1. (30 points)
Apply the method explained in the Example of an Optimization-Oriented Analysis for Truss Design and Safety Analysis to create ... an Optimization-Oriented analysis of the 2D truss shown below.

Joints 1 and 7 are located at coordinates (0,0) and (20,0) meters. Node 1 is pinned in both the X and Y directions. Node 7 is on a “roller support” ... it can move freely horizontally and is bounded from moving in the vertical direction by a reaction force. Node 3 is is loaded in the downward vertical direction with a force $P_3$ and node 3 is is loaded in the downward vertical direction with a force $P_5$. The ten design variables in this problem are the $x$ and $y$ locations of nodes 2, 3, 4, 5, and 6: $(x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6)$.

(a) Complete steps 7 and 8 of section 1 of the example problem
(b) Complete section 2 of the example problem
(c) Complete section 3 of the example problem

2. (20 points)
Apply the method explained in class to determine the most economical balance of electrical power distribution in the power grid shown below. Where the power generating stations are $A$ and $B$ and nodes 1 through 10 are power stations with specified demands, $D_1, ..., D_{10}$. Assume that the power generating stations have no demand of their own.
The power generating stations can produce only up to a maximum of $G_A$ and $G_B$ MW of power. ($G_A$ and $G_B$ are positive values.)

The power arriving at a station must be no less than the demand of the station.

Adopt the convenient convention of nomenclature that power that flows from station $i$ to station $j$ is called $P_{ij}$, and assume that $P_{ij}$ is positive if power flows from $i$ to $j$.

The set of 21 power flows within the branches of the network are the design variables. In your solution, put the variables in this order ...

\[
x = [ P_{A1}, P_{A2}, P_{A9}, P_{A10}, P_{B4}, P_{B5}, P_{B6}, P_{B7},
    P_{1,2}, P_{1,3}, P_{1,10}, P_{2,3}, P_{3,4}, P_{3,5}, P_{5,6},
    P_{6,7}, P_{6,8}, P_{7,8}, P_{8,9}, P_{8,10}, P_{9,10} ]
\]

Power generating station $A$ produces power at $c_A$ $$/MW and that power generating station $B$ produces power at $c_B$ $$/MW. The cost of power generation is

\[f(x) = c^T x\]

(a) Write the 21-element vector $c$ in terms of $c_A$ and $c_B$.

(b) Write two inequalities that are satisfied if the power generating stations stay within their power generating capacities.

(c) Write ten inequalities that are satisfied if the power arriving at the power consuming stations generate no less than the power demanded by the stations.
(d) Express these twelve inequalities in a matrix form ...

\[Ax \leq b\]

by providing the 12-by-21 matrix \(A\) and the 12-element vector \(b\). (The matrix \(A\) is populated with values of 0, 1, or -1. and the vector \(b\) is populated with generation capacities and the demand quantities.

3. (35 points) Consider the water supply system described in the figure below.

Water enters the system through precipitation \(Q_P\) over a watershed and leaves the system through a river \(Q_R\), transpiration \(Q_T\), evaporation \(Q_E\), and the consumption in two communities \(Q_{C1}\) and \(Q_{C2}\). The units of these flows is million gallons per day (Mg/d).

As water flows through the system it is stored in surface water \(V_s\), ground water \(V_g\), reservoir water \(V_r\), and treated water for consumption for the two communities, \(V_1\) and \(V_2\). Water stored is in units of millions of gallons (Mg).

Water flows from storage volume \(i\) to storage volume \(j\) with flow \(Q_{i,j}\). For example, the flow of surface water into the reservoir is \(Q_{s,r}\).
Using a simplifying assumption that the flows are proportional to the connected volumes.

\begin{align*}
Q_T &= \alpha_{s,T} V_s \\
Q_E &= \alpha_{r,E} V_r \\
Q_{s,g} &= \alpha_{s,g} (V_s - V_g) \\
Q_{s,r} &= \alpha_{s,r} V_s \\
Q_{g,r} &= \alpha_{g,r} V_g \\
Q_{r,1} &= \alpha_{r,1} (V_r - V_1) \\
Q_{r,2} &= \alpha_{r,2} (V_r - V_2)
\end{align*}

The coefficients $\alpha_{i,j}$ are positive constants. The external (environmental) factors that drive the system are precipitation, river flow, and consumption.

The key modeling assumption here is that the rate of change of stored water is the difference between the inflow and the outflow. For example,

\[
\frac{dV_s}{dt} = Q_P - Q_T - Q_{s,r} - Q_{s,g}
\]

Carrying out this analysis for the other four tanks, the system of equations may be written in matrix form

\[
\frac{d}{dt} \begin{bmatrix} V_s \\ V_g \\ V_r \\ V_{r,1} \\ V_{r,2} \end{bmatrix} = A \begin{bmatrix} V_s \\ V_g \\ V_r \\ V_{r,1} \\ V_{r,2} \end{bmatrix} + B \begin{bmatrix} Q_P \\ Q_R \\ Q_{c1} \\ Q_{c2} \end{bmatrix}
\]

where $A$ is a 5-by-5 matrix consisting of the $\alpha_{i,j}$ coefficients and $B$ is a 5-by-4 matrix consisting of values of 0, 1, or -1.

The state of the system at any point in time is represented by the values in $x$, the stored volumes. All the flows can be determined from the state $x$ and the inputs $u$.

(a) derive equations for $\frac{d}{dt} V_g$, $\frac{d}{dt} V_r$, $\frac{d}{dt} V_{r,1}$, and $\frac{d}{dt} V_{r,2}$, in terms of the $\alpha$ coefficients, the stored volumes $x$ and the environmental flows $u$.

(b) from these four equations and the equation for $\frac{d}{dt} V_s$, above, determine the matrices $A$ and $B$ in terms of the $\alpha$ coefficients (for $A$) and the values of 0, 1, and -1 (for $B$).