Duke University
Department of Civil and Environmental Engineering
CEE 421L. Matrix Structural Analysis
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The Theorems of Castigliano [1]

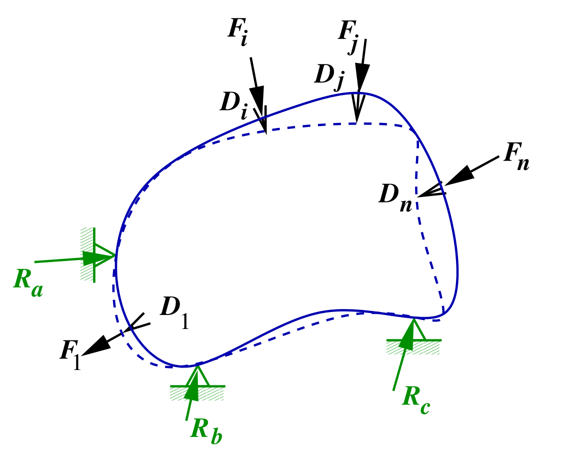
A linear force-displacement relationship between a force, F, and a collocated displacement, D, in statically determinate systems can be determined using the *principle of real work*,

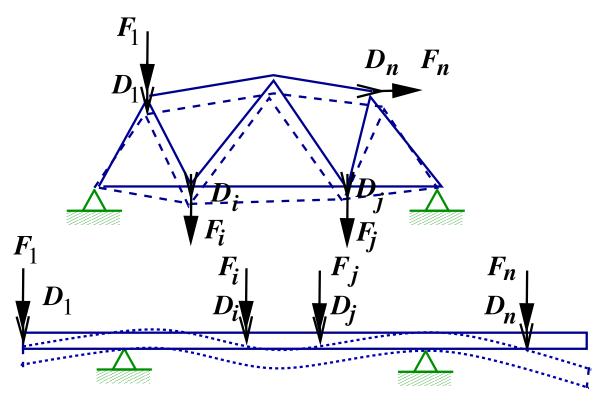
$$W = U \tag{1}$$

$$\frac{1}{2}\, F \cdot D = \frac{1}{2} \int_V \{\sigma\}^T \{\epsilon\}\; dV \; . \tag{2}$$

The force-displacement relationships for systems with multiple external forces or distributed loads, or statically indeterminate systems, involve relationships between multiple forces and displacements. The external work is

$$W = \frac{1}{2} \sum_{i=1}^{n} F_i D_i$$





A set of n force-displacement relationships cannot be found with the single principle of real work equation , $W = U$. Instead, a new method must be developed.

[1] Castigliano, Carlo Alberto, *Intorno ai sistemi elastici*, Ph.D. dissertation, Polytechnic of Turin, 1873.

Castigliano's Theorem - Part I (Force Theorem)

The *strain energy* in any elastic solid subjected to n point forces F_i is a function of the n collocated displacements, D_i.

$$U(D_1, D_2, \cdots, D_n) = \sum_{i=1}^{n} \int_0^{D_i} F_i(D_i)\; dD_i$$

$$\Delta U \approx F_j\; \Delta D_j\;, \qquad F_j \approx \frac{\Delta U}{\Delta D_j}\;, \qquad \Rightarrow \qquad F_j = \frac{\partial U(D)}{\partial D_j} \tag{3}$$

The force, F_j, on an elastic solid is equal to the partial derivative of the *strain energy*, $U(D_1, D_2, \cdots, D_n)$, with respect to the collocated displacement, D_j.

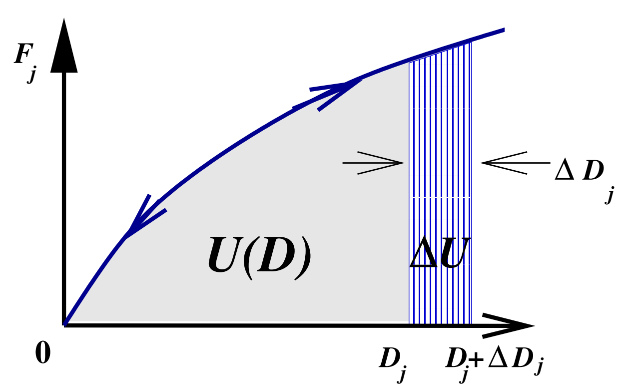
Castigliano's Theorem - Part II (Deflection Theorem)

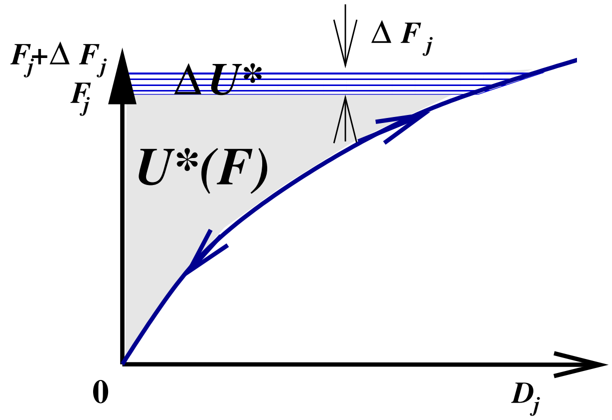
The *complementary strain energy* in any elastic solid subjected to n point forces F_i is a function of the n forces and is the complement of the strain energy.

$$U^*(F_1, F_2, \cdots, F_n) = \sum_{i=1}^{n} F_i D_i - U(D_1, D_2, \cdots, D_n) = \sum_{i=1}^{n} F_i D_i - \sum_{i=1}^{n} \int_0^{D_i} F_i(D_i)\; dD_i$$

$$\Delta U^* \approx D_j\; \Delta F_j\;, \qquad D_j \approx \frac{\Delta U^*}{\Delta F_j}\;, \qquad \Rightarrow \qquad D_j = \frac{\partial U^*(F)}{\partial F_j} \tag{4}$$

The partial derivative of the *complementary strain energy* of an elastic system, $U^*(F)$, with respect to a selected force acting on the system, F_j, gives the displacement of that force along its direction, D_j.

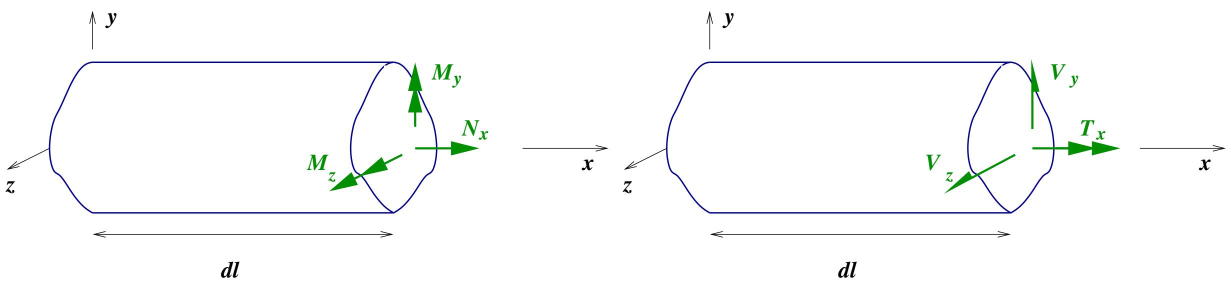




If the solid is linear elastic, then $U^*(F) = U(D)$.

For linear elastic *prismatic* solids in equilibrium,

$$U^*(F) = U(D) = \frac{1}{2}\int_l \frac{N^2}{EA}\,dl + \frac{1}{2}\int_l \frac{M_z^2}{EI_z}\,dl + \frac{1}{2}\int_l \frac{M_y^2}{EI_y}\,dl + \frac{1}{2}\int_l \frac{V_z^2}{G(A/\alpha_z)}\,dl + \frac{1}{2}\int_l \frac{V_y^2}{G(A/\alpha_y)}\,dl + \frac{1}{2}\int_l \frac{T^2}{GJ}\,dl \;. \quad (5)$$



So,

$$\frac{\partial U^*}{\partial F_j} = \frac{\partial U}{\partial F_j} = \int_l \frac{N\frac{\partial N}{\partial F_j}}{EA}\,dl + \int_l \frac{M_z\frac{\partial M_z}{\partial F_j}}{EI_z}\,dl + \int_l \frac{M_y\frac{\partial M_y}{\partial F_j}}{EI_y}\,dl + \int_l \frac{V_z\frac{\partial V_z}{\partial F_j}}{G(A/\alpha_z)}\,dl + \int_l \frac{V_y\frac{\partial V_y}{\partial F_j}}{G(A/\alpha_y)}\,dl + \int_l \frac{T\frac{\partial T}{\partial F_j}}{GJ}\,dl \;. \quad (6)$$

Superposition

Superposition is an extremely powerful method for separating a system with multiple *linear* force-displacement relationships into multiple systems with single *linear* force-displacement relationships.

The principle of superposition states:

Any response of a linear system to multiple inputs can be represented as the sum of the responses to the inputs taken individually.

By "response" we can mean a displacement, a strain, a stress, an internal force, a rotation, an internal moment, etc.

By "input" we can mean an externally applied load, a temperature change, a support settlement, etc.