Real Modes of Vibration of Building Structures

CEE 421L. Matrix Structural Analysis
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1 Mass and Stiffness Matrices

Consider a building frame modeled by a set of rigid, massive floors supported by flexible, massless columns. This provides the simplest representation of a building for the purposes of investigating lateral dynamic responses, as produced by earthquakes or strong winds. The lateral position of the i-th floor with respect to the ground is represented by the variable $r_i(t)$, k_i is the lateral stiffness of all the columns in story i, m_i is the mass of the i-th floor, and f_i is an external force applied to the i-th floor.

For a three-story building, this kind of representation is shown in Figure 1.

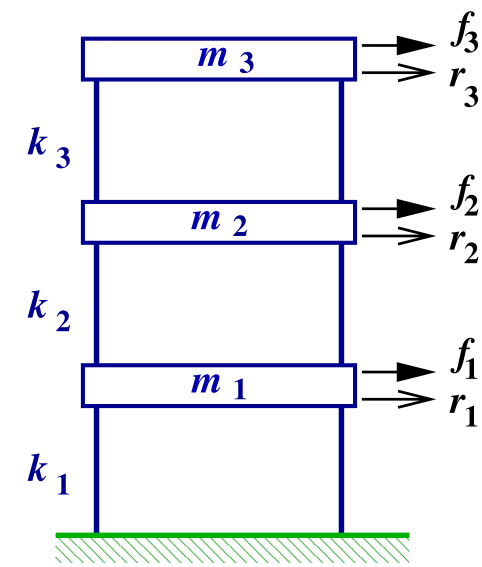

Figure 1. A simplified model of a building frame with massive rigid floors and light flexible columns.

Exercise 1: Draw free-body diagrams of each floor in this building model, including the restoring forces of the deforming columns and the inertial forces in the accelerating floors. Balancing forces in the horizontal direction at each floor will result in three ordinary differential equations, which can be expressed as the matrix form

$$\mathbf{M}\ddot{\mathbf{r}}(t) + \mathbf{K}\mathbf{r}(t) = \mathbf{f}(t) \ , \qquad \mathbf{r}(0) = \mathbf{d_o} \ , \qquad \dot{\mathbf{r}}(0) = \mathbf{v_o} \ , \qquad (1)$$

where $\mathbf{d_o}$ and $\mathbf{v_o}$ are the initial floor displacements and the initial floor velocities. Show that the mass matrix and stiffness matrix for this three-story building are:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} .$$

Convince yourself that each of these three ordinary differential equations involves two or more adjacent floor displacements and because of this, the three differential equations are inter-related or *coupled.*

For an n-story building modeled in this way, the mass matrix is diagonal,

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \vdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & m_n \end{bmatrix} ,$$

and the stiffness matrix is tri-diagonal,

$$\mathbf{K} = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & \cdots & \cdots & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 & \cdots & \cdots & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 & \ddots & \cdots & 0 \\ \vdots & 0 & -k_4 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & -k_{n-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & -k_{n-1} & k_{n-1}+k_n & -k_n \\ 0 & 0 & 0 & \cdots & 0 & -k_n & k_n \end{bmatrix} .$$

2 Modes of Vibration

In the absence of external forcing, one may presume that the natural response of a spring-mass system (such as shown in Figure 1 and modeled by equation (1)) is sinusoidal with frequency $\omega_{\rm n}$ rad/s and a vector of amplitudes $\bar{\mathbf{r}}$, $\bar{\mathbf{r}} = [\bar{r}_1 \ \bar{r}_2 \ \bar{r}_3]^{\sf T}$. Substituting the displacements

$$\mathbf{r}(t) = \bar{\mathbf{r}} \ \sin \omega_{\rm n} t \ ,$$

and accelerations

$$\ddot{\mathbf{r}}(t) = -\bar{\mathbf{r}} \ \omega_{\rm n}^2 \ \sin \omega_{\rm n} t \ ,$$

into equation (1) and eliminating $\sin \omega_{\rm n} t$ we obtain,

$$\mathbf{K}\bar{\mathbf{r}} - \omega_{\rm n}^2 \mathbf{M}\bar{\mathbf{r}} = \mathbf{0} \ ,$$

which may be re-written as the generalized eigenvalue problem,

$$\left[\mathbf{K} - \omega_{\rm n}^2 \mathbf{M}\right] \bar{\mathbf{r}} = \mathbf{0} \ . \qquad (2)$$

The square of the natural frequencies are the eigenvalues and the amplitudes of natural vibration are the associated eigenvectors. As long as $\mathbf{M}$ and $\mathbf{K}$ are positive definite, the eigen-values, $\omega_{\rm n}^2$, are positive. A planar building frame with n rigid floor masses will have n *natural frequencies*, $\omega_{{\rm n}i}$, and n *natural mode shapes*, $\bar{\mathbf{r}}_i$, $i = 1, \ldots, n$. For a natural frequency $\omega_{{\rm n}i}$ and natural mode shape $\bar{\mathbf{r}}_i$ satisfying equation (2), this equation may be pre-multiplied by $\bar{\mathbf{r}}_i^{\sf T}$ to obtain

$$\omega_{{\rm n}i}^2 = \frac{\bar{\mathbf{r}}_i^{\sf T}\mathbf{K}\bar{\mathbf{r}}_i}{\bar{\mathbf{r}}_i^{\sf T}\mathbf{M}\bar{\mathbf{r}}_i} \ ,$$

which is called the *Rayleigh quotient* for the i-th mode.

Mode vectors are orthogonal with respect to the mass and stiffness matrices. This means that

$$\bar{\mathbf{r}}_i^{\sf T}\mathbf{M}\bar{\mathbf{r}}_j = \left\{ \begin{array}{cc} 0 & i \neq j \\ m_i^* & i = j \end{array} \right. \tag{3}$$

and

$$\bar{\mathbf{r}}_i^{\sf T}\mathbf{K}\bar{\mathbf{r}}_j = \left\{ \begin{array}{cc} 0 & i \neq j \\ k_i^* & i = j \end{array} \right. \ , \tag{4}$$

so that $\omega_{{\rm n}i}^2 = k_i^*/m_i^*$. The scalar values m_i^* and k_i^* are called the *modal mass* and *modal stiffness* for the i-th mode. The n natural mode vectors $\bar{\mathbf{r}}_1, \ldots, \bar{\mathbf{r}}_n$ may be arranged column-wise into a *modal matrix*, $\bar{\mathbf{R}}$,

$$\bar{\mathbf{R}} = \left[\begin{array}{ccc} | & & | \\ \bar{\mathbf{r}}_1 & \ldots & \bar{\mathbf{r}}_n \\ | & & | \end{array} \right] \ . \tag{5}$$

Exercise 2: Column vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, have dimension n and matrix $\mathbf{Q}$ is square with dimension n by n. First show that

$$\left[\begin{array}{ccc} | & | & | \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ | & | & | \end{array} \right]^{\sf T} \mathbf{Q} \left[\begin{array}{ccc} | & | & | \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ | & | & | \end{array} \right] = \left[\begin{array}{ccc} \mathbf{a}^{\sf T}\mathbf{Q}\mathbf{x} & \mathbf{a}^{\sf T}\mathbf{Q}\mathbf{y} & \mathbf{a}^{\sf T}\mathbf{Q}\mathbf{z} \\ \mathbf{b}^{\sf T}\mathbf{Q}\mathbf{x} & \mathbf{b}^{\sf T}\mathbf{Q}\mathbf{y} & \mathbf{b}^{\sf T}\mathbf{Q}\mathbf{z} \\ \mathbf{c}^{\sf T}\mathbf{Q}\mathbf{x} & \mathbf{c}^{\sf T}\mathbf{Q}\mathbf{y} & \mathbf{c}^{\sf T}\mathbf{Q}\mathbf{z} \end{array} \right] \ .$$

Next, use this result along with equations (3), (4), and (5) to show that the n by n matrices $\bar{\mathbf{R}}^{\sf T}\mathbf{M}\bar{\mathbf{R}}$ and $\bar{\mathbf{R}}^{\sf T}\mathbf{K}\bar{\mathbf{R}}$ are diagonal matrices with diagonal elements m_i^* and k_i^*.

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3 Damping

The damping forces experienced by a structure arise from material viscosity, air resistance, and friction within connections. Damping forces dissipate energy and cause the free (un-forced) response of a structure to decay to zero. When a structure is excited with sinusoidal forces, $\mathbf{f}(t) = \mathbf{f_o}\cos(\omega t)$, at a forcing frequency near a natural frequency ($\omega \approx \omega_{{\rm n}i}$), a slight amount of damping reduces structural responses significantly. Linear-viscous forces ($f_{{\rm d}i} = c(\dot{r}_{i+1} - \dot{r}_i)$) may be assembled into a damping matrix $\mathbf{C}$. The matrix form of the n coupled second order differential equations including viscous damping effects is:

$$\mathbf{M}\ddot{\mathbf{r}}(t) + \mathbf{C}\dot{\mathbf{r}}(t) + \mathbf{K}\mathbf{r}(t) = \mathbf{f}(t) \ , \qquad \mathbf{r}(0) = \mathbf{d_o} \ , \qquad \dot{\mathbf{r}}(0) = \mathbf{v_o} \ , \tag{6}$$

where **C** is a symmetric non-negative definite damping matrix.

In general, mode vectors that are mass-orthogonal and stiffness-orthogonal are not damping-orthogonal. In many lightly-damped structures, however, the damping may be approximately modeled by a matrix that is proportional to mass and stiffness,

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K} \; . \tag{7}$$

This representation of damping is called *Rayleigh damping* or proportional damping.

Exercise 3: Show that if the units of all terms in **C** are N/mm/s, the units of **M** is tonnes and the units of **K** is N/mm, then the unit of α is (1/seconds) and the unit of β is seconds.

Exercise 4: Show that if the damping matrix is proportional to the mass and stiffness matrices (equation (7)), then

$$\bar{\mathbf{r}}_i^{\mathsf{T}} \mathbf{C} \bar{\mathbf{r}}_j = \left\{ \begin{array}{cc} 0 & i \neq j \\ c_i^* = \alpha m_i^* + \beta k_i^* & i = j \end{array} \right. \; . \tag{8}$$

Exercise 5: Use the MATLAB programs `Rmodes3run.m`, `Rmodes3analysis.m`, and `N_dof_anim.m` , to investigate the effect of different mass and stiffness distributions on the natural mode shapes. For the combinations of mass and stiffness shown in Table 1, use the MATLAB programs to compute natural frequencies and natural modes. Print the mode-shape plots for the six cases shown in Table 1 and breifly describe how changes in mass and stiffness affect natural modes and natural frequencies.

Table 1. Six cases of mass and stiffness distribution.

case:	1	2	3	4	5	6	**units**
m_1	1	1	1	10	1	1	tonne
m_2	1	1	1	1	10	1	tonne
m_3	1	1	1	1	1	10	tonne
k_1	100	1000	1000	1000	1000	1000	N/mm
k_2	1000	100	1000	1000	1000	1000	N/mm
k_3	1000	1000	100	1000	1000	1000	N/mm

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4 Modal Coordinates

At any point in time, the lateral displacement of the floor masses is given by the vector $\mathbf{r}(t)$, $\mathbf{r}(t) = [r_1(t),\; r_2(t), \cdots, r_n(t)]^{\mathsf{T}}$. Because the set of natural mode vectors fills the n-dimensional space of floor displacement vectors, the floor displacement vectors can be written as a weighted sum of the natural mode vectors. See Figure 2.

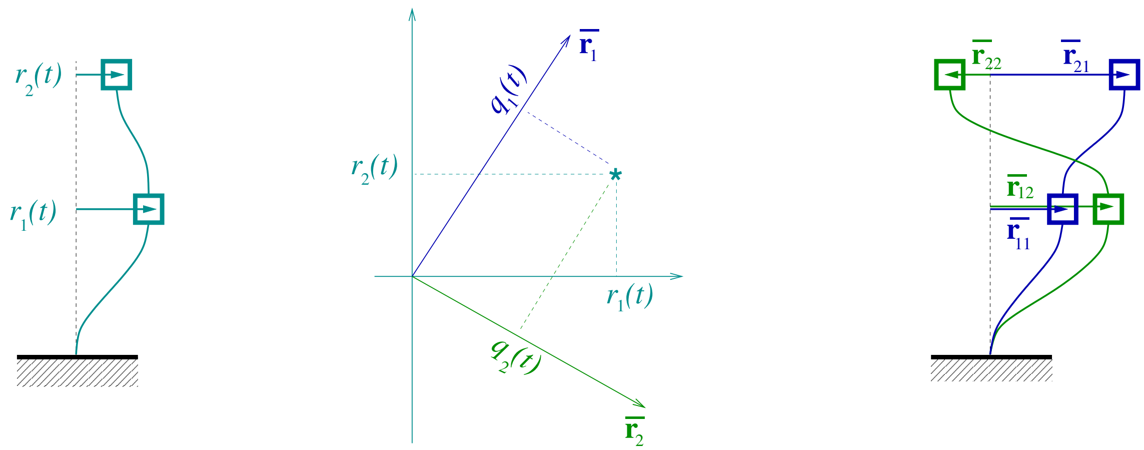

Figure 2. At any point in time, the displacements of a two-coordinate system are $[r_1(t), r_2(t)]$. This point in the $r_1 - r_2$ plane can be represented by the sum of any two (linearly-independent) vectors in the $r_1 - r_2$ plane. The modal vectors $[\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2]$ are shown and the point $[r_1(t), r_2(t)]$ can be expressed as $q_1(t)$ times $\bar{\mathbf{r}}_1$ plus $q_2(t)$ times $\bar{\mathbf{r}}_2$.

$$\mathbf{r}(t) = \bar{\mathbf{r}}_1 \; q_1(t) + \bar{\mathbf{r}}_2 \; q_2(t) + \cdots + \bar{\mathbf{r}}_n \; q_n(t) \; ,$$

or

$$\begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_n(t) \end{bmatrix} = \begin{bmatrix} \bar{r}_{11} \\ \bar{r}_{21} \\ \vdots \\ \bar{r}_{n1} \end{bmatrix} q_1(t) + \begin{bmatrix} \bar{r}_{12} \\ \bar{r}_{22} \\ \vdots \\ \bar{r}_{n2} \end{bmatrix} q_2(t) + \cdots + \begin{bmatrix} \bar{r}_{1n} \\ \bar{r}_{2n} \\ \vdots \\ \bar{r}_{nn} \end{bmatrix} q_n(t) \; ,$$

or

$$\begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_n(t) \end{bmatrix} = \begin{bmatrix} \bar{r}_{11} & \bar{r}_{12} & \cdots & \bar{r}_{1n} \\ \bar{r}_{21} & \bar{r}_{22} & \cdots & \bar{r}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{r}_{n1} & \bar{r}_{n2} & \cdots & \bar{r}_{nn} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix}$$

or

$$\mathbf{r}(t) = \bar{\mathbf{R}} \; \mathbf{q}(t) \tag{9}$$

The vector $\mathbf{q}(t)$ is called the vector of *modal coordinates.*

5 Un-coupled Second Order Differential Equations

Substituting equation (9) into equation (6) results in

$$\mathbf{M}\bar{\mathbf{R}}\ddot{\mathbf{q}}(t) + \mathbf{C}\bar{\mathbf{R}}\dot{\mathbf{q}}(t) + \mathbf{K}\bar{\mathbf{R}}\mathbf{q}(t) = \mathbf{f}(t) \; , \quad \mathbf{q}(0) = \bar{\mathbf{R}}^{-1}\mathbf{d_o} \; , \quad \dot{\mathbf{q}}(0) = \bar{\mathbf{R}}^{-1}\mathbf{v_o} \; .$$

Pre-multiplying both sides of this equation by the transpose of the modal matrix results in:

$$\bar{\mathbf{R}}^{\mathsf{T}}\mathbf{M}\bar{\mathbf{R}}\ddot{\mathbf{q}}(t) + \bar{\mathbf{R}}^{\mathsf{T}}\mathbf{C}\bar{\mathbf{R}}\dot{\mathbf{q}}(t) + \bar{\mathbf{R}}^{\mathsf{T}}\mathbf{K}\bar{\mathbf{R}}\mathbf{q}(t) = \bar{\mathbf{R}}^{\mathsf{T}}\mathbf{f}(t) \; , \quad \mathbf{q}(0) = \bar{\mathbf{R}}^{-1}\mathbf{d_o} \; , \quad \dot{\mathbf{q}}(0) = \bar{\mathbf{R}}^{-1}\mathbf{v_o} \; .$$

Because the modal matrix is mass-orthogonal and stiffness-orthogonal, and assuming the modal matrix is also damping-orthogonal (e.g., the damping is proportional), the equation above may be written

$$\begin{bmatrix} m_1^* & 0 & \cdots & 0 \\ 0 & m_2^* & \vdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & m_n^* \end{bmatrix} \begin{bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \\ \vdots \\ \ddot{q}_n(t) \end{bmatrix} + \begin{bmatrix} c_1^* & 0 & \cdots & 0 \\ 0 & c_2^* & \vdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & c_n^* \end{bmatrix} \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \vdots \\ \dot{q}_n(t) \end{bmatrix} + \begin{bmatrix} k_1^* & 0 & \cdots & 0 \\ 0 & k_2^* & \vdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & k_n^* \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} =$$

$$\begin{bmatrix} \bar{r}_{11} & \bar{r}_{21} & \cdots & \bar{r}_{n1} \\ \bar{r}_{12} & \bar{r}_{22} & \cdots & \bar{r}_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{r}_{1n} & \bar{r}_{2n} & \cdots & \bar{r}_{nn} \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} ,$$

or

$$m_i^* \ddot{q}_i(t) + c_i^* \dot{q}_i(t) + k_i^* q_i(t) = \bar{\mathbf{r}}_i^{\mathsf{T}} \mathbf{f}(t) \tag{10}$$

for each mode $i = 1, \ldots, n$. This represents n *un-coupled* second order differential equations in terms of the modal coordinates $q_i(t)$. All of the solutions pertaining to a single degree of freedom oscillator are relevant to equation (10). Diving both sides of equation (10) by m_i^*,

$$\ddot{q}_i(t) + \frac{c_i^*}{m_i^*} \dot{q}_i(t) + \frac{k_i^*}{m_i^*} q_i(t) = \frac{1}{m_i^*} \bar{\mathbf{r}}_i^{\mathsf{T}} \mathbf{f}(t) ,$$

or

$$\ddot{q}_i(t) + 2\zeta_i \omega_{\mathrm{n}i} \dot{q}_i(t) + \omega_{\mathrm{n}i}^2 q_i(t) = \frac{1}{m_i^*} \bar{\mathbf{r}}_i^{\mathsf{T}} \mathbf{f}(t) ,$$

where ζ_i is the damping ratio associated with mode i, and

$$\zeta_i = \frac{c_i^*}{c_{\mathrm{c}i}^*} = \frac{c_i^*}{2\sqrt{m_i^* k_i^*}} . \tag{11}$$

The analysis of the response $\mathbf{r}(t)$ of an n degree of freedom structural system can be reduced to the analysis of the responses $q_i(t)$ of n single degree of freedom systems, which can then be assembled into the system responses by $\mathbf{r}(t) = \bar{\mathbf{R}}\ \mathbf{q}(t)$. In a free vibration, $\mathbf{q}(t)$ are sinusoidal functions with a single frequency; $q_1(t)$ oscillates only at the first natural frequency, $\omega_{\mathrm{n}1}$, $q_2(t)$ oscillates only at the second natural frequency, $\omega_{\mathrm{n}2}$, and so on. The free vibration of the masses, $\mathbf{r}(t)$, can involve all the modes of vibration, and can oscillate at all of the natural frequencies. The elements of the modal coordinate vector represent the amount of each mode present in the total response.

Exercise 6: Use the Matlab programs `Rmodes3run.m`, `Rmodes3analysis.m`, and `N_dof_anim.m` , to determine values of α and β that will give approximately 5 percent damping in the first mode and approximately 1 percent damping in the third mode for Cases 2, 4, and 6 shown in Table 1. Does increasing α increase the damping in the lower-frequency modes or the higher-frequency modes? Does increasing β increase the damping in the lower-frequency modes or the higher-frequency modes? Using equations (8) and (11) explain how this observed dependence of ζ_i on α and β makes sense.

Exercise 7: For the stiffness and mass values from Case 1 of Table 1, $\mathbf{d_o} = [1, 2, -1]^\mathsf{T}$, $\alpha = 1.0$ and $\beta = 0.0005$, run `Rmodes3run.m` and zoom in on the modal displacement plot at $t = 1$ s and find values for the vector $\mathbf{q}(1)$. Next, zoom in on the floor displacement plot at $t = 1$ s and find values for the vector $\mathbf{r}(1)$. Compute $\bar{\mathbf{R}}\mathbf{q}(1)$ and compare the result to $\mathbf{r}(1)$.

6 Initial Conditions in Modal Coordinates and Free Response

The initial conditions in modal coordinates are

$$\mathbf{q}(0) = \bar{\mathbf{R}}^{-1}\mathbf{d_o} \qquad \text{and} \qquad \dot{\mathbf{q}}(0) = \bar{\mathbf{R}}^{-1}\mathbf{v_o} \ .$$

Because of the mass-orthognality of the natural mode vectors, the inverse modal matrix, $\bar{\mathbf{R}}^{-1}$, can be found without actually inverting $\bar{\mathbf{R}}$.

$$\begin{bmatrix} 1/m_1^* & & \\ & \ddots & \\ & & 1/m_n^* \end{bmatrix} \bar{\mathbf{R}}^T \mathbf{M} \bar{\mathbf{R}} = \mathbf{I}_n \ ,$$

so

$$\bar{\mathbf{R}}^{-1} = \begin{bmatrix} 1/m_1^* & & \\ & \ddots & \\ & & 1/m_n^* \end{bmatrix} \bar{\mathbf{R}}^T \mathbf{M} \ .$$

The initial conditions for the un-coupled second order differential equations in modal coordinates (equation (10)) are therefore

$$q_i(0) = \frac{1}{m_i^*}\bar{\mathbf{r}}_i^\mathsf{T}\mathbf{M}\mathbf{d_o} \qquad \text{and} \qquad \dot{q}_i(0) = \frac{1}{m_i^*}\bar{\mathbf{r}}_i^\mathsf{T}\mathbf{M}\mathbf{v_o} \ . \tag{12}$$

If the initial velocities are all zero and the initial displacements, $\mathbf{d_o}$, are proportional to the i-th natural mode vector, $\bar{\mathbf{r}}_i$, then the free response ensuing from that initial displacement will consist entirely of the i-th mode, and will have no components from other modes. In other words, if $\mathbf{v_o} = \mathbf{0}$ and $\mathbf{d_o} = a\bar{\mathbf{r}}_i$, then $\dot{q}_i(0) = 0$ for all i, $q_i(0) = a$, and $q_j(0) = 0$ for all $j \neq i$.

Exercise 8: Use the MATLAB programs `Rmodes3run.m`, `Rmodes3analysis.m`, and `N_dof_anim.m` , to investigate this property of natural modes. For the stiffness and mass values from Case 1 of Table 1, set the initial displacement proportional to each of the three mode shape vectors, and observe that the free response consists almost entirely of that mode. Now select some other set of initial displacements and observe that the free response contains all three modes. Print a few plots of these mode-shape and free response plots and discuss the results in a short paragraph.

7 Explore!

Exercise 9: Use the MATLAB programs MATLAB programs Rmodes3run.m, Rmodes3analysis.m, and N_dof_anim.m , to explore the effects of very large and very small values of mass, damping, and stiffness. What happens if you increase α and/or β so that the damping is more than 100 percent? What happens if α is positive and β is slightly negative, and vice-versa? What happens if one of the stiffness coefficients is much *much* larger than the other coefficients? What happens if one of the stiffness coefficients is slightly *negative*? What happens if one of the mass coefficients is very *negative*?