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Subspace-based Methods for the Identification of Linear Time-invariant Systems*

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An overview of subspace-based system identification methods is presented. Comparison between different algorithms are given and similarities pointed out.

Key Words—System identification; subspace methods; parameter estimation; multivariable systems; instrumental variable methods.

Abstract—Subspace-based methods for system identification have attracted much attention during the past few years. This interest is due to the ability of providing accurate state-space models for multivariable linear systems directly from input–output data. The methods have their origin in classical state-space realization theory as developed in the 1960s. The main computational tools are the QR and the singular-value decompositions. Here, an overview of existing subspace-based techniques for system identification is given. The methods are grouped into the classes of realization-based and direct techniques. Similarities between different algorithms are pointed out, and their applicability is commented upon. We also discuss some recent ideas for improving and extending the methods. A simulation example is included for comparing different algorithms. The subspace-based approach is found to perform competitive with respect to prediction-error methods, provided the system is properly excited.

1. INTRODUCTION

System identification is generally the art of mathematical modeling, given input–output measurements from a dynamical system. The problem is of interest in a variety of applications, ranging from chemical process simulation and control to identification of vibrational modes in flexible structures. In the classical system identification problem the input (the control signal) is known exactly, whereas the output signal may be corrupted by additive noise. The process can also have external unmeasurable inputs. The focus here is on parametric *model-based* techniques for system identification. The task is then to find a suitable mathematical model for the system from a finite number of data, corrupted by noise and/or process disturbances.

The most important traditional system identification techniques are the prediction error method (PEM) and the instrumental variable method (IVM); see e.g. Ljung (1987) and Söderström and Stoica (1989). A number of algorithms within these two classes have been proposed and found to perform well on both simulated and real data. The statistical properties of the PEM and IVM methods are well known—in particular, so is their relation to maximum-likelihood estimation. The aforementioned 'traditional techniques' offer good solutions to many real-world problems. However, they do have a number of shortcomings, particularly so for systems with many inputs and outputs. The PE and the IV methods are primarily used with so-called black-box model structures. The candidate models then consist of linear difference equations for describing the input–output relations (and, if desired, the noise color). The use of such models is quite cumbersome in the general multivariable case, and the numerical reliability may be poor for large system orders. The preferred model structure for more complex problems is therefore a *state-space model*. In principle, PEMs are easily adapted to work with state-space models. However, in practice this generalization may lead to a huge number of unknowns, rendering the numerical optimization required to calculate the optimal PEM estimate impractical. A canonical parameterization of a general nth-order system with m inputs and l outputs requires $n(2l+m)+ml$ free parameters. In extreme applications the number of inputs and

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outputs are on the order of 10, whereas the number of states can be on the order of 100. Furthermore, the problem of finding a numerically reliable canonical parametrization is also to a large extent unsolved, and in fact it may have no satisfactory solution at all. See for example VanOverbeek and Ljung (1982), Chou (1994) and Mehra (1994) for more details on this problem. While PEMs are essentially independent of the system representation, the extension of IVMs to state-space models is less obvious. It is shown here that subspace-based methods can be viewed as one way of generalizing IVMs in a numerically reliable way.

Subspace-based state-space system identification (4SID, pronounced 'forsid') methods offer numerically reliable state-space models for complex multivariable dynamical systems directly from measured data. No nonlinear search is performed, nor is a canonical parametrization used. The computational complexity is modest compared with PEM, particularly when the number of outputs and states is large. Two review papers have recently been published on the subject. In Rao and Arun (1992) systems without measurable input signals (i.e. time series) are considered. In VanDerVeen *et al.* (1993) a good overview of methods based on state-space realization theory is provided. Both papers emphasize the strong connection to certain problems in signal processing; see further Section 6. Also, the theses by VanOverschee (1995) and McKelvey (1995) contain nice tutorials on subspace-based methods.

The 4SID methods have their origin in state-space realization theory as developed in the 1960s, although they can in fact be traced to principal component factor analysis (Hotelling, 1933). A classical contribution is by Ho and Kalman (1966), where a scheme for recovering the system matrices from impulse response measurements is outlined. Refinements of Ho and Kalman's scheme are reported in Zeiger and McEwen (1974) and Kung (1978), introducing the SVD as a tool to reduce the sensitivity to errors in the measured impulse response. More recently, a number of algorithms further extending these ideas have been proposed (e.g. King *et al.*, 1988; Liu and Skelton, 1991; Ljung, 1991; Bayard, 1992). We shall refer to this class of techniques as the *realization-based* 4SID methods.

A problem with the above-mentioned approach is the difficulty of obtaining a reliable nonparametric estimate of the impulse response. A number of algorithms require special inputs, such as impulse or white noise signals (see e.g. Liu and Skelton, 1992). An alternative approach is to extract the desired information directly from the data, without explicitly forming impulse responses. An early attempt to solve this problem is presented in Gopinath (1969), and more recently developments are due to DeMoor *et al.* (1988), Moonen *et al.* (1989) and Verhaegen (1991). We shall refer to this class of algorithms as *direct* 4SID methods. The basic direct 4SID techniques produce consistent estimates of the system matrices only under very restrictive assumptions on the noise. More recently techniques incorporating so-called *instrumental variables* (Ljung, 1987; Söderström and Stoica, 1989) have been suggested for overcoming this drawback (see Verhaegen, 1993b, 1994; Viberg *et al.*, 1993; VanOverschee and DeMoor, 1994b).

We shall focus on systems driven by both observable and unobservable inputs. The reader is referred for example to Akaike (1974), Faurre (1976), Aoki (1990), Rao and Arun (1992), VanOverschee and DeMoor (1993) and Deistler *et al.* (1994) for 4SID methods and their analysis in the time-series case. We keep in mind, though, that the distinction between inputs and outputs is not so obvious in all applications (Willems, 1986). The discussion is also confined to open-loop systems. An extension to systems operating in closed loop is reported in Verhaegen (1993a). There, a 'global system', with the control signal, the system output and the different error signals as outputs is identified. The open-loop transfer function is then calculated from this global system. One should also consider applying the techniques of Schrama (1991) and VanDenHof and Schrama (1993) to 4SID methods. The idea is to estimate a 'noise-free' input signal, and to apply the open-loop identification method using the reconstructed input. We also tend to ignore the computational aspects of 4SID methods. For efficient implementations of the basic techniques, see Verhaegen and Deprettere (1991) and Cho *et al.* (1994).

The remainder of this paper is organized as follows. Section 2 gives an introduction to state-space realization from impulse response data. Section 3 presents the basic two classes of 4SID methods. In Section 4, a framework for statistical analysis is introduced, and the basic techniques are analyzed with respect to consistency. Section 5 presents extensions of the basic techniques using the concept of instrumental variables. Section 6 gives some details from a related problem in signal processing. In Section 7 some recent research trends are reported, and suggestions for improving the IV-based 4SID metods are made. Section 8 presents the results of a simple computer simulation to illustrate the ideas, and Section 9 concludes the paper.

2. REALIZATION THEORY

Consider a causal linear time-invariant (LTI) system with m input signals collected in the vector u_t, and l output signals denoted by y_t. In the absence of noise, the input–output relation can be described by the convolution sum

$$y_t = \sum_{k=0}^{\infty} h_k u_{t-k}, \tag{1}$$

where h_k denotes the $l \times m$ matrix of impulse responses. That is, the ijth element of h_k is the response in output i at time k, to a unit impulse applied to input j at time 0. Assume that the system is of finite dimension n. An alternative description of the dynamics is then obtained from a state-space realization:

$$x_{t+1} = Ax_t + Bu_t, \tag{2}$$

$$y_t = Cx_t + Du_t. \tag{3}$$

Here, x_t is an n-dimensional state vector, the matrix A is $n \times n$, B is $n \times m$, C is $l \times n$ and D is $l \times m$. The description (2), (3) is not unique. Let T be a nonsingular $n \times n$ matrix. It is then easy to see that

$$z_{t+1} = \bar{A}z_t + \bar{B}u_t,$$

$$y_t = \bar{C}z_t + \bar{D}u_t$$

provides an alternative state-space realization, where $z_t = Tx_t$, $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$ and $\bar{D} = D$. The transfer function is of course the same for different state-space realizations, though. As can easily be checked, $H(z) = C(zI - A)^{-1}B + D = \bar{C}(zI - \bar{A})^{-1}\bar{B} + \bar{D}$.

Classical realization theory deals with the problem of finding a *minimal* state-space model† given the collection of impulse responses $\{h_t\}$, sometimes referred to as the *Markov parameters*. By applying impulse inputs to (2) and (3), one finds immediately the relation

$$h_t = \begin{cases} 0 & (t<0), \\ D & (t=0), \\ CA^{t-1}B & (t>0). \end{cases} \tag{4}$$

Thus, the system matrix D is readily available from h_0. The so-called *Hankel matrix* of order $(n+1) \times (n+1)$ of the system is defined by

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 & \dots & h_{n+1} \\ h_2 & h_3 & h_4 & \dots & h_{n+2} \\ h_3 & h_4 & h_5 & \dots & h_{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n+1} & h_{n+2} & h_{n+3} & \dots & h_{2n+1} \end{bmatrix}$$

Using (4), it is straightforward to verify that $\mathbf{H}$ can be factored as

$$\mathbf{H} = \mathbf{\Gamma}_{n+1}\mathbf{\Omega}_{n+1}, \tag{5}$$

where

$$\mathbf{\Gamma}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{6}$$

is the observability matrix of the system, and

$$\mathbf{\Omega}_n = [B \quad AB \quad \dots \quad A^{n-1}B] \tag{7}$$

is the controllability matrix. For a minimal realization, the latter matrices have full rank (see Kailath, 1980), and hence $\mathbf{H}$ has rank n, equal to the system order. Moreover, if $\mathbf{H} = \bar{\mathbf{\Gamma}}_{n+1}\bar{\mathbf{\Omega}}_{n+1}$ is any given full-rank factorization of the Hankel matrix then $\bar{\mathbf{\Gamma}}_{n+1}$ is the observability matrix and $\bar{\mathbf{\Omega}}_{n+1}$ is the controllability matrix for *some* state-space realization. The algorithm of Ho and Kalman (1966) is based on the above observations. The first step is to compute a factorization (5) of the given Hankel matrix. The system matrices B and C are then read directly from the first block column (first m columns) of $\mathbf{\Omega}_{n+1}$ and the first block row (first l rows) of $\mathbf{\Gamma}_{n+1}$ respectively. It remains to compute A. This can be done, for instance, by observing that $\mathbf{\Gamma}_{n+1}$ has a so-called *shift invariance* structure. More exactly, define the submatrices $\mathbf{\Gamma}_{2:n+1}$ and $\mathbf{\Gamma}_{1:n}$, which are obtained from $\mathbf{\Gamma}_{n+1}$ by deleting the first and the last block rows respectively. It is then easy to see that the system matrix A is the transformation that maps $\mathbf{\Gamma}_{1:n}$ onto $\mathbf{\Gamma}_{2:n+1}$, i.e.

$$\mathbf{\Gamma}_{2:n+1} = \mathbf{\Gamma}_{1:n}A. \tag{8}$$

If we require the realization to be minimal, the observability matrix $\mathbf{\Gamma}_{1:n} = \mathbf{\Gamma}_n$ is nonsingular, and A can be computed from $\mathbf{\Gamma}_{n+1}$ by solving (8):

$$A = \mathbf{\Gamma}_{1:n}^{\dagger}\mathbf{\Gamma}_{2:n+1}, \tag{9}$$

where $(\cdot)^{\dagger}$ denotes the Moore–Penrose pseudo-inverse (note that $\mathbf{\Gamma}_{1:n}$ is, in general, not square). The relation (9) completes the basic deterministic realization scheme. One should note that the particular realization that the method delivers depends on the factorization algorithm used for finding $\mathbf{\Gamma}_{n+1}$ and $\mathbf{\Omega}_{n+1}$ from $\mathbf{H}$. This is an unattractive feature for some applications, where a certain *canonical* realization is desired. We refer the interested reader to linear systems textbooks, such as Kailath (1980), for more details on canonical parametrizations and other aspects of state-space realization theory.

3. SUBSPACE METHODS

Most 4SID (subspace-based state-space system identification) methods suggested to date have a great deal in common with Ho and Kalman's realization algorithm. More specifically, all

† A state-space model is minimal if there exists no other realization of lower degree. Minimal realizations are always both observable and controllable.

methods involve extraction of the extended observability matrix from input–output data, possibly after a first step where the impulse response is estimated. The various techniques differ in the way the observability matrix is estimated, and also how it is used for finding the system matrices. In the vector input–output relation (15) below, the extended observability matrix spans an n-dimensional *signal subspace,* which contains the part of the output that is due to the state variable. This is a proper subspace, since the dimension of the observability matrix is larger than n, the system order. A 'data compression' (or, more precisely, rank reduction) step will therefore be necessary for computing the signal subspace. As will shortly be demonstrated, this data compression is typically performed by the SVD (Golub and VanLoan, 1989). Here, the class of 4SID methods is somewhat loosely defined as those techniques that employ a rank reduction of some quantity estimated from input–output data, arriving at an estimate of an extended observability matrix. The latter matrix can then be used in various ways for obtaining a state-space model (and in some cases a full stochastic model) of the system under study. The 4SID methods are split into two subclasses, where the members of the first class form an explicit estimate of the impulse response, whereas the second type of methods circumvent this step.

3.1. *Realization-based 4SID methods*

A natural idea is to apply a state-space realization technique to an estimated impulse response. This procedure turns realization theory into system identification, but it also raises two important questions.

1. How does one estimate the impulse response given a finite number of noisy input–output data?
2. How does one reduce the effect of errors in the estimated impulse response to arrive at a low-dimensional state-space realization?

The most straightforward way to solve the first problem is to employ a nonparametric technique, since a parametric model is delivered from the realization step. A direct method is to apply impulse input signals in the different input channels, and simply measure the response in each output (Liu and Skelton, 1992). To reduce the effect of noise, the experiment may be repeated and the average output calculated. A less time-consuming approach is to use correlation analysis. Consider the convolution (1), and assume that the system is corrupted by some disturbances sources, whose effect on the output is lumped into the zero-mean additive noise term v_t:

$$y_t = \sum_{k=0}^{\infty} h_k u_{t-k} + v_t. \tag{10}$$

Suppose for the moment that a stationary white noise input signal is selected,† so that

$$E[u_t u_s^{\mathrm{T}}] = r_{uu}\delta_{t,s}, \tag{11}$$

where $\delta_{t,s}$ is the Kronecker delta. Provided u_t and u_s are uncorrelated for all t and s and the system is stable, we get

$$r_{yu}(\tau) = e[y_t u_{t-\tau}^{\mathrm{T}}] = h_\tau r_{uu}. \tag{12}$$

Hence a finite number of impulse response parameters can be consistently estimated from the corresponding nonparametric estimates of the cross-covariance function $r_{yu}(\tau)$ (Ljung, 1987; Söderström and Stoica, 1989). Another nonparametric approach for estimating the impulse response is to use the inverse discrete Fourier transform of a nonparametric transfer function estimate. This idea is pursued in McKelvey *et al.* (1994), and a close link to a method that operates directly in the frequency domain is presented. It is also possible to apply a parametric technique for estimation of the impulse response. The approach of Ljung (1991) is to use a black-box IVM, whereas Bayard (1992) proposes to use an iterative frequency domain technique. In both cases an overparameterized model should be used—a fact that suggests ill-conditioned estimation problems.

Given a number of estimated impulse response parameters, Kung (1978) (see also Juang and Pappa, 1985) proposed to modify Ho and Kalman's algorithm in the following way. Firstly, to reduce the effect of errors in the estimation of A, the system (8) of equations should be overdetermined. Hence a 'large' dimension of the Hankel matrix $\mathbf{H}$ should be used. In the presence of noise, the Hankel matrix will generically be of full rank. An 'optimal' way of reducing the rank‡ of the Hankel matrix, namely to truncate its SVD, is therefore used. Let $\hat{\mathbf{H}}$ be an estimated Hankel matrix of (block) size $\alpha \times \beta$, say, where α and β are both chosen to exceed the largest expected system order. Let the SVD of $\hat{\mathbf{H}}$ be given by

$$\hat{\mathbf{H}} = \hat{\mathbf{Q}}\hat{\mathbf{S}}\hat{\mathbf{V}}^{\mathrm{T}},$$

† Temporal whiteness is no restriction so long as the color of the input signal is known (or estimated). A prefilter can then be applied to both u_t and y_t, rendering the filtered input signal temporally white.

‡ The truncation of the SVD is optimal in the sense of minimizing both the Frobenius norm and the spectral norm of the difference between the original matrix and its low-rank approximation. Note, however, that the reduced-rank matrix has generally lost the block Hánkel structure.

where $\hat{\mathbf{Q}}$ and $\hat{\mathbf{V}}$ are orthogonal matrices, and $\hat{\mathbf{S}}$ is a diagonal matrix with the singular values in nonincreasing order on the diagonal. In the ideal case $\hat{\mathbf{H}}$ is of rank n, so that only the n first singular values are nonzero. In the case of noisy impulse response measurements, all singular values will be positive w.p.1. The user must then decide on the number of 'significant' singular values, which in turn will determine the resulting system order. This is a nontrivial task, for which few theoretical guidelines are available in general. Assuming the system order to be given, the SVD matrices are partitioned into the 'signal' and 'noise' parts

$$\hat{\mathbf{Q}} = [\hat{\mathbf{Q}}_s \quad \hat{\mathbf{Q}}_n],$$
$$\hat{\mathbf{V}} = [\hat{\mathbf{V}}_s \quad \hat{\mathbf{V}}_n],$$

where $\hat{\mathbf{Q}}_s$ and $\hat{\mathbf{V}}_s$ contain the n principal singular vectors, whose corresponding singular values are collected in the $n \times n$ diagonal matrix $\hat{\mathbf{S}}_s$. The 'cleaned' estimates of the extended observability and controllability matrices are then

$$\hat{\mathbf{\Gamma}}_\alpha = \hat{\mathbf{Q}}_s \hat{\mathbf{S}}_s^{1/2},$$
$$\hat{\mathbf{\Omega}}_\beta = \hat{\mathbf{S}}_s^{1/2} \hat{\mathbf{V}}_s^{\mathrm{T}}.$$

As in the classical realization algorithm, the system matrices B and C are read directly from the first block column of $\hat{\mathbf{\Omega}}_\beta$ and the first block row of $\hat{\mathbf{\Gamma}}_\alpha$ respectively, whereas A is obtained by solving the overdetermined system of equations

$$\hat{\mathbf{\Gamma}}_{2:\alpha} \approx \hat{\mathbf{\Gamma}}_{1:\alpha-1} A. \tag{13}$$

A least-squares or a total least-squares (Golub and VanLoan, 1980) solution can be used, both leading to the same first-order sensitivity to errors in the estimated impulse response parameters (Rao and Hari, 1989; Stoica and Viberg, 1995). The particular state-space realization resulting from the procedure outlined above is balanced in the sense that $\hat{\mathbf{\Gamma}}_\alpha$ and $\hat{\mathbf{\Omega}}_\beta$ both have orthogonal columns and

$$\hat{\mathbf{\Gamma}}_\alpha^{\mathrm{T}} \hat{\mathbf{\Gamma}}_\alpha = \hat{\mathbf{\Omega}}_\beta \hat{\mathbf{\Omega}}_\beta^{\mathrm{T}} = \hat{\mathbf{S}}_s.$$

Of the existing alternatives to Kung's approach, we mention the eigensystem realization approach (ERA) due to Juang and Pappa (1985). The close ties to Hankel-norm model reduction (Glover, 1984) also deserve recognition. A more elaborate realization-based approach is obtained by considering not only the impulse response parameters, but also the atuocovariance of the output, $r_y(\tau)$, at a number of lags τ. A realization approach that matches measurements of both these types of parameters to their corresponding models is the Q-Markov COVER (covariance equivalent realization) method (King *et al.*, 1988; Liu and Skelton, 1991; Liu *et al.*, 1992).

3.2. *Direct 4SID methods*

The direct 4SID methods are based on the same geometrical ideas as the realization-based techniques. However, rather than forming the Hankel matrix, a direct input–output relation implies by the description (2), (3) is used. Introduce the $l\alpha$-vector of stacked outputs as

$$\mathbf{y}(t) = [y_t^{\mathrm{T}} \quad y_{t+1}^{\mathrm{T}} \quad \dots \quad y_{t+\alpha-1}^{\mathrm{T}}]^{\mathrm{T}}, \tag{14}$$

where $\alpha > n$ is a user-specified integer, whose role is similar to the number of block rows in the Hankel matrix employed by the realization-based methods. The $m\alpha$-vector of stacked inputs, $\mathbf{u}(t)$, is defined in an analogous fashion. The following input–output equation (Gopinath, 1969; DeMoor, 1988) is then easily derived from the system description (2), (3):

$$\mathbf{y}(t) = \mathbf{\Gamma}_\alpha x_t + \mathbf{\Phi}_\alpha \mathbf{u}(t), \tag{15}$$

where the matrix $\mathbf{\Phi}_\alpha$ is a block lower-triangular and Toeplitz matrix of impulse responses:

$$\mathbf{\Phi}_\alpha = \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ CA^{\alpha-2}B & \dots & CB & D \end{bmatrix}. \tag{16}$$

Suppose we are given observations of y_t and u_t for $t = 1, 2, \dots, N + \alpha - 1$. Using (15), the observations can be modeled by the matrix equation

$$\mathbf{Y} = \mathbf{\Gamma}_\alpha \mathbf{X} + \mathbf{\Phi}_\alpha \mathbf{U}, \tag{17}$$

where the block Hankel matrices of outputs and inputs are defined as

$$\mathbf{Y} = [\mathbf{y}(1) \quad \dots \quad \mathbf{y}(N)], \tag{18}$$

$$\mathbf{U} = [\mathbf{u}(1) \quad \dots \quad \mathbf{u}(N)], \tag{19}$$

and the state trajectory is collected in the matrix

$$\mathbf{X} = [x_1 \quad \dots \quad x_N]. \tag{20}$$

Consider the problem of obtaining an estimate of $\mathbf{\Gamma}_\alpha$ in (17), with knowledge of the data matrices $\mathbf{Y}$ and $\mathbf{U}$. If the impulse response matrix $\mathbf{\Phi}_\alpha$ was known, we could simply subtract the $\mathbf{\Phi}_\alpha \mathbf{U}$ term from $\mathbf{Y}$, followed by a truncated SVD. As $\mathbf{\Phi}_\alpha$ is unknown, it is natural to instead subtract an estimate of it. An unstructured least-squares estimate

$$\min_{\mathbf{\Phi}_\alpha} \|\mathbf{Y} - \mathbf{\Phi}_\alpha \mathbf{U}\|_{\mathrm{F}}^2, \tag{21}$$

where $\|\cdot\|_F$ denotes the Frobenius norm,† leads to the matrix

$$\mathbf{Y} - \hat{\mathbf{\Phi}}_\alpha \mathbf{U} = \mathbf{Y}\mathbf{\Pi}^\perp_{U^T}, \tag{22}$$

where $\mathbf{\Pi}^\perp_{U^T}$ is the orthogonal projection onto the nullspace of $\mathbf{U}$:

$$\mathbf{\Pi}^\perp_{U^T} = \mathbf{I} - \mathbf{U}^T(\mathbf{U}\mathbf{U}^T)^{-1}\mathbf{U}. \tag{23}$$

The indicated inverse exists if the input is persistently exciting and $N > m\alpha$. Since

$$\mathbf{U}\mathbf{\Pi}^\perp_{U^T} = 0,$$

we have in effect removed the part of the output $\mathbf{y}(t)$, (15), that did not emanate from the state x_t. The remaining part is

$$\mathbf{Y}\mathbf{\Pi}^\perp_{U^T} = \mathbf{\Gamma}_\alpha \mathbf{X}\mathbf{\Pi}^\perp_{U^T}, \tag{24}$$

and, provided $\mathbf{X}\mathbf{\Pi}^\perp_{U^T}$ is of full rank n, the signal subspace (i.e. the range space of $\mathbf{\Gamma}_\alpha$) can be recovered from a truncated SVD of $\mathbf{Y}\mathbf{\Pi}^\perp_{U^T}$. The required rank condition can be shown to hold under mild conditions on the input sequence (Liu, 1992). Partition the SVD as

$$\mathbf{Y}\mathbf{\Pi}^\perp_{U^T} = \hat{\mathbf{Q}}_s \hat{\mathbf{S}}_s \hat{\mathbf{V}}_s^T + \hat{\mathbf{Q}}_n \hat{\mathbf{S}}_n \hat{\mathbf{V}}_n^T, \tag{25}$$

where $\hat{\mathbf{Q}}_s$ contains the n principal left singular vectors and the diagonal matrix $\hat{\mathbf{S}}_s$ the corresponding singular values. It is clear that $\hat{\mathbf{S}}_n = 0$ in the absence of noise. However, in practice, a decision on the number of significant singular values must be made. The estimate of the observability matrix (for some state-space realization) is then taken as

$$\hat{\mathbf{\Gamma}}_\alpha = \hat{\mathbf{Q}}_s. \tag{26}$$

A column scaling by, for example, $\hat{\mathbf{S}}^{1/2}$ is often applied to the above, but note that this does not change the estimated system poles. Once $\mathbf{\Gamma}_\alpha$ is known, C can be read from its first block row, whereas A can be computed by solving (13). To unravel B and D, premultiply (17) by $\hat{\mathbf{Q}}_n^T$ and postmultiply by $\mathbf{U}^\dagger = \mathbf{U}^T(\mathbf{U}\mathbf{U}^T)^{-1}$ to yield (in the ideal case)

$$\hat{\mathbf{Q}}_n^T \mathbf{Y}\mathbf{U}^\dagger = \hat{\mathbf{Q}}_n^T \mathbf{\Phi}_\alpha. \tag{27}$$

In the presence of imperfect data, the above yields an overdetermined system of linear equations that can be solved in a least-squares sense with respect to B and D (note that the impulse response matrix $\mathbf{\Phi}_\alpha$, (16), depends linearly on B and D). For details we refer to DeMoor *et al.* (1988) and DeMoor (1988).

† The Frobenius norm of a matrix is the square root of the sum of squared moduli of all elements.

A more efficient implementation of the basic scheme is proposed in Verhaegen (1991) and Verhaegen and Dewilde (1992). Rather than explicitly forming the quantity $\mathbf{Y}\mathbf{\Pi}^\perp_{U^T}$, the following QR factorization (or rather, LQ factorization) is used:

$$\begin{bmatrix} \mathbf{U} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{11} & 0 \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1^T \\ \mathbf{Q}_2^T \end{bmatrix}. \tag{28}$$

Note that the matrix $\mathbf{Q}_2$ spans the part of the nullspace of $\mathbf{U}$ that is not shared by $\mathbf{Y}$. It follows that

$$\mathbf{Y}\mathbf{\Pi}^\perp_{U^T} = \mathbf{Y}\mathbf{Q}_2\mathbf{Q}_2^T = \mathbf{R}_{22}\mathbf{Q}_2^T.$$

Since $\mathbf{Q}_2^T\mathbf{Q}_2 = \mathbf{I}$, the left singular vectors of $\mathbf{R}_{22}$ coincide with those of $\mathbf{R}_{22}\mathbf{Q}_2^T$, and hence also with the left singular vectors of $\mathbf{Y}\mathbf{\Pi}^\perp_{U^T}$. Verhaegen's approach is to compute the SVD

$$\mathbf{R}_{22} = \hat{\mathbf{Q}}_s \hat{\mathbf{S}}_s (\hat{\mathbf{V}}_s^T \mathbf{Q}_2) + \hat{\mathbf{Q}}_n \hat{\mathbf{S}}_n (\hat{\mathbf{V}}_n^T \mathbf{Q}_2)$$

(where the notation is compatible with (25)). The algorithm then continues analogously to DeMoor *et al.* (1988), and consequently produces identical estimates. Another variation of the theme is presented in Liu (1992). There, an eigendecomposition (or SVD) of the matrix

$$\hat{\mathbf{R}} = \hat{\mathbf{R}}_{yy} - \hat{\mathbf{R}}_{yu}\hat{\mathbf{R}}_{uu}^{-1}\hat{\mathbf{R}}_{uy}$$

is performed. Using the natural covariance estimates, we have

$$\begin{aligned} \hat{\mathbf{R}} &= \frac{1}{N}\mathbf{Y}\mathbf{Y}^T - \frac{1}{N}\mathbf{Y}\mathbf{U}^T(\mathbf{U}\mathbf{U}^T)^{-1}\mathbf{U}\mathbf{Y}^T \\ &= \frac{1}{N}\mathbf{Y}\mathbf{\Pi}^\perp_{U^T}\mathbf{Y}^T. \end{aligned} \tag{29}$$

Clearly, the eigenvectors of $\hat{\mathbf{R}}$ coincide with the left singular vectors of $\mathbf{Y}\mathbf{\Pi}^\perp_{U^T}$, and we conclude that also the ORSE (observability range space extraction) method of Liu (1992) computes estimates identical to those of DeMoor *et al.* (1988). The method proposed in Moonen *et al.* (1989) is different (and more computationally demanding) than the basic 4SID scheme. However, similarly to the latter technique (see the next section), the approach by Moonen *et al.* (1989) requires restrictive assumptions on the noise for yielding consistent estimates (Moonen and Moor, 1992). A remedy is possible if the noise color is known, as demonstrated in Moonen and Vandewalle (1990).

4. STATISTICAL FRAMEWORK

Up to this point, we have mainly considered purely deterministic systems. Of course, any practical system is subject to uncertainty and

model mismatch. The question of sensitivity of the different algorithms then naturally arises. We shall focus here on the statistical variability of the estimates, owing to the necessity of processing only a finite number of noisy measurements. Thus assume that the system (2), (3) is corrupted by additive noise. Let the innovations representation (see Kailath, 1980) of the system be

$$x_{t+1} = Ax_t + Bu_t + Ke_t, \tag{30}$$

$$y_t = Cx_t + Du_t + e_t, \tag{31}$$

where e_t is the unmeasurable, temporally white, innovation process and K is the Kalman gain. The transfer functions from u_t and e_t to y_t are denoted by $H(q^{-1})$ and $N(q^{-1})$ respectively, where q^{-1} is the delay operator. Hence an alternative description of (30) and (31) is

$$y_t = H(q^{-1})u_t + N(q^{-1})e_t, \tag{32}$$

where the transfer functions are given by

$$H(q^{-1}) = C(qI - A)^{-1}B + D, \tag{33}$$

$$N(q^{-1}) = C(qI - A)^{-1}K + I. \tag{34}$$

It is possible that the dynamics of (30) are (31) are more complex than those of (2) and (3), i.e. the number of states may be larger. This is apparent from (32), because any dynamics of $N(q^{-1})$ that are not shared by $H(q^{-1})$ will increase the dimension of the noisy state-space description. These extra 'noise states' are cancelled when (33) is computed. In the case $K = 0$ the outputs are corrupted by additive white noise only. Such a model is termed *output error* (Ljung, 1987). In general, the disturbances may be temporally correlated rendering $K \neq 0$. The noise may also share dynamics with the plant—for example in the case where a wind gust acts as a disturbance on an aircraft. The distinction between an output error model and a full noise model is relevant when comparing different algorithms and their usefulness in different applications. With some abuse of notation, we shall let n denote the dimension of the state vector in both cases.

The following further assumptions are introduced.

Assumption 1. The innovation process is assumed to be a stationary, ergodic white random process, with zero mean and positive-definite covariance matrix

$$E[e_t e_s^{\mathrm{T}}] = r_{ee}\delta_{t,s}. \tag{35}$$

Assumption 2. The eigenvalues of A are strictly inside the unit circle. The pair $\{A, C\}$ is observable and $\{A, [B\ K]\}$ is controllable.

Assumption 3. The system input is modeled as an arbitrary, quasistationary (Ljung, 1987) deterministic sequence. The 'covariance matrix' of the input is denoted by

$$r_{uu} = \bar{E}[u_t u_t^{\mathrm{T}}],$$

where the averaging operator $\bar{E}$ is defined by the relation†

$$\bar{E}[z_t] = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E[z_t]. \tag{36}$$

The $m\alpha \times m\alpha$ covariance matrix of the stacked input vectors is similarly defined as

$$\mathbf{R}_{uu} = \bar{E}[\mathbf{u}(t)\mathbf{u}^{\mathrm{T}}(t)].$$

The input is assumed to be persistently exciting in the sense that

$$\mathbf{R}_{uu} > 0. \tag{37}$$

4.1. *Consistency*

For the realization-based 4SID methods, consistency‡ is essentially a question of whether or not a sufficient number of impulse response parameters are consistently estimated, since, whenever a consistent estimate of the Hankel matrix is obtained, any reasonable realization-based method will lead to consistent estimates of the system matrices. However, it should be noted that only the purely deterministic subsystem (i.e. the transfer function $H(q^{-1})$) is identified using this approach. Any additional dynamics due to nonwhite disturbance signals are ignored. The resulting model is therefore always in output-error form, although the 'true' data-generating system may not be so. It is of course possible to apply a stochastic realization algorithm (see e.g. Aoki, 1990; VanOverschee and DeMoor, 1993; Deistler *et al.*, 1994) to the residuals $y_t - \hat{H}(q^{-1})$. A complete deterministic–stochastic state-space model can then be obtained by lumping the deterministic and stochastic states, possibly followed by a model reduction step to identify the common dynamics. However, this way to proceed appears unnecessarily complicated. Therefore the realization-based 4SID methods are recommended only when an output-error model is desired.

† The $\bar{E}$ operator is convenient for simultaneously treating deterministic and stochastic sequences (Ljung, 1987). In practice, the expectation in (36) has no effect when the operator is applied to deterministic quasistationary and/or stochastic ergodic processes.

‡ An estimate is said to be (strongly) consistent if it converges w.p.1 to the true value as the number of data tends to infinity.

Consider next the basic direct 4SID method of Section 3.2. The simplest version to analyze is that of Liu (1992). By ergodicity, and using the first equality in (29), $\hat{\mathbf{R}}$ converges w.p.1 to the limiting matrix

$$\mathbf{R} = \mathbf{R}_{yy} - \mathbf{R}_{yu}\mathbf{R}_{uu}^{-1}\mathbf{R}_{uy} \tag{38}$$

as $N \to \infty$, where $\mathbf{R}_{yu} = \bar{E}[\mathbf{y}(t)\mathbf{u}^{\mathrm{T}}(t)]$. Including the noise terms as in (30) and (31), the input–output relation (15) is replaced by

$$\mathbf{y}(t) = \mathbf{\Gamma}_\alpha x_t + \mathbf{\Phi}_\alpha \mathbf{u}(t) + \mathbf{\Psi}_\alpha \mathbf{e}(t), \tag{39}$$

where the noise vector $\mathbf{e}(t)$ is defined analogously to (14), and $\mathbf{\Psi}_\alpha$ is the matrix of impulse responses from e_t to y_t, i.e. the lower block-triangular and block Toeplitz matrix

$$\mathbf{\Psi}_\alpha = \begin{bmatrix} I & \dots & 0 & 0 \\ CK & \ddots & 0 & 0 \\ \vdots & \ddots & I & 0 \\ CA^{\alpha-2}K & \dots & CK & I \end{bmatrix}.$$

Using (38) and (39), a straightforward calculation shows that

$$\mathbf{R} = \mathbf{\Gamma}_\alpha (r_{xx} - \mathbf{r}_{xu}\mathbf{R}_{uu}^{-1}\mathbf{r}_{ux})\mathbf{\Gamma}_\alpha^{\mathrm{T}} + \mathbf{\Psi}_\alpha \mathbf{R}_{ee} \mathbf{\Psi}_\alpha^{\mathrm{T}},$$

with obvious definitions of the various covariance matrices. Now, partition the eigendecomposition of $\mathbf{R}$ as

$$\mathbf{R} = \mathbf{Q}_s \mathbf{L}_s \mathbf{Q}_s^{\mathrm{T}} + \mathbf{Q}_n \mathbf{L}_n \mathbf{Q}_n^{\mathrm{T}},$$

where $\mathbf{Q}_s$ contains the n principal eigenvectors, and the diagonal matrix $\mathbf{L}_s$ the corresponding eigenvalues. It is not difficult to see that the range spaces of $\mathbf{\Gamma}_\alpha$ and $\mathbf{Q}_s$ coincide if†

$$r_{xx} - \mathbf{r}_{xu}\mathbf{R}_{uu}^{-1}\mathbf{r}_{ux} > 0, \tag{40}$$

$$\mathbf{\Psi}_\alpha \mathbf{R}_{ee} \mathbf{\Psi}_\alpha^{\mathrm{T}} = \sigma^2 \mathbf{I} \tag{41}$$

for some scalar σ^2. It is also easy to see that (40) is equivalent to

$$\bar{E}\begin{bmatrix} x_t \\ \mathbf{u}(t) \end{bmatrix}\begin{bmatrix} x_t \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} > 0.$$

While (40) can be shown to hold under mild conditions (see Liu, 1992), the requirement (41) imposes a serious restriction, pointed out in DeMoor (1988) Verhaegen (1991) and Viberg *et al.* (1991). The condition (41) holds essentially only for output-error systems with r_{ee} proportional to the identity matrix.

† There are other possibilities; for example, $\mathbf{\Psi}_\alpha \mathbf{R}_{ee} \mathbf{\Psi}_\alpha^{\mathrm{T}}$ can contain an additive term orthogonal to $\mathbf{\Gamma}_\alpha$. However, since we wish to keep the requirements on the noise independent of the system matrices, (41) is regarded as a necessary and sufficient condition.

In summary, we have the following result.

Theorem 1. Assume that $K = 0$, $r_{ee} = \sigma^2 I$ and that (40) holds true. Then the basic 4SID scheme produces a consistent estimate of the input–output transfer function $H(q^{-1})$.

An expression for the variance of the pole estimates is derived in Viberg *et al.* (1991). If (41) does not hold, the basic 4SID method can yield accurate estimates of the system matrices only if the signal-to-noise ratio (SNR) is high enough. Hence the technique is not applicable to more general systems.

5. INSTRUMENTAL VARIABLE METHODS

During the past few years, a number of 4SID techniques that are similar in spirit to the IVM have been proposed. The term 'instrumental variables' was mentioned in connection with 4SID methods in Aoki (1990), VanOverschee *et al.* (1991) and Verhaegen (1991). More recent developments are presented in Verhaegen (1993b, 1994). In Viberg *et al.* (1993) an IV interpretation of (a variation of) the N4SID method of VanOverschee and DeMoor (1994b) is presented, although less explicit related ideas are also provided in Larimore (1983, 1990). The referenced techniques all operate directly on the input–output equation (39). Consequently, this type of methods also belong to the class of direct 4SID methods.

The problem with the basic direct 4SID methods is that the geometrical properties of (15) are lost in the presence of noise (39). Therefore the IVs are used as 'instruments' for removing the effects of the noise term. The informative part of the signal term must, however, be left intact. Letting $\zeta(t)$ denote the vector of instruments, we should have

$$\bar{E}[\mathbf{e}(t)\zeta^{\mathrm{T}}(t)] = 0, \tag{42}$$

$$\operatorname{rank}(\bar{E}[x_t \zeta^{\mathrm{T}}(t)]) = n. \tag{43}$$

The latter requirement is necessary for guaranteeing that $\mathbf{\Gamma}_\alpha$ can be recovered from an estimate of $\bar{E}[\mathbf{y}(t)\zeta^{\mathrm{T}}(t)]$. A natural candidate instrument is the input signal, which is uncorrelated with the noise (in open-loop applications), and which should be sufficiently correlated with the state. However, the problem here is quite different from that of traditional IVMs, which only demand conditions similar to (42) and (43). In order to enable a direct estimate of $\mathbf{\Gamma}_\alpha$, the IVs should also be orthogonal to the input in (39)! Clearly, this is incompatible with the requirement (43). A solution to this dilemma is to partition the

output into two parts, termed the *past* and the *future*:

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{y}_p(t) \\ \mathbf{y}_f(t) \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} \mathbf{Y}_p \\ \mathbf{Y}_f \end{bmatrix}. \tag{44}$$

The number of block rows in $\mathbf{y}_p(t)$ and $\mathbf{Y}_p$ is denoted by β, and the number of block rows in $\mathbf{y}_f(t)$ and $\mathbf{Y}_f$ is then $\gamma = \alpha - \beta$. The quantities β and γ are user-defined and can be chosen quite arbitrarily. However, γ will be the number of block rows in the estimated observability matrix; thus $\gamma > n$ is required. As will be seen later, $\beta \geq n/(m+l)$ is a necessary condition for recovering $\mathbf{\Gamma}_\gamma$. Note that in Verhaegen (1993b, 1994) and VanOverschee and DeMoor (1994b) only the case $\gamma = \beta$ is considered. The model for the future outputs is

$$\mathbf{Y}_f = \mathbf{\Gamma}_\gamma \mathbf{X}_f + \mathbf{\Phi}_\gamma \mathbf{U}_f + \mathbf{\Psi}_\gamma \mathbf{E}_f, \tag{45}$$

where $\mathbf{X}_f = [x_{\beta+1} \quad \dots \quad x_{\beta+N}]$, and where $\mathbf{U}$ and the matrix of innovations $\mathbf{E}$ have been partitioned conformably with (44). Now, a possible choice of instrumental variable is the past inputs $\mathbf{u}_p(t)$. For simultaneously removing the $\mathbf{U}_f$ term of (45) and decorrelating the noise, Verhaegen (1993b) proposed to consider the quantity

$$\mathbf{Y}_f \mathbf{\Pi}^\perp_{U_f^T} \mathbf{U}_p^T = \mathbf{\Gamma}_\gamma \mathbf{X}_f \mathbf{\Pi}^\perp_{U_f^T} \mathbf{U}_p^T + \mathbf{\Psi}_\gamma \mathbf{E}_f \mathbf{\Pi}^\perp_{U_f^T} \mathbf{U}_p^T, \tag{46}$$

where $\mathbf{\Pi}^\perp_{U_f^T}$ denotes the orthogonal projector onto the nullspace of $\mathbf{U}_f$, i.e.

$$\mathbf{\Pi}^\perp_{U_f^T} = \mathbf{I} - \mathbf{U}_f^T (\mathbf{U}_f \mathbf{U}_f^T)^{-1} \mathbf{U}_f. \tag{47}$$

It is not difficult to see that

$$\lim_{N \to \infty} \frac{1}{N} \mathbf{E}_f \mathbf{\Pi}^\perp_{U_f^T} \mathbf{U}_p^T = 0, \quad \text{w.p.1},$$

and, by definition, $\mathbf{U}_f \mathbf{\Pi}^\perp_{U_f^T} = 0$. The rank condition (43) is, however, more elaborate. Indeed, the rank of the matrix

$$\frac{1}{N} \mathbf{X}_f \mathbf{\Pi}^\perp_{U_f^T} \mathbf{U}_p^T$$

is, under fairly general conditions (Verhaegen, 1993b), equal to the number of purely deterministic states. In other words, with only past inputs as IVs, we can only identify the transfer function $H(q^{-1})$. Any additional dynamics due to colored disturbances are lost in the IV correlation. The PI-MOESP (past inputs multivariable output-error state-space) method of Verhaegen (1993b) consists in first applying an SVD to the matrix (46). The latter matrix is calculated using a QR factorization similar to (28). The principal left singular vectors form the estimate of $\mathbf{\Gamma}_\gamma$, from which A and C are obtained using its shift invariance. The system matrices B and D are finally obtained by solving an overdetermined system of equations, analogous to (27).

In some applications, a complete state-space model is desired, incorporating both deterministic and stochastic states. To fulfill (43) with n equal to the dimension of the complete state, we must also involve the past outputs as instruments. The approach of Verhaegen (1994) and VanOvershee and DeMoor (1994b) is to use past inputs and outputs as IVs. The first step of Verhaegen's PO (past outputs) MOESP algorithm is to compute the following QR factorization:

$$\begin{bmatrix} \mathbf{U}_f \\ \mathbf{U}_p \\ \mathbf{Y}_p \\ \mathbf{Y}_f \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{11} & 0 & 0 & 0 \\ \mathbf{R}_{21} & \mathbf{R}_{22} & 0 & 0 \\ \mathbf{R}_{31} & \mathbf{R}_{32} & \mathbf{R}_{33} & 0 \\ \mathbf{R}_{41} & \mathbf{R}_{42} & \mathbf{R}_{43} & \mathbf{R}_{44} \end{bmatrix} \mathbf{Q}^T,$$

where the $\mathbf{R}_{ii}$ are lower-triangular and the column-unitary matrix $\mathbf{Q}$ is partitioned conformably with the lower-triangular matrix as

$$\mathbf{Q} = [\mathbf{Q}_1 \quad \mathbf{Q}_2 \quad \mathbf{Q}_3 \quad \mathbf{Q}_4].$$

The next step is to compute the SVD

$$[\mathbf{R}_{42}, \mathbf{R}_{43}] = \hat{\mathbf{Q}}_s \hat{\mathbf{S}}_s \hat{\mathbf{V}}_s^T + \hat{\mathbf{Q}}_n \hat{\mathbf{S}}_n \hat{\mathbf{V}}_n^T. \tag{48}$$

The extended observability matrix is then estimated by $\hat{\mathbf{Q}}_s$, and A and C are extracted in the usual way. For finding B and D, Verhaegen argues that the least-squares solution to the overdetermined system of equations

$$[\mathbf{R}_{31} \quad \mathbf{R}_{42}] \approx \mathbf{\Phi}_\gamma [\mathbf{R}_{11} \quad \mathbf{R}_{22}]$$

provides a consistent estimate of $\mathbf{\Phi}_\gamma$, from which B and D are easily calculated. To connect (48) to IVs, observe first from the QR factorization that

$$[\mathbf{R}_{42} \quad \mathbf{R}_{43}] = \mathbf{Y}_f \mathbf{Q}_{2:3},$$

where $\mathbf{Q}_{2:3}$ denotes the compound matrix $[\mathbf{Q}_2 \ \mathbf{Q}_3]$. Next, note that the left singular vectors of $\mathbf{Y}_f \mathbf{Q}_{2:3}$ coincide with those of $\mathbf{Y}_f \mathbf{Q}_{2:3} \mathbf{Q}_{2:3}^T$, because $\mathbf{Q}_{2:3}$ is column-unitary. Consider now the projections

$$\mathbf{Y}_f \mathbf{\Pi}^\perp_{U_f^T} = \mathbf{Y}_f \mathbf{Q}_{2:4} \mathbf{Q}_{2:4}^T,$$

$$\mathbf{Y}_f \mathbf{\Pi}^\perp_{Z^T} = \mathbf{Y}_f \mathbf{Q}_4 \mathbf{Q}_4^T,$$

where

$$\mathbf{Z} = \begin{bmatrix} \mathbf{U}_f \\ \mathbf{U}_p \\ \mathbf{Y}_p \end{bmatrix}.$$

The above relations show that

$$\begin{aligned} \mathbf{Y}_f \mathbf{Q}_{2:3} \mathbf{Q}_{2:3}^T &= \mathbf{Y}_f (\mathbf{\Pi}^\perp_{U_f^T} - \mathbf{\Pi}^\perp_{Z^T}) \\ &= \mathbf{Y}_f (\mathbf{\Pi}_{Z^T} - \mathbf{\Pi}_{U_f^T}). \end{aligned}$$

The projection decomposition theorem implies that

$$\mathbf{\Pi}_{Z^T} = \mathbf{\Pi}_{U_f^T} + \mathbf{\Pi}_{\tilde{Z}^T},$$

where

$$\tilde{\mathbf{Z}}^T = \mathbf{\Pi}_{U_f^T}^{\perp}\mathbf{P}^T,$$

and where the matrix $\mathbf{P}$ represents the past:

$$\mathbf{P} = \begin{bmatrix} \mathbf{U}_p \\ \mathbf{Y}_p \end{bmatrix}.$$

It follows that the observability matrix estimate obtained in the PO-MOESP method can be found equivalently from the SVD of the matrix

$$\mathbf{Y}_f\mathbf{\Pi}_{\tilde{Z}^T} = \mathbf{Y}_f\mathbf{\Pi}_{U_f^T}^{\perp}\mathbf{P}^T(\mathbf{P}\mathbf{\Pi}_{U_f^T}^{\perp}\mathbf{P}^T)^{-1}\mathbf{P}\mathbf{\Pi}_{U_f^T}^{\perp}. \quad (49)$$

A derivation similar to the above is presented in VanOverschee and DeMoor (1994a). A natural interpretation of (49) is that $\mathbf{Y}_f\mathbf{\Pi}_{U_f^T}^{\perp}$ represents the 'data equation' (or, rather, the residual of a least-squares problem similar to (21)), $\mathbf{P}$ is the matrix of instrumental variables, whereas $(\mathbf{P}\mathbf{\Pi}_{U_f^T}^{\perp}\mathbf{P}^T)^{-1}\mathbf{P}\mathbf{\Pi}_{U_f^T}^{\perp}$ represents a column weighting of the IVs. We shall elaborate somewhat on this interpretation in Section 7, where consistency will also be investigated. Let us, however, mention that a slightly different point of view is pursued in Viberg (1994), where $\tilde{\mathbf{Z}}$ is considered the 'instrumental variable matrix'.

The subspace estimate of VanOverschee and DeMoor (1994b) is very similar. They proposed to use an *oblique projection* along $\mathbf{U}_f^T$ on $\mathbf{P}^T$. Hence their N4SID (numerical algorithms for 4SID) method uses the SVD of the matrix

$$\mathbf{Y}_f\mathbf{\Pi}_{P^T/U_f^T}, \quad (50)$$

where the oblique projection can be expressed as

$$\mathbf{\Pi}_{P^T/U_f^T} = \mathbf{\Pi}_{U_f^T}^{\perp}\mathbf{P}^T(\mathbf{P}\mathbf{\Pi}_{U_f^T}^{\perp}\mathbf{P}^T)^{-1}\mathbf{P}.$$

As noted in VanOverschee and DeMoor (1994a), the different projections are related by

$$\mathbf{\Pi}_{\tilde{Z}^T} = \mathbf{\Pi}_{P^T/U_f^T}\mathbf{\Pi}_{U_f^T}^{\perp}.$$

In the above-mentioned IV interpretation, the PO-MOESP and N4SID methods use identical instruments, but different weighting matrices. The difference is only the extra projection $\mathbf{\Pi}_{U_f^T}^{\perp}$ in (49). The resulting subspace estimates should therefore have very similar properties. However, the scheme proposed in VanOverschee and DeMoor (1994b) for unraveling the systemmatrices is quite different. It is based upon reconstructing the state variables and generating an overdetermined system of equations, which delivers A, and C directly and B and D from a relation similar to (27). An interesting property of their method is that it also delivers estimates of the noise covariance matrices. The property that all relevant quantities are obtained in 'one shot' is quite attractive, at least from a computational point of view. However, it has been observed in simulation studies that the pole estimates obtained by applying the usual shift-invariance technique to the estimated observability matrix appears to yield the same accuracy (up to first order) as the N4SID pole estimates (see Viberg *et al.*, 1993).

6. A RELATED SIGNAL-PROCESSING PROBLEM

As an inspiration for further advancing the 4SID methods and their relation to more traditional techniques, we direct readers' attention to the area of sensor array signal processing. In this field, subspace methods have been successfully used since Pisarenko, (1973) and Schmidt (1979), and the relation to ML estimation is well established (see Stoica and Nehorai, 1989; Stoica and Sharman, 1990; Ottersten *et al.*, 1992). However, the array problem is considerably simpler than the one considered here, at least from a theoretical point of view. Consider an array of m sensors, receiving narrowband planar wavefronts from d far-field emitters. The complex-valued array output $\mathbf{y}(t)$ (which consists of quadrature-sampled induced voltages in the sensor elements) is modeled by the relation

$$\mathbf{y}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{x}(t) + \mathbf{e}(t). \quad (51)$$

Here the d-vector $\mathbf{x}(t)$ contains the transmitted signals, $\mathbf{e}(t)$ is the measurement noise and

$$\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1) \quad \ldots \quad \mathbf{a}(\theta_d)] \quad (52)$$

is the so-called *array propagation matrix*. The columns of $\mathbf{A}(\boldsymbol{\theta})$ are the *array propagation vectors*. The kth element of $\mathbf{a}(\theta_l)$ models the gain and phase of a narrowband signal at the kth sensor, relative to some reference point, assuming the signal arrives from the DOA (direction-of-arrival) θ_l. Given a batch of data, $\mathbf{y}(1), \ldots, \mathbf{y}(N)$, the problems of interest in sensor array signal processing include

(i) determination of d, the number of signals;

(ii) estimation of the DOAs $\boldsymbol{\theta} = [\theta_1 \ \ldots \ \theta_d]$;

(iii) reconstruction of the signal waveforms $\mathbf{x}(t)$.

We observe that problem 1 corresponds to the model order determination problem, problem 2 is related to system modeling and problem 3 is essentially Kalman filtering.

Let us now compare the data model (51) with the 'static' relation (15). Apparently, $\mathbf{A}(\boldsymbol{\theta})$ corresponds to the extended observability matrix and $\mathbf{x}(t)$ is the state variable. However, (51) has

no term corresponding to the input. This is a blessing in that the computation of the signal subspace estimate is greatly simplified. However, it is also a curse because problems 1–3 cannot be solved by any algorithm without additional assumptions. For Gaussian data, all information is conveyed by the array convariance matrix

$$\mathbf{R}_{yy} = \mathrm{E}[\mathbf{y}(t)\mathbf{y}^*(t)] = \mathbf{A}(\boldsymbol{\theta})\mathbf{R}_{xx}\mathbf{A}^*(\boldsymbol{\theta}) + \mathbf{R}_{ee}. \quad (53)$$

Clearly, if $\mathbf{R}_{ee}$ were not restricted in any way, we would get a perfect data fit by taking $d = 0$ and $\hat{\mathbf{R}}_{yy} = \mathbf{R}_{ee}$, where

$$\hat{\mathbf{R}}_{yy} = \frac{1}{N}\sum_{t=1}^{N} \mathbf{y}(t)\mathbf{y}^*(t) \quad (54)$$

is the sample covariance matrix. The usual assumption for allowing an identifiable parameterization of the data is to model $\mathbf{e}(t)$ as being spatially white, i.e.

$$\mathbf{R}_{ee} = \sigma^2\mathbf{I}. \quad (55)$$

This assumption is reasonable—at least in situations where the array output can be measured in the absence of signals. The noise covariance matrix can then be estimated, and the noise in (51) can be prewhitened through multiplication by $\mathbf{R}_{ee}^{-1/2}$.

If (55) holds and in addition $\mathbf{R}_{xx}$ has full rank then the signal subspace, i.e. the range space of $\mathbf{A}(\boldsymbol{\theta})$, can be recovered from the d principal eigenvectors of $\mathbf{R}_{yy}$. Partition the eigendecomposition as

$$\mathbf{R}_{yy} = \mathbf{E}_s\boldsymbol{\Lambda}_s\mathbf{E}_s^* + \mathbf{E}_n\boldsymbol{\Lambda}_n\mathbf{E}_n^*, \quad (56)$$

where the $m \times d$ matrix $\mathbf{E}_s$ contains the principal eigenvectors. Then it is easy to show that

$$\boldsymbol{\Lambda}_n = \sigma^2\mathbf{I}, \quad (57)$$

$$\mathbf{E}_n \perp \mathbf{A}(\boldsymbol{\theta}), \quad (58)$$

$$\mathrm{span}\,(\mathbf{E}_s) = \mathrm{span}\,(\mathbf{A}(\boldsymbol{\theta})). \quad (59)$$

The relation (57) is useful for determining d, simply by performing a statistical test of the multiplicity of the smallest eigenvalue (Anderson, 1984; Wax and Kailath, 1985). Either of the relations (58) or (59) can be used for estimating the DOAs. The so-called weighted subspace fitting (WSF) method (Stoica and Sharman, 1990; Viberg and Ottersten, 1991) exploits (59) as follows; since (59) implies that $\mathbf{E}_s = \mathbf{A}(\boldsymbol{\theta})\mathbf{T}$ for some $\mathbf{T}$, consistent DOA estimates are obtained by solving

$$\{\hat{\boldsymbol{\theta}}, \hat{\mathbf{T}}\} = \arg\min_{\boldsymbol{\theta},\mathbf{T}} \|\hat{\mathbf{E}}_s\mathbf{W}^{1/2} - \mathbf{A}(\boldsymbol{\theta})\mathbf{T}\|_{\mathrm{F}}^2, \quad (60)$$

where $\hat{\mathbf{E}}_s$ contains the d principal eigenvectors of the sample covariance matrix and $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. The weighting matrix $\mathbf{W}$ is chosen as (a consistent estimate of)

$$\mathbf{W} = (\boldsymbol{\Lambda}_s - \sigma^2\mathbf{I})^2\boldsymbol{\Lambda}_s^{-1}. \quad (61)$$

This choice is dictated by the statistical variability of the estimated eigenvectors in $\hat{\mathbf{E}}_s$. It is shown in Stoica and Sharman (1990) and Ottersten *et al.* (1992) that the asymptotic covariance matrix of the WSF estimates coincide with the Cramér–Rao lower bound. Consequently, the subspace-based method is a large sample realization of the ML method for the problem at hand. Although the computation of the estimates (60) requires a d-dimensional nonlinear optimization (note that the problem is separable in the linear parameter $\mathbf{T}$), the form of the criterion function is considerably simpler than that of the ML method. If an efficient technique for computing the eigendecomposition is employed (Xu and Kailath, 1994), the computational requirements may be significantly less when using the WSF method. Indeed, for a linear array of identical and equispaced sensors, the search procedure can be reduced to solving two least-squares problems (see Stoica and Sharman, 1990).

7. CURRENT TRENDS

The area of subspace-based system identification is still rather immature, and a number of interesting questions remain unsolved. One of the most challenging issues is the estimation accuracy. This topic can be split into two parts: one concerns the statistical performance, whereas the other is the ability of approximating general dynamics with finite-dimensional realizations. Of particular interest is the relation of the 4SID methods to more traditional identification techniques, such as IVMs and PEMs. The PEM is optimal in the sense of providing minimum variance estimates, and its approximation performance (i.e. the bias distribution) is well known (Wahlberg and Ljung, 1986; Ljung, 1987). In some sense, the connection of 4SID methods to IVMs is established through the observations presented in the previous section. While the statistical relation of IVMs to the optimal PEM is well known, less is known about its asymptotic bias in the case of undermodeling.

Initial statistical results are presented in Viberg *et al.* (1991, 1993). Variance expressions for the pole estimates of the methods of DeMoor *et al.* (1988) and VanOverschee and DeMoor (1994b) are derived. However, the resulting expressions are quite involved, and offer little insight on how different user's choices influence

the performance. No attempt is made to derive an optimal method, like (60). Since subspace-based methods involve two steps, one could consider optimizing these steps separately. The first is the estimation of the extended observability matrix. In light of the observations in Section 5, the subspace estimate can be generalized by introducing row and column weighting matrices. We thus define the weighted IV-4SID estimate as

$$\mathbf{W}_r\mathbf{Y}_f\mathbf{\Pi}_{U_f^T}^{\perp}\mathbf{P}^T\mathbf{W}_c = \hat{\mathbf{Q}}_s\hat{\mathbf{S}}_s\hat{\mathbf{V}}_s^T + \hat{\mathbf{Q}}_n\hat{\mathbf{S}}_n\hat{\mathbf{V}}_n^T \quad (62)$$

$$\hat{\mathbf{\Gamma}}_\gamma = \mathbf{W}_r^{-1}\hat{\mathbf{Q}}_s. \quad (63)$$

See also VanOverschee and DeMoor (1994a), where a slightly different definition of the column weighting is used. From the previous discussion, we have the following result.

Theorem 2. Assume that the matrix

$$\lim_{N\to\infty}\frac{1}{N}\mathbf{X}_f\mathbf{\Pi}_{U_f^T}^{\perp}\mathbf{P}^T \quad (64)$$

exists and has full rank n. Then any choice of positive-definite weighting matrices $\mathbf{W}_r$ and $\mathbf{W}_c$ in (62) and (63) results in a consistent estimate of $\mathbf{\Gamma}_\gamma$.

As in previous schemes, the system matrices A and C can be consistently estimated from $\hat{\mathbf{\Gamma}}_\gamma$, whereas the matrices A and C can be consistently estimated from $\hat{\mathbf{\Gamma}}_\gamma$, whereas the matrices B and D can be estimated similarly to (27). An alternative is to apply the method proposed in VanOverschee and Demoor (1994b), which would yield the whole stochastic representation (30), (31).

More explicit requirements on the input sequence for guaranteeing (64) to hold appear not to be available, although one could loosely state that the relation is 'generically true'. An obvious requirement is that the number of rows in $\mathbf{P}$ be at least n, which yields $\beta \geq n/(m+l)$. It is easy to see that, for example, a white noise input fulfills (64) under Assumptions 1 and 2. Recently, the problem of how to choose the weighting matrices in (62) has attracted some attention, although no optimal choice is available. Following the general theory of IVMs (see e.g. Söderström and Stoica, 1989), natural candidates for the weighting matrices are

$$\mathbf{W}_r = (\mathbf{Y}_f\mathbf{\Pi}_{U_f^T}^{\perp}\mathbf{Y}_f^T)^{-1/2}, \quad (65)$$

$$\mathbf{W}_c = (\mathbf{P}\mathbf{P}^T)^{-1/2}. \quad (66)$$

This choice is motivated by the desire to prewhiten the data as well as the instruments. There is also a clear relation to canonical analysis (see Larimore, 1990; VanOverschee and DeMoor, 1994a).

Another possibility for relating 4SID methods to classical techniques is suggested in Jansson and Wahlberg (1994). The idea is essentially to replace the unknown state in (39) by a reconstructed state from an optimal observer, i.e. a Kalman filter. A similar interpretation is given in VanOverschee and DeMoor (1994b). Replacing x_t in (39) by an estimate derived from the known inputs and outputs (but keeping the observer gain a free parameter) results in a linear regression equation. A 'standard' least-squares technique or an IVM can be applied to this equation, leading to a method with more familiar structure than previous subspace methods.

Regarding optimization of the second step, namely estimation of the system matrices, an initial attempt is presented in Ottersten and Viberg (1994). The inspiration is taken from the structure of the WSF criterion (60). However, a vectorized version of the 'regression equation' $\hat{\mathbf{Q}}_s \approx \mathbf{\Gamma}_\gamma\mathbf{T}$ is considered, because of the more complicated nature of the errors in $\hat{\mathbf{Q}}_s$. Clearly, this expansion of the dimensions significantly increases the computational complexity of the resulting algorithm. Furthermore, only estimation of the system poles is considered in Ottersten and Viberg (1994). This brings up a dilemma when analyzing the performance of 4SID methods. One attractive feature of these techniques is that no canonical parameterization of the system matrices is employed. However, when comparing the performance of different algorithms, such a parameterization is indeed necessary. The simplest possible canonical parameters are the system poles, and accurate pole estimation is certainly relevant, for instance for control design. However, there are other important aspects not captured by the system poles, and a more complete analysis is of interest. A promising possibility is to consider the accuracy in the frequency domain. For this purpose, a frequency-domain method as proposed in McKelvey *et al.* (1994) is a natural choice. We expect an interesting development along these lines in the near future.

Deriving formulas for the bias distribution and in particular instruments for affecting this distribution is perhaps an even more challenging problem. An interesting interpretation that connects row weighting of the subspace with prefiltering is presented in VanOverschee and DeMoor (1994a). Promising connections to Hankel-norm model reduction are also pointed

out. However, the bias distribution is of course also better described in the frequency domain (Ljung, 1987). Thus also this aspect may be easier to answer for methods that operate directly in this domain.

8. EXAMPLES

In this section the results of some computer simulations are presented to illustrate the performance of the varius 4SID methods, and to compare with PEM's. The fifth-order SISO system depicted in Fig. 1 is used as a benchmark plant. The noise sequences w_t and v_t are zero-mean, white and Gaussian, with variances r_{ww} and r_{vv}. The numerator and denominator polynomials are

$$B(q) = q^{-4}(0.0275 + 0.0551q^{-1}),$$

$$A(q) = 1 - 2.3443q^{-1} + 3.0810q^{-2} - 2.5274q^{-3} + 1.2415q^{-4} - 0.3686q^{-5}.$$

This system has a zero at -2 and poles at $\{0.9, 0.8e^{\pm j}, 0.8e^{\pm 1.2j}\}$. The simulation results are based on 100 independent runs, each using a batch of $N = 1000$ samples of input–output data.

Example 8.1: *White noise input.* In the first example, u_t is chosen as a zero-mean white Gaussian noise with variance $r_{uu} = 1$. The noise variances are $r_{ww} = 0.1$ and $r_{vv} = 0.03$. In Fig. 2 the pole estimates for 100 different input and noise realizations are plotted. The methods considered are basic 4SID, PO-MOESP, weighted IV-4SID with weights according to (65) and (66), and the system identification toolbox routine *armax* (Ljung, 1992), which implements a prediction-error method. The time delay in $B(q)$ is assumed known when employing the PEM method, whereas the only information used by the 4SID techniques is the system order. For the latter techniques, the number of rows in the past and future input–output matrices are chosen as $\gamma = \beta = 10$. The N4SID estimates are not included in the plot, since they are virtually identical to those of the MOESP method in this example. As expected, the basic 4SID estimates show a large bias due to the presence of w_t. The MOESP and weighted IV-4SID estimates perform similarly. Surprisingly, the pole estimates from the PEM method show a larger variability in this example. We also tried initializing the armax routine using the N4SID estimates, but without affecting the result significantly.

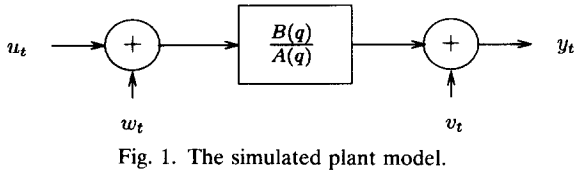

Fig. 1. The simulated plant model.

Example 8.2: *Sinusoidal input.* For comparison, the input is in this example instead choosen as a sum of six sinusoids

$$u_t = A \sum_{i=1}^{6} \sin \omega_i t,$$

where the frequencies ω_i are uniformly spaced in the interval $(0.1, 3)\,\mathrm{rad\,s^{-1}}$, and where the amplitude A is adjusted to yield $r_{uu} = 1$. Note that at least five sinusoids are necessary to enable consistent estimation of a fifth-order system. Since we require (37) to hold, the maximum allowable $\alpha = \gamma + \beta$ is 12. Thus, we choose $\gamma = \beta = 6$ for the 4SID methods. The noise variances are taken as $r_{ww} = r_{vv} = 10^{-4}$. The results for the N4SID, MOESP, weighted IV-4SID and PEM methods are displayed in Fig. 3. As seen in the figure, the 4SID methods perform considerably worse than the optimal PEM technique for this poorly exciting input signal.† Although not shown here, for increasing numbers of sinusoids the 4SID estimates become more accurate, and for 10 sinusoids the difference between 4SID and PEM is again negligible.

9. CONCLUSIONS

The aim of this paper has been to give a guided tour through the somewhat scattered land of subspace-based methods for system identification. In an attempt to organize efforts, the different methods have been grouped into the *realization-based* and the *direct* 4SID methods. The members of the former class explicitly form estimates of the impulse response parameters, whereas the direct methods are based on geometrical properties of the input–output relation (15). A number of similarities between different methods are pointed out. In particular, the basic 4SID methods considered in Section 3.2 are all found to produce identical estimates. Connections between the instrumental-variable-based subspace methods have also been presented, and a suggestion for improving the accuracy outlined. Some trends in the current research efforts have been briefly discussed.

In our opinion, the most promising type of methods for identification of general state-space models with both deterministic and stochastic

† A modification of the N4SID method has been developed, resulting in improved performance for this example—but still far from that of the PEM method (VanOverschee, 1995).

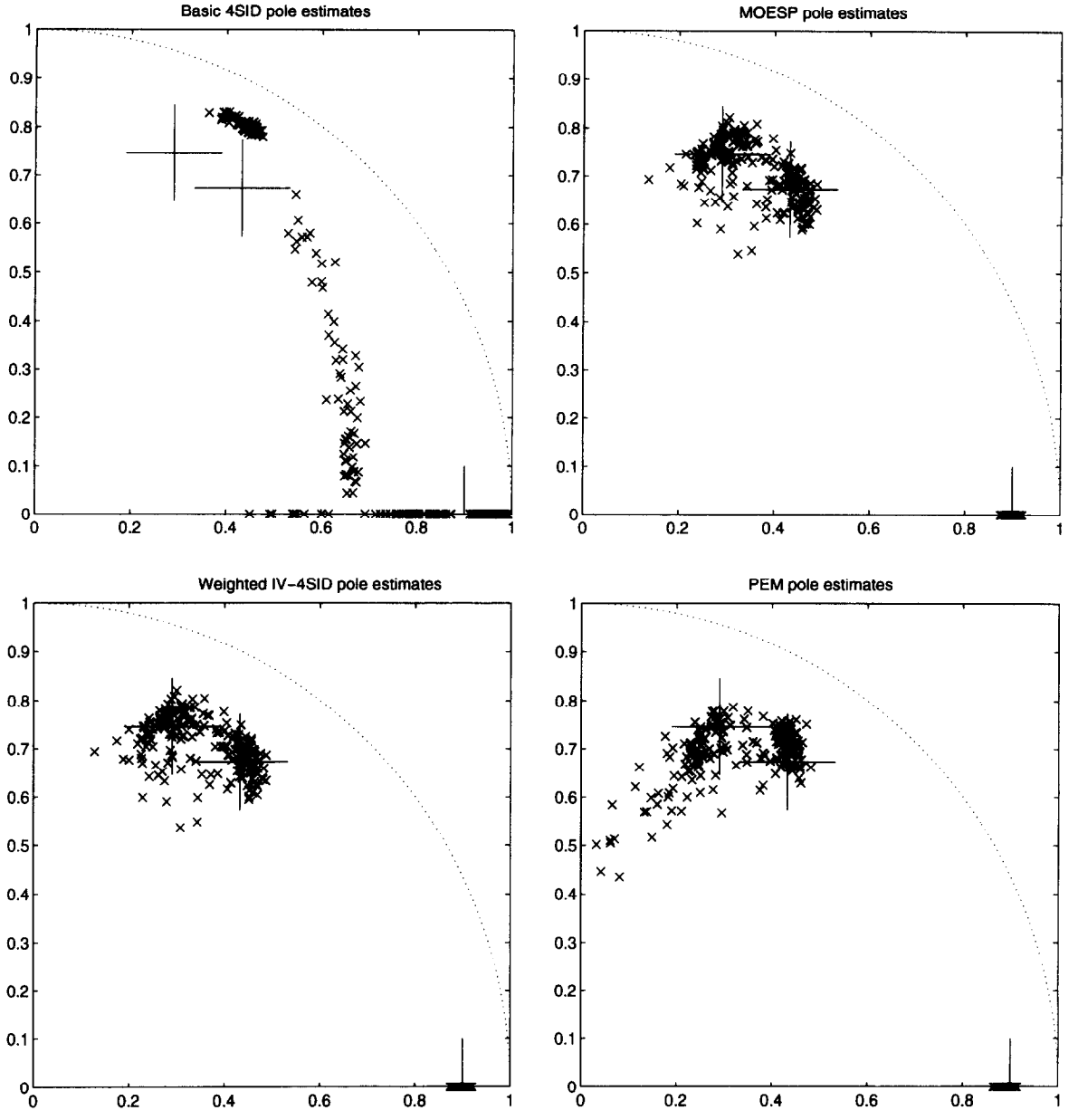

Fig. 2. Scatter plot of pole estimates for various methods, with a white noise input signal. True pole locations are indicated by crosses.

states appears to be the instrumental-variable-based techniques. If an output-error model is desired, a frequency-domain implementation of a realization-based method is a strong candidate. It should be stressed that the field is still quite immature, and further insights that disqualify the opinions presented here can be expected. The popularity of the subspace methods is to a great extent due to their success in a number of real-world applications (see e.g. Liu and Skelton, 1991; Liu *et al.*, 1992; VanOverschee and DeMoor, 1994b; Abrahamsson *et al.*, 1994; Zhu *et al.*, 1994). The underlying theory is not yet fully understood, and much work in terms of analysis and performance optimization remains to be done. In the simulation scenario of Section 8, the IV-based 4SID methods are clearly competitive compared with the 'optimal' PEM method when the input signal is chosen as white noise. However, if the input is more poorly exciting (six sinusoids in Example 2), the performance of the 4SID techniques deteriorates significantly. This observation suggests that the 4SID methods are more sensitive to poor excitation of the system than are the traditional identification methods. Further work for pin-pointing the problem and suggesting remedies is necessary.

The connections to the sensor array signal processing problem have been pointed out. The array problem is considerably simpler than the one considered here. With reference to the

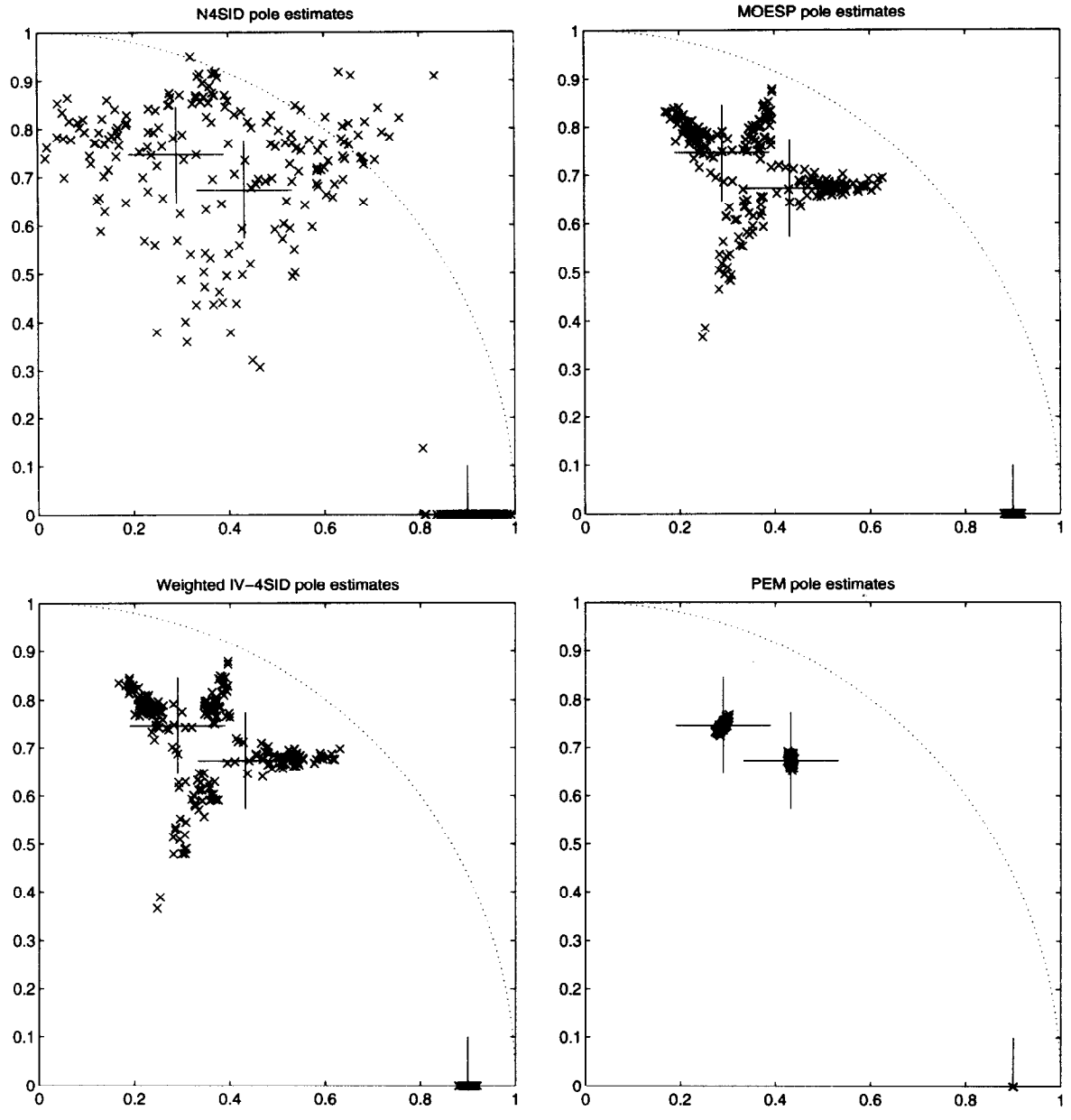

Fig. 3. Scatter plot of pole estimates for various methods. The input signal is a sum of sinusoids. True pole locations are indicated by crosses.

input–output relation (39), the state variable x_t can be assumed temporally white in the array case, the input $\mathbf{u}(t)$ does not appear, and the noise term $\mathbf{\Psi}_\alpha \mathbf{e}(t)$ is temporally and spatially white. For this problem, subspace-based methods providing minimum-variance estimates are known (see Section 6). It remains to be discovered whether or not similar results exist for the more complicated multivariable system identification problem.

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