1,N}^T = P_x \tag{23}$$

be the covariance matrix of the state vector $\mathbf{x}_1$, *that is* $E[\mathbf{x}_1 \mathbf{x}_1^T]$, *then*,

$$\lim_{N\to\infty} \frac{1}{N} R_{22}^N (R_{22}^N)^T = \Gamma_i P_x \Gamma_i^T \tag{24}$$

Proof: For the proof, see § A.2 of the Appendix. □

The above theorem reveals that when $\rho(\Gamma_i P_x \Gamma_i^T) = \rho(\Gamma_i X_{1,N}) = n$ (for a suitably chosen N), the column space of the matrix R_{22}^N obtained in (22) approximately equals the column space of Γ_i. This observation only holds in the limit for $N \to \infty$. However, recalling (A 6) from the proof of Theorem 1 given in § A.2 of the Appendix, namely

$$\frac{1}{N} R_{22}^N (R_{22}^N)^T = \Gamma_i \left[\frac{1}{N} X_{1,N} X_{1,N}^T - \varepsilon_N^3 E_N^3 \left(\frac{1}{N} R_{11}^N (R_{11}^N)^T \right)^{-1} \varepsilon_N^3 (E_N^3)^T \right] \Gamma_i^T$$

this shows that it also holds for the finite data length case. In the next subsection, we demonstrate that it holds for the arbitrary input case as well. In this subsection we will also comment on the crucial condition $\rho(\Gamma_i X_{1,N}) = n$.

3.3. *The arbitrary input signal case*

Consider again an RQ factorization of the pair

$$\begin{bmatrix} U_{1,i,N} \\ \hline Y_{1,i,N} \end{bmatrix}$$

for fixed N now denoted as:

$$\begin{bmatrix} U_{1,i,N} \\ \hline Y_{1,i,N} \end{bmatrix} = \begin{matrix} mi \\ li \end{matrix} \left[\begin{array}{c|c} \overset{mi}{R_{11}} & \overset{li}{0} \\ \hline R_{21} & R_{22} \end{array}\right] \begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix} \tag{25}$$

When the input u_k is arbitrary and the Markov parameters in the matrix H_i are known, then from (8):

$$\Gamma_i X_{1,N} = Y_{1,i,N} - H_i U_{1,i,N} \tag{26}$$

or with the factorizations given in (25):

$$\Gamma_i X_{1,N} = [R_{21} - H_i R_{11} | R_{22}] \begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix} \tag{27}$$

Now we can derive from the matrix product $\Gamma_i X_{1,N}$ the column space of Γ_i or the row space of $X_{1,N}$ only when $\rho(\Gamma_i X_{1,N}) = n$. A crucial role in meeting the latter condition is played by the condition:

$$\rho \begin{bmatrix} U_{1,i,N} \\ \hline X_{1,N} \end{bmatrix} = mi + n \tag{28}$$

which is therefore stated first.

In the literature, various studies have been performed on the requirements on u_k to satisfy (28). For a study in a statistical framework, we refer to Gopinath (1969). For a study in an algebraic framework similar to the one used in this paper, we refer to De Moor (1988). Both studies state that, for most choices of u_k, condition (28) will 'almost surely' (Gopinath 1969) be satisfied.

In this paper, we further complete these studies for the case where u_k is a periodic signal. For this class of widely used test input sequences, we will derive precise requirements on u_k in order to assure (28). This is done in the next theorem for a particular class of periodic signals, the analysis of other periodic signals families being similar.

Theorem 2: *Let,*

(1) *the plant P, represented* (1)–(2), *be operating from time instant* $k = -\infty$ *on,*

(2) *the plant P be asymptotically stable, that is all eigenvalues of the A matrix in* (1) *be within the unit circle,*

(3) *the input* u_k *be a periodic sequence with period* $2(i + j)$ *for* $j \geq n$ *satisfying* $u_k = -u_{k+i+j}$ *and*

(4) $\rho(U_{-j+1,i+j,N}) = m(i + j)$

then the condition

$$\rho \begin{bmatrix} U_{1,i,N} \\ \hline X_{1,N} \end{bmatrix} = mi + n$$

holds.

Proof: For the proof, see § A.3 of the Appendix. □

When condition (28) is satisfied, the matrix pair

$$\begin{bmatrix} U_{1,i,N} \\ \hline X_{1,N} \end{bmatrix}$$

has an RQ factorization given as:

$$\left[\begin{array}{c} U_{1,i,N} \\ \hline X_{1,N} \end{array}\right] = \begin{array}{c} mi \\ n \end{array}\left[\begin{array}{c|c} \overset{mi}{R_{11}} & \overset{n}{0} \\ \hline R_{x1} & R_{x2} \end{array}\right]\left[\begin{array}{c} Q_1 \\ \hline Q_x \end{array}\right] \tag{29}$$

with R_{11} and R_{x2} square invertible matrices. With this notation, we can express $\Gamma_i X_{1,N}$ as:

$$\Gamma_i X_{1,N} = \Gamma_i(R_{x1}Q_1 + R_{x2}Q_x) \tag{30}$$

Hence, when $i \geqslant n$ and under the minimality assumption of the underlying plant, $\rho(\Gamma_i) = n$ and since $\rho(R_{x2}Q_x) = n$, we conclude according to Lemma 1 that $\rho(\Gamma_i X_{1,N}) = n$.

Combining the above expression for $\Gamma_i X_{1,N}$ with the one given in (27) produces:

$$\Gamma_i R_{x1}Q_1 + \Gamma_i R_{x2}Q_x = (R_{21} - H_i R_{11})Q_1 + R_{22}Q_2 \tag{31}$$

Hence, since $Q_2 Q_1^{\mathrm{T}} = 0$, $Q_x Q_1^{\mathrm{T}} = 0$ and $Q_1 Q_1^{\mathrm{T}} = I_{mi}$, we obtain

$$\Gamma_i R_{x1} = R_{21} - H_i R_{11} \tag{32}$$

and

$$\Gamma_i R_{x2} Q_x = R_{22} Q_2 \tag{33}$$

The latter relationship demonstrates that, again by application of Lemma 1, $\rho(R_{22}Q_2) = n$. Furthermore, since $R_{x2}Q_x \in \mathbb{R}^{n \times N}$, the column space of $R_{22}Q_2$ equals the column space of Γ_i! This observation forms the key to the ordinary MOESP algorithm, further outlined in § 5.

Remark 1: When the error-free acquisition of the input signal u_k does not hold, the RQ factorization in the data pre-processing of the MOESP algorithm, outlined in § 4.1, should be substituted or followed by a SVD in order to assure an unbiased realization. For example, consider only white noise errors on both input and output signals, a straightforward SVD of the matrix

$$\left[\begin{array}{c} U_{1,i,N} \\ \hline Y_{1,i,N} \end{array}\right]$$

is then necessary

$$\left[\begin{array}{c} U_{1,i,N} \\ \hline Y_{1,i,N} \end{array}\right] = \left[\begin{array}{c} U_u \\ \hline U_y \end{array}\right] \Sigma V^{\mathrm{T}} \tag{34}$$

Hence we obtain,

$$\Gamma_i X_{1,N} = (U_y - H_i U_u)\Sigma V^{\mathrm{T}} \tag{35}$$

In a similar way to the compact algorithm based on the RQ factorization, described in § 4, it can again be shown that only the factors U_u and U_y are required to calculate a state-space realization. That this would yield an unbiased estimate can be demonstrated similarly to the asymptotic analysis made in § 8, of Verhaegen and Dewilde (1992). For the general case of arbitrary stochastic perturbations on the input and output quantities (of known colouring), the GSVD (Golub and Van Loan 1989) would be required. □

4. An implementation of the elementary MOESP algorithm

The description of an implementation of the elementary MOESP algorithm for the arbitrary input signal case follows. It allows calculation of a quadruple of system matrices $[A_T, B_T, C_T, D]$ from a set of input–output data sequences and a restricted number of Markov parameters contained in the matrix H_i of (8). The latter data sequences are taken from the linear time-invariant system given in (1)–(2). The implementation consists of a data pre-processing step done by means of an RQ factorization, then followed by a SVD and the solution of an overdetermined set of equations.

4.1. *The data pre-processing*

The pre-processing step first partitions the data matrix

$$\left[\begin{array}{c} U_{1,i,N} \\ \hline Y_{1,i,N} \end{array}\right]$$

as follows:

$$\left[\begin{array}{c} U_{1,i,N} \\ \hline\hline Y_{1,i,N} \end{array}\right] = \begin{array}{r} (i-1)m \\ m \\ (i-1)l \\ l \end{array}\left[\begin{array}{c} U_{1,i-1,N} \\ \hline U_{i,1,N} \\ \hline\hline Y_{1,i-1,N} \\ \hline Y_{i,1,N} \end{array}\right] = \begin{array}{r} m \\ (i-1)m \\ l \\ (i-1)l \end{array}\left[\begin{array}{c} U_{1,1,N} \\ \hline U_{2,i-1,N} \\ \hline\hline Y_{1,1,N} \\ \hline Y_{2,i-1,N} \end{array}\right] \tag{36}$$

Next, an RQ factorization of this matrix pair is computed. This is denoted in accordance with the above partitioning as:

$$\left[\begin{array}{c|c} R_{11} & 0 \\ \hline\hline R_{21} & R_{22} \end{array}\right]\left[\begin{array}{c} Q_1 \\ \hline Q_2 \end{array}\right] = \begin{array}{r} (i-1)m \\ m \\ (i-1)l \\ l \end{array}\left[\begin{array}{c|c} \overset{mi}{R_{11}^1} & \overset{li}{0} \\ \hline \star & 0 \\ \hline\hline R_{21}^1 & R_{22}^1 \\ \hline \star & \star \end{array}\right]\left[\begin{array}{c} Q_1 \\ \hline Q_2 \end{array}\right]$$

$$= \begin{array}{r} m \\ (i-1)m \\ l \\ (i-1)l \end{array}\left[\begin{array}{c|c} \overset{mi}{r_u} & \overset{li}{0} \\ \hline R_{11}^2 & 0 \\ \hline\hline r_{y1} & r_{y2} \\ \hline R_{21}^2 & R_{22}^2 \end{array}\right]\left[\begin{array}{c} Q_1 \\ \hline Q_2 \end{array}\right] \tag{37}$$

4.2. *Realization of a quadruple of system matrices $[A_T, B_T, C_T, D]$*

In § 3.3, following the discussion of (29), it was demonstrated that when

condition (28) is satisfied, $\rho(\Gamma_i X_{1,N}) = n$ and hence $[(R_{21} - H_i R_{11})|R_{22}]$ is of rank n. Based on this insight, we formulate the following theorem.

Theorem 3: *Let,*

(1) *the input* u_k *be such that condition* (28) *is satisfied,*

(2) $i > n$

(3) *the RQ factorization of the pair of Hankel matrices*

$$\left[\frac{U_{1,i,N}}{Y_{1,i,N}}\right]$$

be given and partitioned as in (37),

(4) *the Markov parameters in the Toeplitz matrix* H_i *of* (8) *be exactly known*

(5) *the SVD of the matrix* $[(R_{21} - H_i R_{11})|R_{22}]$ *be:*

$$[(R_{21} - H_i R_{11})|R_{22}] = U_n S_n V_n^{\mathrm{T}} \tag{38}$$

with $U_n \in \mathbb{R}^{li \times n}$, S_n *an* $n \times n$ *diagonal matrix and* $V_n \in \mathbb{R}^{(m+l)i \times n}$.

then,

(1) *the submatrix composed of the first* $(i-1)l$ *rows of the matrix* U_n, *denoted by* $U_n^{(1)}$, *has a pseudo-inverse* $(U_n^{(1)})\dagger$ *such that* $(U_n^{(1)})\dagger U_n^{(1)} = I_n$;

(2) *the matrix* $(U_n^{(1)})\dagger \Gamma_{i-1}$ *is a square invertible matrix*;

(3) *the quadruple of system matrices* $[A_T, B_T, C_T, D]$, *which satisfies the following set of overdetermined equations:*

$$\begin{bmatrix} (U_n^{(1)})\dagger(R_{21}^2 - H_{i-1}R_{11}^2) & (U_n^{(1)})\dagger R_{22}^2 \\ r_{y1} & r_{y2} \end{bmatrix} =$$

$$\begin{bmatrix} A_T & B_T \\ C_T & D \end{bmatrix} \begin{bmatrix} (U_n^{(1)})\dagger(R_{21}^1 - H_{i-1}R_{11}^1) & (U_n^{(1)})\dagger R_{22}^1 \\ r_u & 0 \end{bmatrix} \tag{39}$$

is similar to the original quadruple of system matrices $[A, B, C, D]$.

Proof: For the proof, see § A.4 of the Appendix. □

The crucial part in the MOESP algorithm consists of the overdetermined set of equations, given by (39). Owing to errors in the original data as well as due to the incompatibility between the model and the true system, this set of equations generally has no compatible solution. In that case, a possible solution consists of solving (39) in least-squares sense, denoted as:

$$\min_{[A_T, B_T, C_T, D]} \left\| \begin{bmatrix} (U_n^{(1)})\dagger(R_{21}^2 - H_{i-1}R_{11}^2) & (U_n^{(1)})\dagger R_{22}^2 \\ r_{y1} & r_{y2} \end{bmatrix} - \begin{bmatrix} A_T & B_T \\ C_T & D \end{bmatrix} \begin{bmatrix} (U_n^{(1)})\dagger(R_{21}^1 - H_{i-1}R_{11}^1) & (U_n^{(1)})\dagger R_{22}^1 \\ r_u & 0 \end{bmatrix} \right\|_F \tag{40}$$

where $\|M\|_F$ denotes the Frobenius-norm of the matrix M (Golub and Van Loan 1989).

It should be remarked that when the set of equations (39) has no compatible solution, solving the set of equations in least-squares sense does not restore the

lost 'shift invariance' property. The shift invariance property now should be interpreted in the less restrictive sense as pointed out in § 3.1. By forcing a solution in least-squares sense, we only obtain an approximate solution in a similar way as is done with most existing algebraically inspired realization techniques, such as Kung (1978), Zeiger and McEwen (1974) and Moonen *et al.* (1989). Restoring the shift invariance structure would require an analysis as given in Adamjan *et al.* (1971). The value of the approximation (or model reduction) capabilities of MOESP is analysed in § 10 of Part 2.

4.3. *The elementary MOESP1 algorithm*

The exposition in the previous subsections can be summarized into the following algorithm, which is referred to as the elementary MOESP1 algorithm.

The elementary MOESP1 algorithm

Given:

(i) an estimate of the underlying system order n;

(ii) a dimension parameter i, satisfying:

$$i > n \text{ (and } i \geqslant \kappa) \tag{41}$$

where κ, defined later in Lemma 3 of Verhaegen and Dewilde (1992), is yet of no importance;

(iii) the first i (estimated) Markov parameters $h_s = CA^{s-1}B$ for $s = 1, \ldots, i-1$ and $h_0 = D$;

(iv) the input and output data sequences:

$$[u_1, u_2, \ldots, u_{N+i-1}] \quad \text{and} \quad [y_1, y_2, \ldots, y_{N+i-1}]$$

with $N \gg m.i$.

Do the following:

Step 1. Construct the Hankel matrices $U_{1,i,N}$ and $Y_{1,i,N}$ and the lower triangular Toeplitz matrix H_i, all defined in (7) and (8).

Step 2. Achieve a data compression via an RQ factorization, of which the R-factor is partitioned according to (37).

Step 3. Compute the SVD of the matrix given in (38) of Theorem 3.

Step 4. Solve the set of equations in (40).

The compactness of the MOESP1 algorithm stems from the fact that calculating the system matrices $[A_T, B_T, C_T, D]$ (in (39)) only requires knowledge of the lower triangular factor obtained in the pre-processing steps, in addition to the restricted set of Markov parameters in the matrix H_i. From an implementation point of view, it is important that the orthogonal transformations in this RQ factorization need not be accumulated explicitly.

5. An implementation of the ordinary MOESP algorithm

5.1. *Realization of a quadruple of system matrices $[A_T, B_T, C_T, D]$.*

The observation made in the paragraph before Remark 1 in § 3.3 is used in this section to derive the ordinary MOESP algorithm. This algorithm can again be retrieved from the following theorem.

Theorem 4: *Let*

(1) *the input* u_k *be such that condition* (28) *is satisfied,*

(2) $i > n$

(3) *The RQ factorization of the pair of Hankel matrices*

$$\begin{bmatrix} U_{1,i,N} \\ \hline Y_{1,i,N} \end{bmatrix}$$

be given as in (25),

(4) *the SVD of the matrix* $R_{22} \in \mathbb{R}^{li \times li}$ *in* (25) *is*

$$R_{22} = \begin{matrix} \\ li \end{matrix} \begin{bmatrix} \overset{n}{U_n} & \overset{li-n}{U_n^{\perp}} \end{bmatrix} \begin{bmatrix} \overset{n}{S_n} & \overset{li-n}{0} \\ \hline 0 & S_2 \end{bmatrix} \begin{bmatrix} V_n^{\mathrm{T}} \\ (V_n^{\perp})^{\mathrm{T}} \end{bmatrix} \tag{42}$$

(5) $$\rho(U_n^{\perp}(l(i-1)+1{:}li, :)) = l$$

then, the system matrices denoted by A_T, B_T, C_T *and* D *can be computed from the following set of equations:*

$$U_n^{(1)} A_T = U_n^{(2)} \tag{43}$$

$$C_T = U_n(1{:}l, :) \tag{44}$$

$$\begin{matrix} li-n \\ li-n \\ \vdots \\ li-n \end{matrix} \begin{bmatrix} \overset{m}{\Xi(:, 1{:}m)} \\ \Xi(:, m+1{:}2m) \\ \vdots \\ \Xi(:, m(i-1)+1{:}mi) \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} \overset{l}{U_n^{\perp}(1{:}l, :)^{\mathrm{T}}} & \dots & \overset{l}{U_n^{\perp}(l(i-1)+1{:}li, :)^{\mathrm{T}}} \\ U_n^{\perp}(l+1{:}2l, :)^{\mathrm{T}} & \dots & 0 \\ \vdots & \overset{0}{\ddots} & 0 \\ U_n^{\perp}(l(i-1)+1{:}li, :)^{\mathrm{T}} & 0 \quad \dots & 0 \end{bmatrix} \begin{bmatrix} I_l & 0 \\ \hline 0 & U_n^{(1)} \end{bmatrix} \begin{bmatrix} D \\ B_T \end{bmatrix}} \tag{45}$$

with $U_n^{(1)}$, $U_n^{(2)}$ *as defined in Theorem 3 and* $\Xi := (U_n^{\perp})^{\mathrm{T}} R_{21} R_{11}^{-1}$.

Proof: For the proof, see § A.5 of the Appendix. □

Condition (5) in the above theorem is necessary in the computation of the matrix pair $(D^{\mathrm{T}} \quad (B_{\mathrm{T}})^{\mathrm{T}})^{\mathrm{T}}$. Therefore, it has to be checked prior to solving the set of equations (45). When it does not hold for a particular example, a possible cure is to increase the dimension parameter i. This assertion can formally be proved when the underlying system is asymptotically stable based on Lemma 4 of Part 2.

5.2. *The ordinary MOESP algorithm*

Theorem 4 now gives rise to the following algorithm.

The ordinary MOESP algorithm

Given:

(i) An estimate of the underlying system order n.

(ii) A dimension parameter i, satisfying:

$$i > n$$

(iii) The input and output data sequences:

$$[u_1, u_2, \ldots, u_{N+i-1}] \quad \text{and} \quad [y_1, y_2, \ldots, y_{N+i-1}]$$

with $N \gg m.i$.

Do the following:

Step 1. Construct the Hankel matrices $U_{1,i,N}$ and $Y_{1,i,N}$ defined in (7).

Step 2. Achieve a data compression via an RQ factorization, of which the R-factor is partitioned as in (25)

Step 3. Compute the SVD of the matrix R_{22} as given in (42) of Theorem 4.

Step 4. Solve the set of equations (43) and (45).

Again, when the latter set of equations has no compatible solution, it is possible to force a solution by solving them in least-squares sense.

As demonstrated in § 3.3 and Theorem 4, the ordinary MOESP algorithm resulted in a straightforward way from studying the realization capabilities of the elementary MOESP scheme. As such, it has been derived independently from the related solution presented in De Moor and Vandewalle (1987). The algorithm summarized above is, however, more attractive than the solution presented in De Moor and Vandewalle (1987) because of the use of the RQ factorization. The fact that one only needs to use the lower triangular factor obtained in this factorization makes our solution especially appealing from an implementation and computational point of view. In addition, further attention to this scheme in Verhaegen (1991 a) has demonstrated that it is capable of handling a realistic version of the collective identification problem.

6. Concluding remarks

In the first part of this series of papers, a novel state-space realization approach is presented. The approach is indicated by the multiple-input, multiple-output output-error state space (MOESP) approach.

In the elementary MOESP scheme we realize a quadruple of system matrices $[A_T, B_T, C_T, D]$ making use of the available input–output data and a restricted set of Markov parameters. It has been demonstrated that when the necessary Markov parameters are known, the above calculations only require a single RQ factorization followed by a SVD and the solution of a least squares problem. The presented implementation is compact mainly because only the R factor is necessary in the RQ factorization and subsequent calculations.

Due to the use of a restricted set of Markov parameters, the elementary

MOESP algorithm is closely related to the classical Ho and Kalman (1966) approach based on Markov parameters only. Therefore, in Part 2 (Verhaegen and Dewilde, 1992) we perform a detailed comparison study between both algorithms for a simplified version of the collective identification problem.

Although we require only input–output data in the ordinary MOESP scheme, the algorithmic steps are similar to those in the elementary MOESP scheme. In an identification context, the capabilities of both implementations still have to be investigated. In Part 2 (Verhaegen and Dewilde 1992) we analyse the elementary scheme and in the third part (Verhaegen 1991 a, 1993) we analyse the ordinary scheme.

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Appendix

Proofs of the Lemma and Theorems of Part 1

A.1. **Proof of Lemma 2:** The solution to the difference equation (1), given that the state vector at $k = 0$ is x_0, is:

$$x_k = A^k x_0 + \sum_{i=1}^{k} A^{i-1} B u_{k-i} \tag{A 1}$$

Hence, since $\mathbf{x}_0$ is independent of $\mathbf{u}_k$, since $\mathbf{u}_k$ is zero-mean white noise and since $l \geqslant k \geqslant 0$:

$$E[\mathbf{x}_k \mathbf{u}_l^{\mathrm{T}}] = A^k \mathrm{E}[\mathbf{x}_0 \mathbf{u}_l^{\mathrm{T}}] + \sum_{i=1}^{k} A^{i-1} B E[\mathbf{u}_{k-i} \mathbf{u}_l^{\mathrm{T}}] = 0 \qquad \square$$

A.2. **Proof of Theorem 1:** From (22) we obtain:

$$U_{1,i,N} = R_{11}^N Q_1^N \tag{A 2}$$

Based on the white noise property of u_k, and take N such that the condition in (12) holds, then since the matrix R_{11}^N is square, the latter matrix cannot be singular (Lemma 1). Introducing the data equation (8), we find:

$$Y_{1,i,N} = \Gamma_i X_{1,N} + H_i U_{1,i,N} = R_{21}^N Q_1 + R_{22}^N Q_2 \tag{A 3}$$

and hence with (47) we obtain:

$$\Gamma_i X_{1,N} - R_{22}^N Q_2^N = (R_{21}^N - H_i R_{11}^N) Q_1^N \tag{A 4}$$

Since the input u_k is ergodic, (21) of Lemma 2 can be denoted as:

$$\frac{1}{N} X_{1,N} U_{1,i,N}^{\mathrm{T}} = \varepsilon_N^3 E_N^3$$

with ε_N^3 a sequence of real numbers such that $\lim_{N \to \infty} \varepsilon_N^3 = 0$ and E_N^3 a matrix of appropriate dimensions with $\|E_N^3\| \leqslant 1$. Substituting the factorization of $U_{1,i,N}$ given in (A 2) yields:

$$\varepsilon_N^3 E_N^3 = \frac{1}{\sqrt{N}} X_{1,N} (Q_1^N)^{\mathrm{T}} \frac{1}{\sqrt{N}} (R_{11}^N)^{\mathrm{T}}$$

Since for finite and sufficiently large N the matrix $1/\sqrt{N}(R_{11}^N)^T$ remains invertible, we obtain:

$$\frac{1}{\sqrt{N}} X_{1,N}(Q_1^N)^T = \varepsilon_N^3 E_N^3 \left(\frac{1}{\sqrt{N}}(R_{11}^N)^T\right)^{-1}$$

and therefore:

$$\frac{1}{\sqrt{N}} \Gamma_i X_{1,N}(Q_1^N)^T = \Gamma_i \varepsilon_N^3 E_N^3 \left(\frac{1}{\sqrt{N}}(R_{11}^N)^T\right)^{-1} \tag{A 5}$$

Since, $Q_2^N(Q_1^N)^T = 0$ and $Q_1^N(Q_1^N)^T = I_{m,i}$, multiplication of (A 4) on the right by $1/\sqrt{N}(Q_1^N)^T$ yields:

$$\frac{1}{\sqrt{N}} (R_{21}^N - H_i R_{11}^N) = \Gamma_i \varepsilon_N^3 E_N^3 \left(\frac{1}{\sqrt{N}} (R_{11}^N)^T\right)^{-1}$$

and hence again by (A 4)

$$\frac{1}{\sqrt{N}} R_{22}^N Q_2^N = \Gamma_i \left(\frac{1}{\sqrt{N}} X_{1,N} - \varepsilon_N^3 E_N^3 \left(\frac{1}{\sqrt{N}}(R_{11}^N)^T\right)^{-1} Q_1^N\right)$$

Multiplying each side of this equation by its transpose and making use of (A 5) yields:

$$\frac{1}{N} R_{22}^N (R_{22}^N)^T = \Gamma_i \left[\frac{1}{N} X_{1,N} X_{1,N}^T - \varepsilon_N^3 E_N^3 \left(\frac{1}{N} R_{11}^N (R_{11}^N)^T\right)^{-1} \varepsilon_N^3 (E_N^3)^T\right] \Gamma_i^T \tag{A 6}$$

Based on the white noise property in (14) we deduce that $\lim_{N\to\infty}(1/N(R_{11}^N(R_{11}^N)^T))^{-1} = \lim_{N\to\infty}(1/N(U_{1,i,N}U_{1,i,N}^T))^{-1} = \sigma_u^{-2} I_{m.i}$. Making use of the condition in (23) of the theorem and taking the limit $\lim_{N\to\infty}$ of both sides of (A 6) we obtain the result of the theorem. □

A.3. **Proof of Theorem 2:** From relationship (A 1) for $x_0 \equiv 0$ and the sum going to ∞, we obtain:

$$X_{1,N} = [B \;\; AB \;\; A^2B \;\; \ldots] \begin{bmatrix} u_0 & u_1 & \ldots & u_{N-1} \\ u_{-1} & & \ldots & u_{N-2} \\ u_{-2} & & & \\ \vdots & & & \vdots \end{bmatrix} \tag{A 7}$$

Since

$$\rho\begin{bmatrix} U_{1,i,N} \\ \hline X_{1,N} \end{bmatrix} = \rho\left[\begin{array}{cccc} u_i & u_{i+1} & \ldots & u_{N+i-1} \\ u_{i-1} & u_i & \ldots & u_{N+i-2} \\ \vdots & & & \vdots \\ u_1 & & \ldots & u_N \\ \hline x_1 & x_2 & \ldots & x_N \end{array}\right]$$

we will investigate the rank of the latter matrix.

Let Δ denote the matrix $[B \;\; AB \;\; \ldots \;\; A^{j-1}B]$, substitute (A 7) and use condition (3), to find,

$$\begin{matrix} \\ mi \\ \\ \\ n \end{matrix}\left[\begin{array}{ccc} u_i & \cdots & u_{N+i-1} \\ u_{i-1} & \cdots & u_{N+i-2} \\ \vdots & & \vdots \\ u_1 & \cdots & u_N \\ \hline x_1 & \cdots & x_N \end{array}\right] =$$

$$\left[\begin{array}{c|c||c|c||c} I_{mi} & 0 & 0 & 0 & 0 \\ \hline 0 & \Delta & \cdots & A^{j+i}\Delta & \cdots \end{array}\right] \left[\begin{array}{ccc} u_i & \cdots & u_{N+i-1} \\ u_{i-1} & \cdots & u_{N+i-2} \\ \vdots & & \vdots \\ u_1 & \cdots & u_N \\ \hline u_0 & \cdots & u_{N-1} \\ u_{-1} & \cdots & u_{N-2} \\ \vdots & & \vdots \\ u_{-j+1} & \cdots & u_{N-j} \\ \hline\hline -u_i & \cdots & -u_{N+i-1} \\ -u_{i-1} & \cdots & -u_{N+i-2} \\ \vdots & & \vdots \\ -u_1 & \cdots & -u_N \\ \hline -u_0 & \cdots & -u_{N-1} \\ -u_{-1} & \cdots & -u_{N-2} \\ \vdots & \cdots & \vdots \\ -u_{-j+1} & \cdots & -u_{N-j} \\ \hline\hline \vdots & & \vdots \end{array}\right] \begin{matrix} \\ mi \\ \\ \\ mj \\ \\ \\ \\ mi \\ \\ \\ mj \\ \\ \\ \\ \vdots \end{matrix}$$

The right hand side of the above equation can be written as:

$$= \left[\begin{array}{c|c||c|c||c} I_{mi} & 0 & 0 & 0 & 0 \\ \hline 0 & \Delta & \cdots & A^{j+i}\Delta & \cdots \end{array}\right] \left[\begin{array}{c} \begin{matrix} \\ mi \\ mj \end{matrix}\left[\begin{array}{c|c} \overset{mi}{R_{11}} & \overset{mj}{0} \\ \hline \star & R_{22} \end{array}\right] \\ \hline\hline \begin{matrix} \\ mi \\ mj \end{matrix}\left[\begin{array}{c|c} \overset{mi}{-R_{11}} & \overset{mj}{0} \\ \hline \star & -R_{22} \end{array}\right] \\ \hline\hline \vdots \end{array}\right] Q$$

Explicit multiplication of the above pair of matrices yields:

$$= \left[\begin{array}{c|c} R_{11} & 0 \\ \hline \star & [I_n - A^{j+i} + (A^{j+i})^2 - \ldots]\Delta R_{22} \end{array}\right] Q$$

Hence

$$\rho\left[\begin{array}{c} U_{1,i,N} \\ \hline X_{1,N} \end{array}\right] = \underbrace{\rho(R_{11})}_{mi} + \rho([I_n - A^{j+i} + (A^{j+i})^2 - \cdots]\Delta R_{22})$$

Since, $\rho(U_{-j+1,i+j,N}) = m(i+j)$ the submatrices R_{11} and R_{22} are square invertible matrices and since A has its eigenvalues within the unit circle $[I_n - A^{j+i} + (A^{j+i})^2 - \cdots] = (I_n + A^{j+i})^{-1}$, an application of Lemma 1 completes the proof. □

A.4. **Proof of Theorem 3:** With the partitioning in (37) and condition (4) of the theorem, we can express $\Gamma_{i-1}X_{1,N}$ and $\Gamma_{i-1}X_{2,N}$ analogously to (27) as:

$$\Gamma_{i-1}X_{1,N} = (R_{21}^1 - H_{i-1}R_{11}^1)Q_1 + R_{22}^1 Q_2$$

and

$$\Gamma_{i-1}X_{2,N} = (R_{21}^2 - H_{i-1}R_{11}^2)Q_1 + R_{22}^2 Q_2 \tag{A 8}$$

A combination of the SVD in (38) and the expression of $\Gamma_i X_{1,N}$, given by (27), yields a SVD of the latter matrix given as:

$$\Gamma_i X_{1,N} = U_n S_n V_n^{\mathrm{T}} \left[\begin{array}{c} Q_1 \\ \hline Q_2 \end{array}\right] =: U_n S_n \bar{V}_n^{\mathrm{T}} \tag{A 9}$$

Hence, $\Gamma_{i-1}X_{1,N}$ can be expressed as:

$$\Gamma_{i-1}X_{1,N} = U_n^{(1)} S_n \bar{V}_n^{\mathrm{T}} \tag{A 10}$$

or with the expression for $X_{1,N}$ obtained in (30), as $\Gamma_{i-1}X_{1,N} = \Gamma_{i-1}(R_{x1}Q_1 + R_{x2}Q_x)$. Since $i > n$, $\rho(\Gamma_{i-1}) = n$ and since condition (28) is satisfied, $\rho(R_{x2}Q_x) = n$, Lemma 1 shows that $\rho(\Gamma_{i-1}X_{1,N}) = n$. Hence, since $U_n^{(1)} \in \mathbb{R}^{(i-1)l \times n}$, $S_n \in \mathbb{R}^{n \times n}$ and $\bar{V}_n \in \mathbb{R}^{N-n}$, another application of Lemma 1 teaches that the rank of these factors of $\Gamma_{i-1}X_{1,N}$ has to be n. Therefore, $U_n^{(1)}$ has full column rank and hence has a left inverse, denoted by $(U_n^{(1)})\dagger$. This proves the first part of the theorem.

The matrix product $(U_n^{(1)})\dagger\Gamma_{i-1}$ is a square n by n matrix. Denote this matrix product by T, then (A 10) becomes:

$$TX_{1,N} = S_n \bar{V}_n^{\mathrm{T}} \tag{A 11}$$

Since $T \in \mathbb{R}^{n \times n}$, $X_{1,N} \in \mathbb{R}^{n \times N}$ and $S_n \bar{V}_n^{\mathrm{T}}$ obviously is of rank n, Lemma 1 again demonstrates that T is full rank and hence invertible. This proves the second part of the Theorem. Hence,

$$X_{1,N} = T^{-1} S_n \bar{V}_n^{\mathrm{T}}$$

or with (A 8) and (A 10), this is equivalent to

$$X_{1,N} = T^{-1}(U_n^{(1)})\dagger[(R_{21}^1 - H_{i-1}R_{11}^1)|R_{22}^1]\begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix}$$

Applying the same transformation $(U_n^{(1)})\dagger$ on the left of the expression for $\Gamma_{i-1}X_{2,N}$ in (A 8) yields:

$$X_{2,N} = T^{-1}(U_n^{(1)})\dagger[(R_{21}^2 - H_{i-1}R_{11}^2)|R_{22}^2]\begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix}$$

Substituting the latter two expressions for $X_{1,N}$ and $X_{2,N}$ into the set of equations:

$$\begin{bmatrix} x_2 & x_3 & \cdots & x_{N+1} \\ y_1 & y_2 & \cdots & y_N \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x_1 & x_2 & \cdots & x_N \\ u_1 & u_2 & \cdots & u_N \end{bmatrix}$$

yields,

$$\begin{bmatrix} T^{-1}(U_n^{(1)})\dagger(R_{21}^2 - H_{i-1}R_{11}^2) & T^{-1}(U_n^{(1)})\dagger R_{22}^2 \\ r_{y1} & r_{y2} \end{bmatrix}\begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix} =$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} T^{-1}(U_n^{(1)})\dagger(R_{21}^1 - H_{i-1}R_{11}^1) & T^{-1}(U_n^{(1)})\dagger R_{22}^1 \\ r_u & 0 \end{bmatrix}\begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix}$$

From which we easily deduce the third part of the theorem. □

A.5. **Proof of Theorem 4:** Recall (33):

$$\Gamma_i R_{x2} Q_x = R_{22} Q_2$$

Condition (1) of the theorem implies (see the paragraph following (33)) that $\rho(R_{22}) = n$. Hence, it has a SVD given as:

$$R_{22} = U_n S_n V_n^{\mathrm{T}} \tag{A 12}$$

Furthermore, as in the above mentioned paragraph, we conclude that the column space of R_{22} equals the column space of Γ_i. By (A 12), we deduce that the column space of R_{22} equals U_n, hence there exists a non-singular $n \times n$ transformation matrix T, such that:

$$\Gamma_i T = U_n \tag{A 13}$$

With the definition of Γ_i following (7), this can explicitly be denoted as:

$$\begin{bmatrix} (CT) \\ (CT)(T^{-1}AT) \\ \vdots \\ (CT)(T^{-1}A^{i-1}T) \end{bmatrix} = \begin{bmatrix} C_T \\ C_T A_T \\ \vdots \\ C_T A_T^{i-1} \end{bmatrix} = U_n$$

The shift-invariance property of the matrix U_n then gives rise to (43) and (44).

(43) is solvable, since by condition (2) of the theorem, the matrix $U_n^{(1)} = \Gamma_{i-1}T$ has full column rank.

Recall (A 3) i.e. $Y_{1,i,N} = \Gamma_i X_{1,N} = R_{21}Q_1 + R_{22}Q_2$. Substitute on its right-hand side equality, the expressions for $U_{1,i,N}$ in (A 2), for R_{22} in (A 12) and for Γ_i in (A 13), then we obtain:

$$U_n T^{-1} X_{1,N} + H_i R_{11} Q_1 = R_{21} Q_1 + U_n S_n V_n^{\mathrm{T}} Q_2 \qquad \text{(A 14)}$$

Since $(U_n^{\perp})^{\mathrm{T}} U_n = 0$, multiplication of (A 14) on the left by $(U_n^{\perp})^{\mathrm{T}}$ yields $((U_n^{\perp})^{\mathrm{T}} H_i R_{11} - (U_n^{\perp})^{\mathrm{T}} R_{21}) Q_1 = 0$ or,

$$(U_n^{\perp})^{\mathrm{T}} H_i R_{11} - (U_n^{\perp})^{\mathrm{T}} R_{21} = 0$$

Since, condition (28) implies that $\rho(U_{1,i,N}) = mi$, and hence the matrix R_{11} is again invertible, the above relationship can be transformed into:

$$(U_n^{\perp})^{\mathrm{T}} H_i - (U_n^{\perp})^{\mathrm{T}} R_{21} R_{11}^{-1} = 0 \qquad \text{(A 15)}$$

With the definition of H_i following (7) and the definition of Ξ equal to $(U_n^{\perp})^{\mathrm{T}} R_{21} R_{11}^{-1}$, this equation can explicitly be denoted as in (45).

By condition (5) of the theorem, the left matrix in the underbraced matrix product in (45) has (full column) rank li. Subsequently, by condition (2) of the theorem we have that $\rho(U_n^{(1)} = \Gamma_{i-1}\mathrm{T}) = n$, therefore, via an application of Lemma 1 the latter condition demonstrates that the rank of the underbraced matrix product in (45) is $l + n$. Hence, the latter matrix product has a left inverse and we can solve for the matrix pair $(D^{\mathrm{T}} \ (B_T)^{\mathrm{T}})^{\mathrm{T}}$. □

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