Design of a two-bar truss: optimization and uncertainty propagation

CEE 201L. Uncertainty, Design, and Optimization
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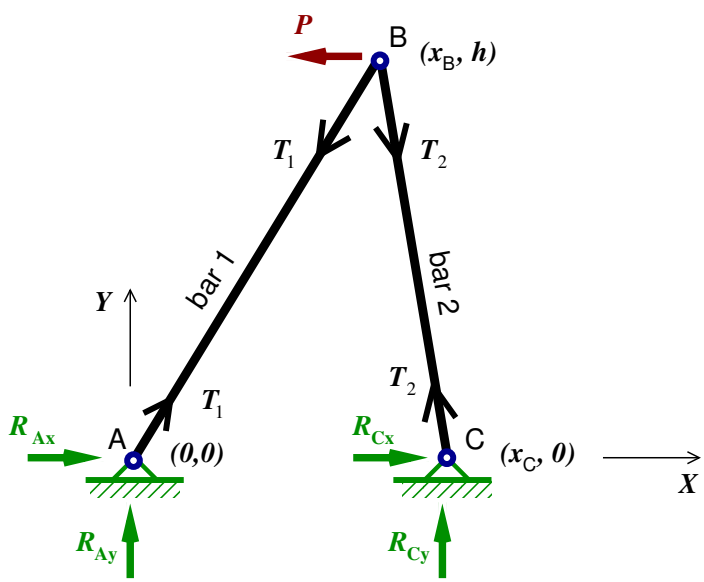
Consider a statically-determinate two-bar truss loaded by a horizontal load P. The lengths of bars 1 and 2 depend on the locations of joints A, B, and C.

$$
\begin{aligned}
L_1 &= \sqrt{X_1^2 + Y_1^2} = \sqrt{(x_\mathsf{B} - x_\mathsf{A})^2 + (y_\mathsf{B} - y_\mathsf{A})^2} = \sqrt{x_\mathsf{B}^2 + h^2} \\
L_2 &= \sqrt{X_2^2 + Y_2^2} = \sqrt{(x_\mathsf{C} - x_\mathsf{B})^2 + (y_\mathsf{B} - y_\mathsf{C})^2} = \sqrt{(x_\mathsf{C} - x_\mathsf{B})^2 + h^2}
\end{aligned}
$$

where X_i and Y_i are the positive-valued x-axis and y-axis projections of the inclined bars. The positive-valued cosine and sine of the angle θ between the bar and the horizontal for each bar i, are $\cos\theta_i = c_i = X_i/L_i$ and $\sin\theta_i = s_i = Y_i/L_i$.



Design Analysis

In truss analysis, there are $2J$ independent equilibrium equations and $B + R$ unknowns, where J is the number of joints, B is the number of bars, and R is the number of external reaction force components. If $2J = B + R$, the truss is statically determinate. (The number of equilibrium equations equals the number of unknowns.)

The equilibrium equations at joint A are

$$\begin{aligned}\sum^{+\rightarrow} F_x = 0: \quad c_1 T_1 + R_{\mathsf{A}x} &= 0 \ , \\ \sum^{+\uparrow} F_y = 0: \quad s_1 T_1 + R_{\mathsf{A}y} &= 0 \ ,\end{aligned}$$

The equilibrium equations at the loaded joint (joint B) are

$$\begin{aligned}\sum^{+\rightarrow} F_x = 0: \quad -c_1 T_1 + c_2 T_2 &= P \ , \\ \sum^{+\uparrow} F_y = 0: \quad -s_1 T_1 - s_2 T_2 &= 0 \ .\end{aligned}$$

The equilibrium equations at joint C are

$$\begin{aligned}\sum^{+\rightarrow} F_x = 0: \quad -c_2 T_2 + R_{\mathsf{C}x} &= 0 \ , \\ \sum^{+\uparrow} F_y = 0: \quad s_2 T_2 + R_{\mathsf{C}y} &= 0 \ .\end{aligned}$$

The bar tension and reaction forces may be found by solving the matrix equation

$$\begin{bmatrix} c_1 & 0 & 1 & 0 & 0 & 0 \\ s_1 & 0 & 0 & 1 & 0 & 0 \\ -c_1 & c_2 & 0 & 0 & 0 & 0 \\ -s_1 & -s_2 & 0 & 0 & 0 & 0 \\ 0 & -c_2 & 0 & 0 & 1 & 0 \\ 0 & s_2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ R_{\mathsf{A}x} \\ R_{\mathsf{A}y} \\ R_{\mathsf{C}x} \\ R_{\mathsf{C}y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that the first B columns of the matrix sum to zero. This is true for any truss analyzed in this way and is simply a statement that each bar of the truss is in equilibrium with itself.

The reaction forces may also be found from equilibrium equations for the entire truss, and can be used as a check on the solution.

$$\begin{aligned} R_{\mathsf{A}x} + R_{\mathsf{C}x} &= P \\ R_{\mathsf{A}y} + R_{\mathsf{C}y} &= 0 \\ R_{\mathsf{A}y}\, x_{\mathsf{C}} &= Ph \\ R_{\mathsf{C}y}\, x_{\mathsf{C}} &= -Ph \end{aligned}$$

These reaction force equations are simply a linear combination of two or more of the other equations (as long as no errors were made in deriving the three pairs of joint equilibrium equations).

Design Goal

Given fixed values for the material properties (E, S_{y}), the load P, the height h, the bar cross section (A_1, I_1, A_2, I_2), determine the values of parameters x_{B} and x_{C} that minimize the total weight of the truss, such that yielding and buckling are prevented, with some degree of safety. In other words,

$$\min J = f(x_{\mathsf{B}}, x_{\mathsf{C}}) = A_1 L_1 + A_2 L_2 = A_1\sqrt{x_{\mathsf{B}}^2 + h^2} + A_2\sqrt{(x_{\mathsf{C}} - x_{\mathsf{B}})^2 + h^2}$$

subject to the safety conditions

$$\begin{aligned}
g_1(x_{\mathsf{B}}, x_{\mathsf{C}}) &= \quad (|T_1(x_{\mathsf{B}}, x_{\mathsf{C}})|\ \phi_L)\,/\,(A_1 S_{\mathrm{y}}\ \phi_R) - 1 \quad &\leq 0 \quad &\text{(bar 1 yields)}\\
g_2(x_{\mathsf{B}}, x_{\mathsf{C}}) &= \quad (|T_2(x_{\mathsf{B}}, x_{\mathsf{C}})|\ \phi_L)\,/\,(A_2 S_{\mathrm{y}}\ \phi_R) - 1 \quad &\leq 0 \quad &\text{(bar 2 yields)}\\
g_3(x_{\mathsf{B}}, x_{\mathsf{C}}) &= \quad (-T_1(x_{\mathsf{B}}, x_{\mathsf{C}})\ \phi_L)\,/\,\left(\tfrac{\pi^2 E I_1}{L_1^2}\ \phi_R\right) - 1 \quad &\leq 0 \quad &\text{(bar 1 buckles)}\\
g_4(x_{\mathsf{B}}, x_{\mathsf{C}}) &= \quad (-T_2(x_{\mathsf{B}}, x_{\mathsf{C}})\ \phi_L)\,/\,\left(\tfrac{\pi^2 E I_2}{L_2^2}\ \phi_R\right) - 1 \quad &\leq 0 \quad &\text{(bar 2 buckles)}
\end{aligned}$$

where ϕ_L is a safety factor on the load effects and ϕ_R is a safety factor on the resistance to the loads ($\phi_L > 1$ and $\phi_R < 1$). Note that the non-dimensionalized form of the constraint equations, $(T\phi_L)/(R\phi_R) - 1 \leq 0$, is equivalent to $(T\phi_L) - (R\phi_R) \leq 0$.

Characteristics of the design problem

To investigate the characteristics of this optimal design problem, consider a case in which both bars are made of mild steel. The elastic modulus, E, is 200×10^6 kN/m^2, and the yield stress, S_{y}, is 250×10^3 kN/m^2. Further, fix the height h to be 5 m, and the diameters of bars 1 and 2 to be 30 mm and is 5 mm, respectively. The safety factors are $\phi_L = 1.2$ and $\phi_R = 0.9$. These safety factors imply that there is a degree of uncertainty in the actual loads that may be applied to the system (the operating environment) and in the ability of the system to withstand a given load (the quality of the materials and workmanship).

The objective function, $f(x_{\mathsf{B}}, x_{\mathsf{C}})$, and constraint equations, $g_j(x_{\mathsf{B}}, x_{\mathsf{C}})$ can be thought of as surfaces over the $x_{\mathsf{B}} - x_{\mathsf{C}}$ plane. Many contours of $f(x_{\mathsf{B}}, x_{\mathsf{C}})$ and the contour line $g_j(x_{\mathsf{B}}, x_{\mathsf{C}}) = 0$ for design loads of 3.5 kN, 4.0 kN, and 4.5 kN are shown in the next three figures. The thin ellipsoidal lines are the contours of the cost function $f(x_{\mathsf{B}}, x_{\mathsf{C}})$ and the thicker lines show the constraints bounding the feasible design variables.

Bar 1 yields at all three load levels for any design with $-0.23 \leq x_{\mathsf{C}} \leq 0.23$ (with joint 1 very near joint 3). The yielding of bar 1 depends on x_{C}, but only very slightly.

Bar 2 buckles at all three load levels for any design with $x_{\mathsf{C}} \leq 0$, (with joint 1 to the left of joint 3). Recall that Bar 2 is much smaller in diameter than bar 1.

Values of $(x_{\mathsf{B}}, x_{\mathsf{C}})$ below the thick green line labeled Bar 2 Yields correspond to yielding of bar 2. Values of $(x_{\mathsf{B}}, x_{\mathsf{C}})$ within the region below the thick red line labeled Bar 1 Buckles and above $x_{\mathsf{C}} = 0$ correspond to buckling of bar 1. To prevent yielding and buckling in both bars, the design values $(x_{\mathsf{B}}, x_{\mathsf{C}})$ must lie above both the red and green curves.

The optimal design $(x_{\mathsf{B}}^*, x_{\mathsf{C}}^*)$ gives the lightest truss that is safe against yielding and buckling.

The safety constraint equations, $g_j(x_\mathsf{B}, x_\mathsf{C})$ can be plotted as a function of x_C for a particular values of x_B. The following figures show zoomed-in plots for the four constraint equations plotted for $x_\mathsf{B} = 2$ m. For all three load values $g_1 > 0$ (not ok) only in a narrow band around $x_\mathsf{C} = 0$, and $g_4 \leq 0$ (ok) for any positive value of x_C.

Optimize for $P = 3.5$ kN.

For a design load of 3.5 kN, the lightest truss has optimal dimensions $(x^*_\mathsf{B}, x^*_\mathsf{C}) = (0.751, 9.970)$ m and uses 3780 cm^3 of steel. This optimal design is constrained by the yielding of bar 2. A small increase in the load P, would cause failure through the yielding of bar 2. Moving the design point in either direction along the green curve "Bar 2 Yields" would increase the weight of the truss.

Optimize for $P = 4.0$ kN.

Increasing the load to 4 kN decreases the feasible region, and requires a heavier truss. For a design load of 4 kN, the lightest truss has optimal dimensions $(x^*_\mathsf{B}, x^*_\mathsf{C}) = (2.193, 11.060)$ m, and uses 4060 cm^3 of steel. This design is constrained by yielding of bar 2 *and* buckling of bar 1.

Optimize for $P = 4.5$ kN.

For loads greater than 4.45 kN, there is no feasible design. Given the limitations on the material properties and the cross-section areas of the bars, the truss will fail regardless of the values of x_B and x_C.

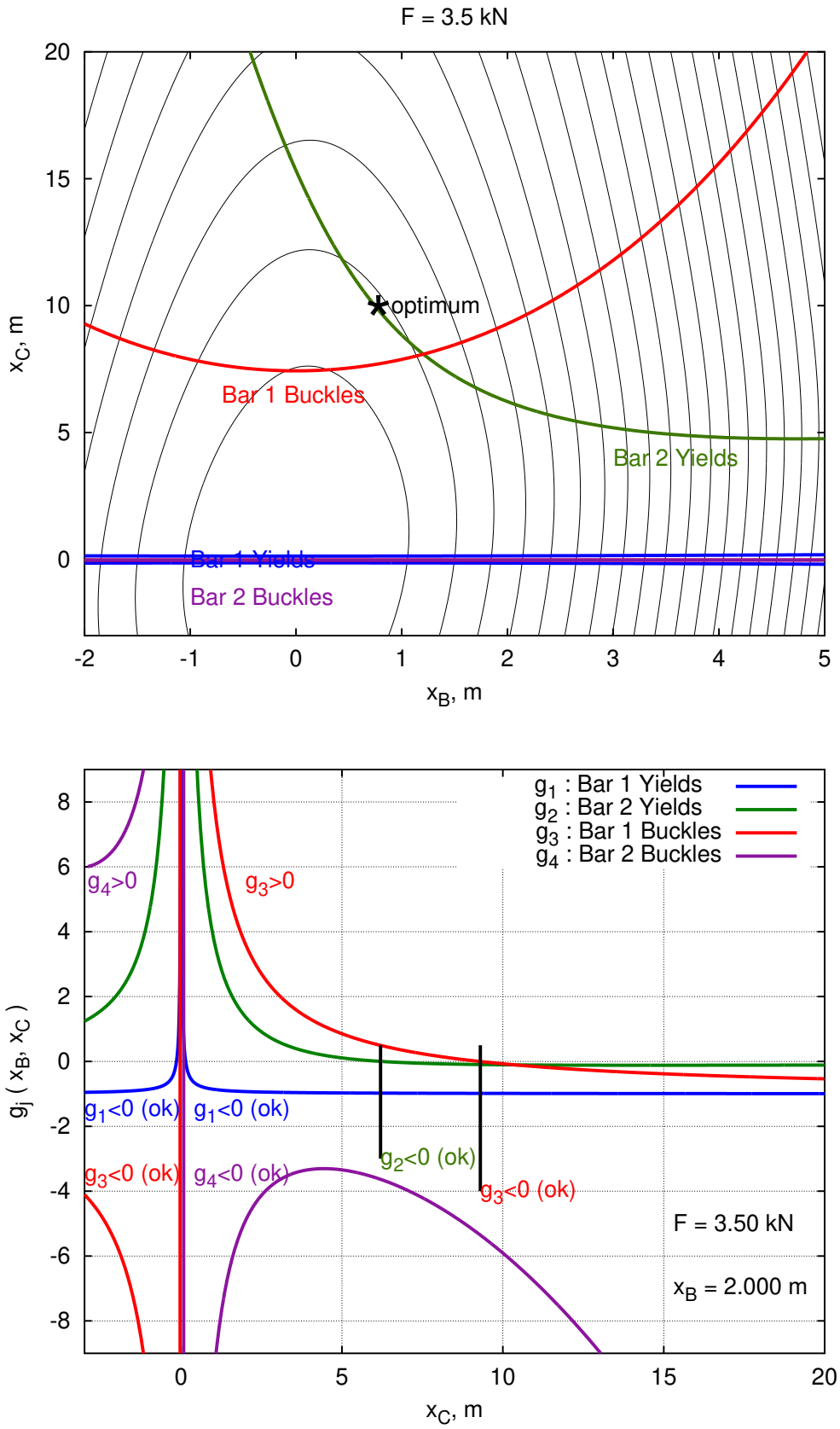


Figure 1. The constrained design optimization for $P = 3.5$ kN. Top: contours of $f(x_\mathrm{B}, x_\mathrm{C})$ and $g_j(x_\mathrm{B}, x_\mathrm{C}) = 0$. Bottom: $g_j(2.0, x_\mathrm{C})$ vs. x_3.

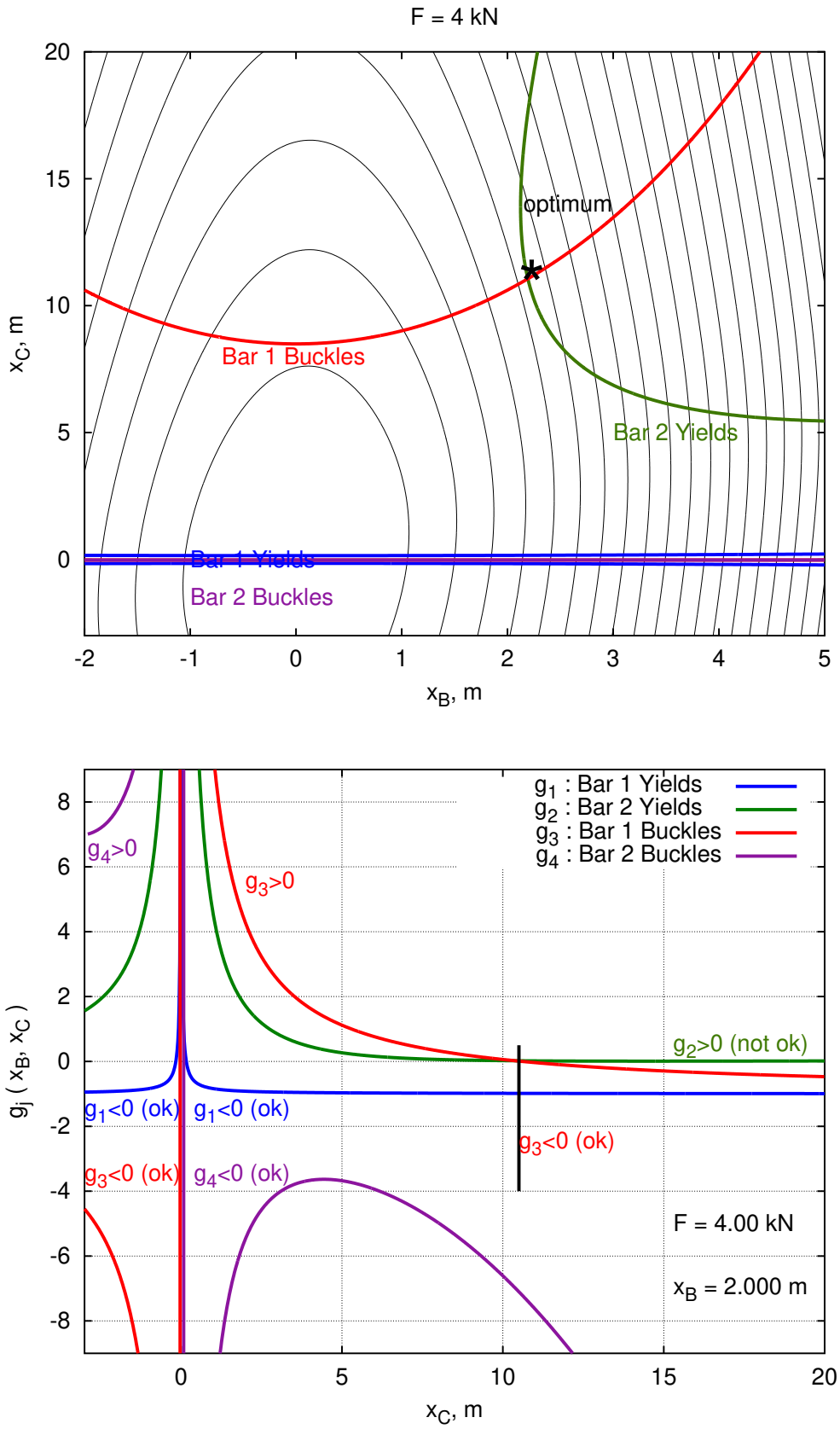


Figure 2. The constrained design optimization for $P = 4.0$ kN. Top: contours of $f(x_\mathrm{B}, x_\mathrm{C})$ and $g_j(x_\mathrm{B}, x_\mathrm{C}) = 0$. Bottom: $g_j(2.0, x_\mathrm{C})$ vs. x_C.

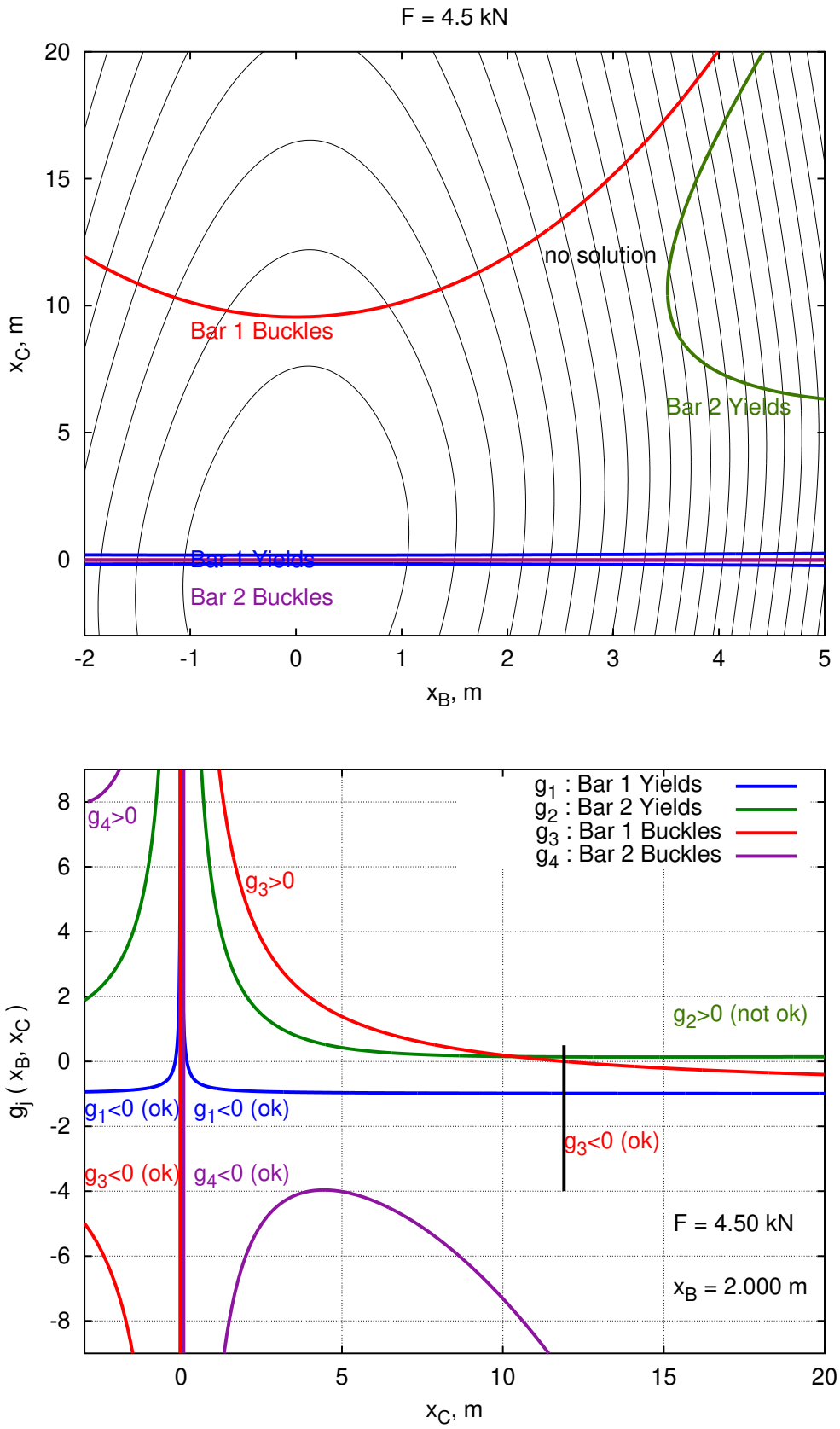


Figure 3. The constrained design optimization for $P = 4.5$ kN. Top: contours of $f(x_B, x_C)$ and $g_j(x_B, x_C) = 0$. Bottom: $g_j(2.0, x_C)$ vs. x_C.

Safety Analysis

The safety constraints used in this design optimization incorporate a safety factor ϕ_L for the truss bar loads and a safety factor ϕ_R for the resistance of the truss bars to these loads. A load factor $\phi_L > 1$ makes the design safer by designing for larger-than-expected loads. A resistance factor $\phi_R < 1$ makes the design safer by designing for weaker-than-expected strength/performance.

In most design situations the designer is uncertain as to the future "as-built" quality and the operating environment of the designed system. The designer may also be uncertain as to the as-built quality of the designed system, when it is eventually produced. If the designer can estimate:

- an expected, average, or *nominal* load level (mean load, μ_L)
- the variability in the loads (standard deviation of the loads, σ_L)
- an expected, average, or *nominal* level of as-built quality (mean of the performance or resistance μ_R), and
- the variability in the as-built quality (standard deviation of the performance or resistance, σ_R)

then the designer can carry out a safety analysis to determine the probability of a product failure. To do so, the safety of the optimized design is re-evaluated hundreds or thousands of times, each time using different randomly-generated values for the loading environment and the strength characteristics. These randomly-generated loads and strengths will conform to the designer's expectation regarding the variability of the load and resistance.

For example, in this two-bar truss problem, a design is optimized using a load value of 3.5 kN and a strength value of 250×10^3 kN/m^2. These values represent the designer's expected (or mean or average) load, μ_L, and strength, μ_R, values. In the design of an adequately-safe system, truss bar forces calculated on the basis of this expected load are multiplied by a factor ϕ_L of 1.2, and truss bar strengths calculated on the basis of this expected yield stress are multiplied by a factor ϕ_R of 0.9.

With an assessment of the variability in the loading environment, σ_L, and the strength, σ_R, the probability of failure of the optimized design $(x^*_{\mathsf{B}}, x^*_{\mathsf{C}})$ can be carried out as follows:

1. Generate random values for the parameters, x_{B} and x_{C}, the load, P, and yield strength, S_{y}. These random values will have mean values and variances corresponding to the designer's expectation of the loading environment, the strength properties of the manufactured product, and the manufacturing precision. The mean design parameters should be the optimized values. The coefficient of variation for the design parameters can be small (less than 5 percent).

2. Analyze the safety of the optimized design with the random values of x_{B}, x_{C}, P and S_{y} using $\phi_L = 1$ and $\phi_R = 1$ in the safety constraints (g_j).

3. Save the values of the random design parameters, the random environmental variables, and the associated safety constraints. ($g_j \leq 0$ is o.k., $g_j > 0$ represents a failure).
4. After repeating steps 1, 2 and 3 N times ($10^2 < N < 10^9$), count the number, n, of analyses resulting in one or more violated constraint.
5. The probability of failure is n/N.
6. Plot probability distributions of the safety constraints, if desired.
7. The correlation between random design parameters, random environmental variables, and safety criteria may be easily computed and scatter-plots may be made, if desired.

In this two-bar truss design problem, a safety analysis may be carried out by assuming that the variability in the loads is twenty percent of the mean load and that the variability in the strength is ten percent of the mean strength,

$$P = \mu_P(1 + 0.2Z) \ , \qquad \text{and} \qquad S_{\rm y} = \mu_{S_{\rm y}}(1 + 0.1Z) \ ,$$

where Z is a normally-distributed random number with a mean of zero and a standard deviation of unity. A different value for Z is used for each realization of each random variable. The safety criteria (g_1, g_2, g_3, g_4) are evaluated for each of N analyses. The probability of failure is the number of evaluations resulting in a failure, n, (buckling of bar 1, yielding of bar 1, *or* yielding of bar 2) divided by the total number of evaluations, N,

$$P_f = n/N.$$

Design Optimization Results

load	P	3.5	4.0	kN
yield stress	$S_{\rm y}$	250×10^3	250×10^3	kN/m^2
load factor	ϕ_L	1.2	1.2	
resistance factor	ϕ_R	0.9	0.9	
optimum $x_{\rm B}$ value	$x^*_{\rm B}$	0.751	2.193	m
optimum $x_{\rm C}$ value	$x^*_{\rm C}$	9.970	11.060	m
amount of material	J	3780	4060	cm^3

Safety Analysis Results

mean load	μ_P	3.5	4.0	kN
c.v. load	c_P	0.2	0.2	.
mean strength	$\mu_{S_{\rm y}}$	250×10^3	250×10^3	kN/m^2
c.v. strength	$c_{S_{\rm y}}$	0.1	0.1	.
mean construction	$\mu_{x_{\rm B}}, \ \mu_{x_{\rm C}}$	0.751, 9.970	2.193, 11.060	m
c.v. construction	$c_{x_{\rm B}}, c_{x_{\rm C}}$	0.01	0.01	.
probability of failure	P_f	8.3%	9.3%	

If an eight or nine percent probability of failure is not acceptable, then a re-design would be carried out using a larger ϕ_L and/or a smaller ϕ_R.

Correlation Analysis Results

	x_B	x_C	P	S_y	g_1	g_2	g_3	g_4
x_B	1.00	0.00	-0.05	-0.01	-0.04	-0.05	-0.03	0.08
x_C	0.00	1.00	-0.01	-0.04	-0.03	0.01	-0.06	-0.07
P	-0.05	-0.01	1.00	0.03	0.89	0.89	1.00	-1.00
S_y	-0.01	-0.04	0.03	1.00	-0.42	-0.42	0.03	-0.02
g_1	-0.04	-0.03	0.89	-0.42	1.00	1.00	0.89	-0.88
g_2	-0.05	0.01	0.89	-0.42	1.00	1.00	0.89	-0.89
g_3	-0.03	-0.06	1.00	0.03	0.89	0.89	1.00	-0.99
g_4	0.08	-0.07	-1.00	-0.02	-0.88	-0.89	-0.99	1.00

- Parameters x_B and x_C are most strongly correlated to g_4 (buckling of bar 2), but this correlation is weak and g_4 is not active in the design, so small random variations in the design parameters do not affect safety.
- Load P is highly correlated to yielding failures, and is perfectly correlated to buckling failures. Increasing P makes both bars more likely to yield, bar 1 more likely to buckle, and bar 2 less likely to buckle.
- Strength S_y is moderately negatively correlated to yielding failures. As the yield strength increases, yielding failures tend to decrease.
- Constraints g_1 and g_2 (yielding of bars 1 and 2) are perfectly positively- correlated.
- Constraints g_3 and g_4 (buckling of bars 1 and 2) are perfectly negatively-correlated.

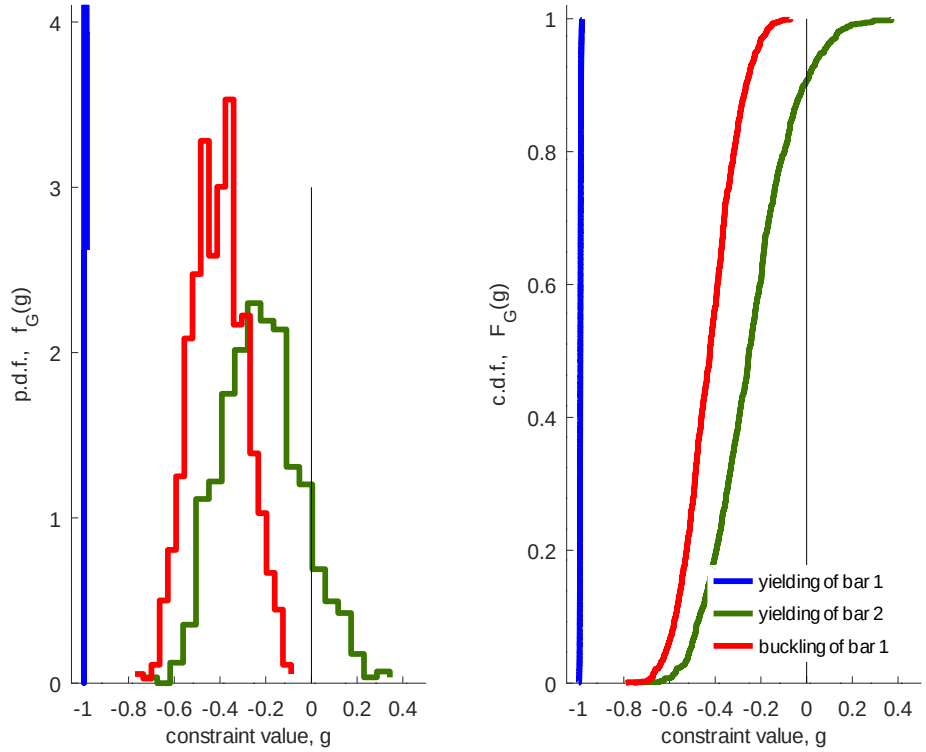


Figure 4. Probability distribution plots of three of the safety criteria for $P = 3.5$ kN.

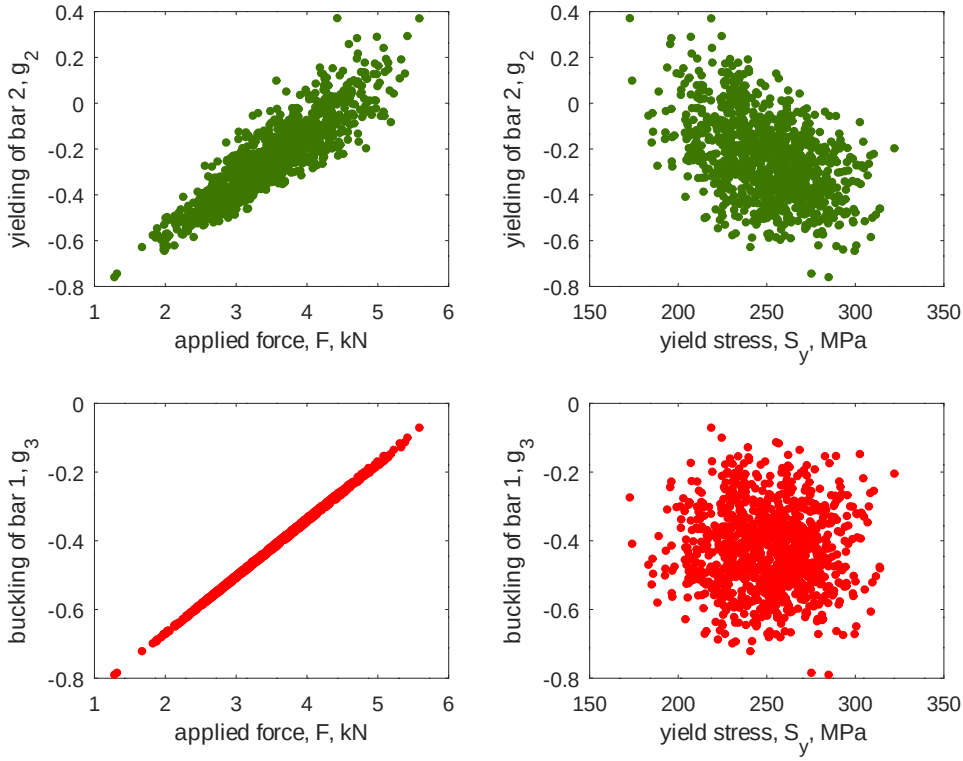


Figure 5. Scatter plots of the failure criteria with respect to random variables for $P = 3.5$ kN.

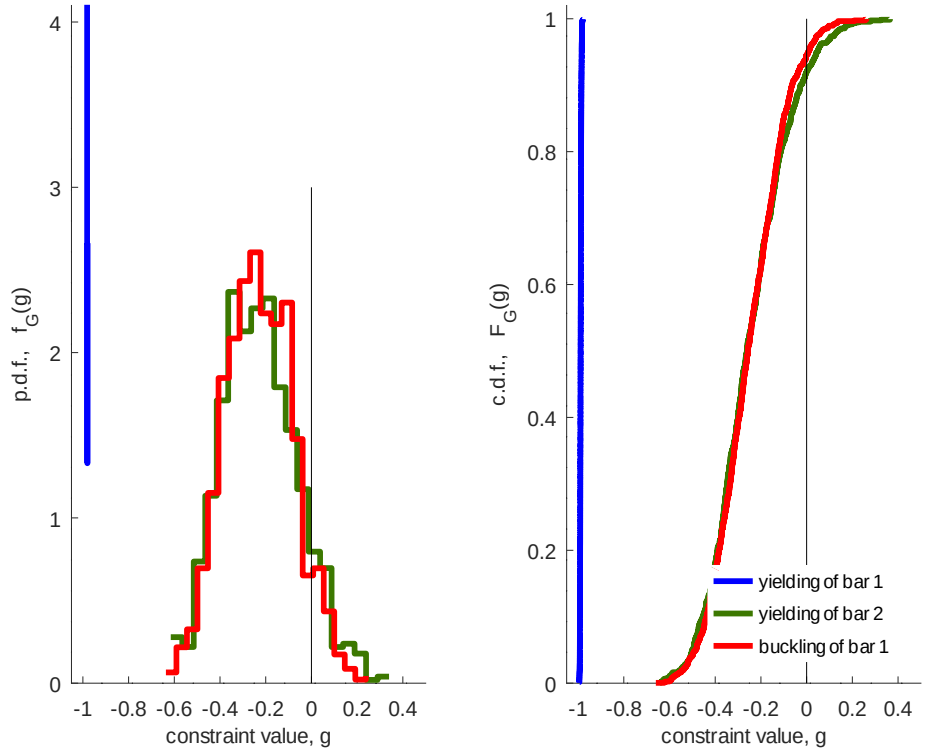


Figure 6. Probability distribution plots of three of the safety criteria for $P = 4.0$ kN.

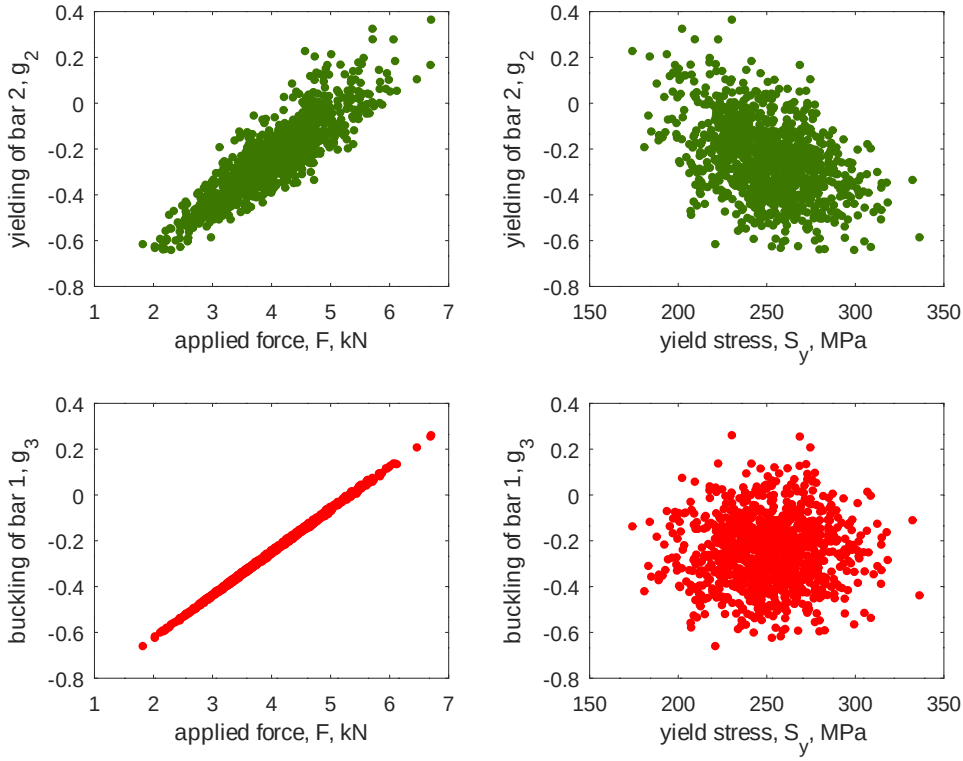


Figure 7. Scatter plots of the failure criteria with respect to random variables for $P = 4.0$ kN.

Questions

1. Check the equilibrium equations on pages 1 and 2 by drawing a free-body-diagram for each of the three trusses in the joint.
2. Make a scaled drawing of the truss optimized for $P = 3.5$ kN and for $P = 4.0$ kN.

   Why are the bar angles more efficient for the 4.0 kN-optimized truss?

   If this geometry is more efficient for the 4.0 kN-optimized truss, why not use it for the 3.5 kN-optimized truss as well?
3. What's the deal with the constraints at $x_{\mathsf{C}} = 0$?
4. For $x_{\mathsf{C}} < 0$, why does it make sense that bar 2 will buckle and bar 3 will not? As part of your answer, draw a sketch of the truss in this configuration, indicate which bar is in tension, and which is in compression, and make use of the known (given) bar diameters.
5. Why does it make sense that bar 2 will never buckle if $x_{\mathsf{C}} > 0$?
6. At an optimal design point constrained by a single inequality (e.g., the design optimization for $P = 3.5$ kN), the contour of $g = 0$ and the contour of the cost function, f, are tangent to each other. This is a mathematical condition for the constrained optimum. (Moving along the constraint curve from the optimum point can only increase the cost.) Looking at the top part of Figure 1, does this appear to be the case in this problem?
7. For the design optimized for $P = 4.0$ kN, which Lagrange multiplier would you expect to be larger, λ_2 or λ_3? Explain your answer using information from Figure 2.
8. The constraint for bar 1 yielding, g_1, is nearly -1 for almost every possible design. Looking at the equations for the inequality constraints on page 3, could g_1 possibly ever be less than -1? Physically, what does it mean that g_1 is practically equal to -1?
9. The constraint for bar 2 buckling, g_4, becomes very negative for $x_{\mathsf{C}} > 0$. What does a very negative value for g_4 represent, physically?
10. When carrying out a safety analysis to compute the probability of failure, why should safety constraints (g_j) be analyzed with safety factors set equal to 1 ($\phi_L = 1; \phi_R = 1$)?
11. Figures 4 and 6 show three distributions, as histograms and CDFs. Why do the distributions for the yielding of bar 1 (the blue line) have values only for $g \approx -1$.
12. For the design optimized for $P = 3.5$ kN, is the truss more likely to fail from bar 2 yielding or bar 1 buckling? Explain your answer using information from Figure 1.
13. For the design optimized for $P = 4.0$ kN, is the truss more likely to fail from bar 2 yielding or bar 1 buckling? Explain your answer using information from Figure 2.
14. Which change would reduce the failure probability for the $P = 4.0$ design more:
    (a) re-optimizing with $\phi_L = 0.90$ and $\phi_R = 1.44$, or
    (b) re-optimizing with $\phi_L = 0.81$ and $\phi_R = 1.20$?
    Why? Use information from Figure 7 in your answer.
15. In this problem, as in *the real world*, why is uncertainty greater on the yield strength than on the elastic modulus? If a failure were to occur, why would a yielding failure be preferable to a buckling failure? Using the responses to the previous two questions, why is the safety factor on bukling resistance greater than that for yielding resistance?