Limit State, Reliability Index, and Safety Factors

CEE 251L. Uncertainty, Design, and Optimization

Duke University

March 9, 2026

The statistics of safety can be quantified in terms of the statistics of uncertain capacity $\mathcal{C}$ and the effects of random demands $\mathcal{D}$.

- The uncertain *limit state function* represents an uncertain *capacity gap* and quantifies how much the demand exceeds the capacity.
  - For normally distributed demand and capacity, the capacity gap is $\mathcal{G} = \mathcal{D} - \mathcal{C}$, and the failure probability is $\mathsf{P}[\mathcal{G} > 0]$. The safety constraint is $\mathcal{G} \leq 0$.
  - For log-normally distributed demand and capacity, the capacity gap is $\mathcal{G} = \mathcal{D}/\mathcal{C}$, and the failure probability is $\mathsf{P}[\mathcal{G} > 1]$. The safety constraint is $\mathcal{G} - 1 \leq 0$.
- The *reliability index* β is defined as as the number of standard deviations of $\mathcal{G}$ that the mean of $\mathcal{G}$ is away from the critical safety boundary, which is 0 for normal variables and 1 for log-normal variables.
  For normally distributed demand and capacity, $\beta = -\mu_{\mathcal{G}}/\sigma_{\mathcal{G}}$.
  For log-normally distributed demand and capacity, $\beta = -\mu_{\ln \mathcal{G}}/\sigma_{\ln \mathcal{G}}$.

  The reliability index can be expressed in terms of the mean capacity μ_C, the mean demand μ_D, the standard deviation of capacity σ_C, and the standard deviation of demand σ_D.

  $$\text{for normal variables: ... } \beta = -\frac{\mu_{\mathcal{G}}}{\sigma_{\mathcal{G}}} = -\frac{\mu_{\mathcal{D}} - \mu_{\mathcal{C}}}{\sqrt{\sigma^2_{\mathcal{D}} + \sigma^2_{\mathcal{C}}}}$$

  $$\text{for lognormal variables: ... } \beta = -\frac{\mu_{\ln \mathcal{G}}}{\sigma_{\ln \mathcal{G}}} = -\frac{\mu_{\ln \mathcal{D}} - \mu_{\ln \mathcal{C}}}{\sqrt{\sigma^2_{\ln \mathcal{D}} + \sigma^2_{\ln \mathcal{C}}}}$$
- The *probability of failure* $\mathsf{P_F}$ is the area under the probability density function of $\mathcal{G}$ for $\mathcal{G} \geq 0$. and equals standard normal CDF of $-\beta$

  $$\mathsf{P_F} = \Phi(-\beta) \qquad\qquad \beta = -\Phi^{-1}(\mathsf{P_F})$$

failure probability	$\mathsf{P_F}$	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
reliability index	β	1.28	2.33	3.09	3.71	4.26	4.75	5.19	5.62	5.99

- In traditional deterministic design, *safety factors* $\phi_{\mathcal{C}}$ and $\phi_{\mathcal{D}}$ decrease the nominal value of capacity C and increase the nominal value of demand D in order to constrain the dessign to a desired level of reliability corresponding to a maximum allowable failure probability $\mathsf{P}^*_{\mathsf{F}}$.

  $$\phi_{\mathcal{D}} D - \phi_{\mathcal{C}} C \leq 0 \qquad \text{or} \qquad \frac{\phi_{\mathcal{D}} D}{\phi_{\mathcal{C}} C} - 1 \leq 0 \tag{1}$$

  Safety factors result in designs that will work in worse-than-expected situations, in which the capacity of the system is less than expected, and the demands on the system are more than expected. So, the capacity safety factor is less than one, $\phi_{\mathcal{C}} < 1$, and the demand safety factor is greater than one, $\phi_{\mathcal{D}} > 1$. The *nominal values* of capacity and demand are set by authorities based on empirical data and prior experience. For convenience, they are rounded to two or three significant figures. Nominal values are related to mean values by *mean to nominal ratios*, which typically take values between 0.8 and 1.2. The reliability of candidate designs may also be constrained to be at least as large as a target reliability β^*

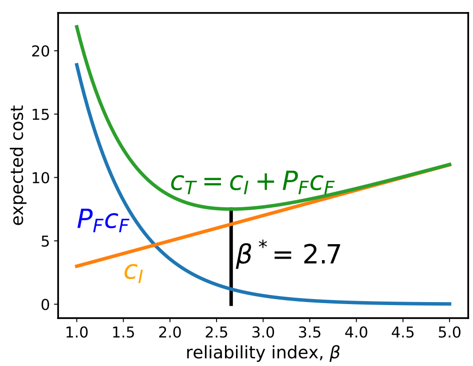
$$\beta^* = -\Phi^{-1}(\mathsf{P}_\mathsf{F}^*) \leq \frac{\mu_{\mathcal{C}} - \mu_{\mathcal{D}}}{\sqrt{\sigma_{\mathcal{C}}^2 + \sigma_{\mathcal{D}}^2}} \tag{2}$$

where the mean capacity $\mu_{\mathcal{C}}$ is a result of a design optimization and the capacity variance $\sigma_{\mathcal{C}}^2$ can be obtained separately, via monte carlo simulation or empirical data.

- In a *reliability based design optimization* (RBDO), in which both the inequality constraint on nominal capacity and demand (1) and the constraint on the reliability index (2) are included. The safety factors themselves can be design variables. Making safety factors design variables in RBDO empowers the design process to discover the optimal balance between safety, cost, and performance while respecting reliability targets. Aditionally, if costs of the candidate design can be assigned to (a) design and construction c_I and (b) failure c_F, and if the probability of failure P_F can also be assessed, then the total expected cost

$$c_\mathsf{T} = c_\mathsf{I} + \mathsf{P}_\mathsf{F} c_\mathsf{F}$$

can be the design objective and the target reliabilty β^* can also be included as a design variable. These approaches turns safety factors and target reliability levels from fixed conservative assumptions into dynamic, data-driven decisions within the optimization.



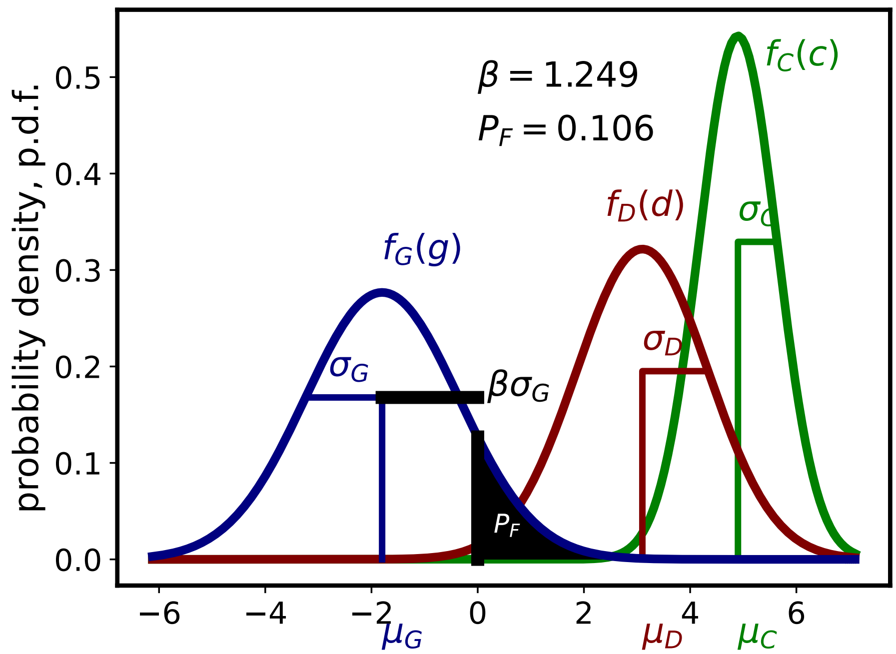
An RBDO example related to *popping balloons* is implmented in `balloon_rbdo.py`

Problems

1. If the reliability index β equals 1.6, what is the value of the probability of failure P_F?
2. Assume that the expected value of the demand increases by 10% and the variability of the demand increases by 20%. Without changing the value of the expected capacity to meet those demands, how must the variability in the capacity (e.g., the quality of manufacturing or construction) change in order to maintain the same level of reliability?

Probability density functions of capacity, $\mathcal{C}$, demand, $\mathcal{D}$, and the capacity gap $\mathcal{G}$ are shown below. These figures are generated with `BetaPlot.py`

- normal variables



- lognormal variables