This course considers \textit{engineering design} as an application of \textit{engineering analysis} in which the design specification (i.e., design plan) is quantified by a set of \textit{design variables} 
\[ \mathbf{x} = [x_1, x_2, x_3, \cdots, x_n]^T \] and through which performance, utility, safety, equity, and sustainability can be quantified and optimized or constrained.

A design that meets performance, utility, safety, equity, and sustainability requirements can be called \textit{functional}. In the approach adopted for this course one of these design requirements is selected to be the primary design objective, and is to be optimized. This is quantified by an \textit{objective function}, which is a function of the design variables.

\[ f(x_1, x_2, \cdots, x_n) \quad \text{or, more compactly,} \quad f(\mathbf{x}) . \]

A common objective is to simply minimize the total cost of the design, possibly including externalities. In other design problems the objective could be to maximize an aspect of performance.

The functionality of a design depends on more than the primary objective (performance, profits, etc.). Most designs must also meet a number of other criteria (e.g., reliable enough, safe enough, stable enough, strong enough, equitable enough, sustainable enough). These criteria also depend upon the values of the design variables, and are expressed as inequalities. In general, a set of \( m \) inequality constraints can be written as

\[
\begin{align*}
g_1(x_1, x_2, x_3, \cdots, x_n) & \leq 0 \\
g_2(x_1, x_2, x_3, \cdots, x_n) & \leq 0 \\
& \vdots \\
g_m(x_1, x_2, x_3, \cdots, x_n) & \leq 0
\end{align*}
\]

by collecting the \( m \) individual inequalities into a single vector inequality. Design constraints confine the design variables to domains (or sub-spaces) of \textit{admissible} alternatives.

In this framework, the admissible and optimal design \( \mathbf{x}^* \) minimizes (or maximizes) the primary design objective while satisfying the design constraints:

\[ \min_{\mathbf{x}} f(\mathbf{x}) \quad \text{such that} \quad g(\mathbf{x}) \leq 0 . \]

In certain rare cases, we can write simple equations for \( f(\mathbf{x}) \) and \( g(\mathbf{x}) \) and use calculus to derive equations for the constrained optimum. In the vast majority of practical problems, however, these equations are much too complicated to be solved with pencil and paper. In such cases computer-aided \textit{analysis} can automate the evaluation of the objective and admissibility of a particular trial design. Further, computer-aided \textit{optimization} allows designers to automatically iterate on candidate designs in order to converge rapidly to an admissible and possibly “optimal” solution. But don’t expect too much from optimization. In very challenging design optimization problems it can be computationally impractical to converge to an admissible or feasible design that is perfectly optimal \textit{and} robust to uncertainties.

Sometimes feasibility suffices. Sometimes just being ok is enough.
1. engineering ethics (5 points):

Engineering Ethics Question 3

2. quadratic programming (30 points):

This question involves matrix mathematics. Re-read “A little matrix math” and sections 2 and 4 of “Quadratic Programming and Lagrange Multipliers” for background on this question.

Consider a quadratic cost function in terms of three variable parameters $x_1, x_2, x_3,$

$$f(x_1, x_2, x_3) = 40x_1^2 + 16x_2^2 + 103x_3^2 - 22x_1x_2 - 52x_1x_3 - 23x_2x_3 + 60x_1 - 111x_2 - 340x_3 + 268$$

(a) Derive three equations for $\partial f/\partial x_1, \partial f/\partial x_2,$ and $\partial f/\partial x_3.$

(b) This cost function may be written in a form

$$f(x) = \frac{1}{2}x^T H x + c^T x + d$$

where $H$ is a $3 \times 3$ Hessian matrix, $c$ is a $3 \times 1$ vector and $d$ is a constant. The Hessian matrix represents the curvature of the objective function.

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j}$$

Find the numerical values for the matrix $H$ and the vector $c$ corresponding to the cost function $f(x)$, above.

(c) Is $H$ always symmetric ($H_{ij} = H_{ji}$), ($H = H^T$)?

(d) Show that in this example (as in all quadratic problems) the vector

$$\begin{bmatrix}
\partial f/\partial x_1 \\
\partial f/\partial x_2 \\
\partial f/\partial x_3
\end{bmatrix}$$

is the same as $Hx + c$.

(e) Compute the values of $x_1^*, x_2^*, x_3^*$ that minimize $f([x_1, x_2, x_3])$. This is an unconstrained minimization. What is the value of the cost function $f(x^*)$? You may use MATLAB to solve this linear system of three equations and three unknowns.

(f) Suppose that in addition to minimizing $f$, the parameter values must also satisfy two equality constraints:

$$g_1(x_1, x_2, x_3) = 6x_1 - 4x_2 + 2x_3 - 8 = 0$$

and

$$g_2(x_1, x_2, x_3) = -7x_1 + 5x_2 - 3x_3 + 9 = 0$$

Find the numerical values of the matrix $A$ ($2 \times 3$) and the vector $b$ ($2 \times 1$) representing these two equality constraints as $Ax - b = 0$.

Show that $A_{ij} = \partial g_i/\partial x_j$. 
(g) Using a column-vector of Lagrange multipliers, $\lambda$, and the augmented cost function,

$$f_\lambda(x, \lambda) = f(x) + \lambda^T g(x) = \frac{1}{2} x^T H x + c^T x + d + \lambda^T (A x - b) \quad (2)$$

show that the two necessary conditions for optimality

$$\left. \frac{\partial f_\lambda}{\partial x} \right|_{x = x^*, \lambda = \lambda^*} = 0^T \quad \text{and} \quad \left. \frac{\partial f_\lambda}{\partial \lambda} \right|_{x = x^*, \lambda = \lambda^*} = 0^T$$

may be expressed in the following KKT matrix equation,

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}.$$ 

Note that since $f_\lambda$ is a scalar

$$f_\lambda = f_\lambda^T = \frac{1}{2} x^T H x + x^T c + d + (x^T A^T - b^T) \lambda$$

(h) Using the numerical values for $H$, $c$, $A$, and $b$, compute values for the optimal parameter vector $x^*$ and the Lagrange multipliers $\lambda^*$. You may use MATLAB to solve this linear system of five equations and five unknowns. Confirm that $A x^* - b = 0$. Are both Lagrange multipliers positive? What is the value of the cost function $f(x^*)$ for which $A x^* - b = 0$?

(i) Change $b_2$ by 1 and repeat part (h) above. What is the new value of the cost function, $f(x^*)$? Compare the difference between $f(x^*)$ in part (h) and $f(x^*)$ in part (i) to the average of $\lambda^*_2$ in parts (h) and (i). Did you find that $f^*_2(h) - f^*_2(i) \approx (\lambda^*_2(h) + \lambda^*_2(i))/2$?

(j) Now resetting $b_2$ to the value from part (h), change $b_1$ by 1 and repeat part (i) above. Did you find that $f^*_1(h) - f^*_1(i) \approx (\lambda^*_1(h) + \lambda^*_1(i))/2$?

(k) Are you getting convinced that the values of the Lagrange multipliers indicate the sensitivity of the cost function to changes in the constraints?
3. fit a constrained function to data (20 points):

Many constrained optimization problems can be solved in one step by using the KKT equations instead of using numerical optimization. Curve fitting is one such example.

(a) Re-read “Constrained Linear Least Squares”
(b) Download constrained_least_squares.m and confirm that it runs without error.
(c) Revise constrained_least_squares.m in order to fit the equation

\[ \hat{y}(t; a) = a_1 + a_2/(t + 1) + a_3 \cos(t/\pi) + a_4 \sin(\pi t) + a_5 \tanh((t/\pi)) \]

to a set of simulated noisy data over the domain \(0 \leq t \leq 10\) such that \(\hat{y}(0; a) = 0\) and \(\hat{y}'(10; a) = -15.4\)

To generate the measured data, use “true” coefficients
\((a_1 = 1, a_2 = -6, a_3 = 5, a_4 = -4, a_5 = 3)\).

To solve this problem, you need only edit the eight lines marked with \% *

(d) Include the numerical results and a plot of figure 1 generated by the program
(e) Run constrained_least_squares.m a few times.

i. Which coefficient for the unconstrained fit seems most variable?
   How variable is it?

ii. Which coefficient for the constrained fit seems most variable?
   How variable is it?
4. minimize the total potential energy ($\Pi$) subject to constraints (30 points):

Two discs of mass $m_1$ and $m_2$ can roll along ramps with slopes $a_1 < 0$ and $a_2 > 0$ and are connected by a spring with spring constant $k$. The position coordinates of the disk centers are $(x_1, y_1)$ and $(x_2, y_2)$. (For the moment, disregard the existence of the ramps. The ramps will come into play as constraints on the heights $y_1$ and $y_2$ of the two disks.)

The total potential energy of the system is the elastic potential energy in the spring $\frac{1}{2}kd^2$ plus the gravitational potential energy of the disks $m_1gy_1 + m_2gy_2$. Assuming the unstretched length of the spring is zero, the stretch of the spring is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

(a) Derive an expression for the total potential energy $\Pi$ of the system, which is the sum of the elastic potential energy and the gravitational potential energy. The equilibrium configuration of this system minimizes the total potential energy subject to the constraints that $y_1 \geq a_1 x_1$ and $y_2 \geq a_2 x_2$.

(b) Presuming that it is logical that both constraints bind the positions of the masses, write the augmented total potential energy function, $\Pi_A$, which in this case is the total potential energy plus the constraints individually multiplied by Lagrange multipliers $\lambda_1$ and $\lambda_2$.

(c) Defining the coordinate vector $q$ of this system as

$$ q = [x_1, \ y_1, \ x_2, \ y_2]^T $$

derive the six KKT equations ...

$$ \frac{\partial \Pi_A}{\partial q} \bigg|_{q=q^*} = \mathbf{0}^T \quad \text{and} \quad \frac{\partial \Pi_A}{\partial \lambda} \bigg|_{\lambda=\lambda^*} = \mathbf{0}^T $$

(d) Write the KKT matrix equation for this problem

(e) Using numerical values of $m_1 = 3$ kg, $m_2 = 2$ kg, $a_1 = -3$ m/m, $a_2 = 4$ m/m, $k = 20$ N/m and $g = 9.81$ m/s$^2$, solve the system of six equations and six unknowns for (optimal) equilibrium values of $x_1^*$, $y_1^*$, $x_2^*$, $y_2^*$, $\lambda_1^*$ and $\lambda_2^*$.

(f) What is the force in the spring?

(g) Does the solution satisfy the constraints $y_1 \geq a_1 x_1$, $y_2 \geq a_2 x_2$? Are both Lagrange multipliers positive? Are both constraints binding? The constraint forces are $A^T \lambda$. Do these forces act perpendicularly to the ramps?

This method finds the static equilibrium forces and the coordinate displacements of elastic systems that are statically determinate or statically indeterminate without the need to derive equilibrium equations (like $\sum F_x = 0$ or $\sum F_y = 0$). ... cool? ... you’re welcome.