Beam Element Stiffness Matrices

CEE 421L. Matrix Structural Analysis
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Truss elements carry only axial forces. Beam elements carry shear forces and bending moments. Frame elements carry shear forces, bending moments, and axial forces. This document presents the development of beam element stiffness matrices in local coordinates.

1 A simply supported beam carrying end-moments

Consider a simply supported beam resisting moments M_1 and M_2 applied at its ends.

The flexibility relates the end rotations $\{\theta_1, \theta_2\}$ to the end moments $\{M_1, M_2\}$:

$$\left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\} = \left[\begin{array}{cc} F_{11} & F_{12} \\ F_{21} & F_{22} \end{array} \right] \left\{ \begin{array}{c} M_1 \\ M_2 \end{array} \right\} .$$

The flexibility coefficients, F_{ij}, may be obtained from Castigliano's 2nd Theorem, $\theta_i = \partial U^*(M_i)/\partial M_i$.

First Column

Second Column

The applied moments M_1 and M_2 are in equilibrium with the reactions forces V_1 and V_2; $V_1 = (M_1 + M_2)/L$ and $V_2 = -(M_1 + M_2)/L$

$$V(x) = \frac{M_1 + M_2}{L} \qquad\qquad M(x) = M_1\left(\frac{x}{L} - 1\right) + M_2\frac{x}{L}$$

The total potential energy of a beam with these forces and moments is:

$$U = \frac{1}{2}\int_0^L \frac{M^2}{EI}\,dx + \frac{1}{2}\int_0^L \frac{V^2}{G(A/\alpha)}\,dx$$

By Castigliano's Theorem,

$$\begin{aligned}
\theta_1 &= \frac{\partial U}{\partial M_1} \\
&= \int_0^L \frac{M(x)\frac{\partial M(x)}{\partial M_1}}{EI}\,dx + \int_0^L \frac{V(x)\frac{\partial V(x)}{\partial M_1}}{G(A/\alpha)}\,dx \\
&= \left(\int_0^L \frac{\left(\frac{x}{L}-1\right)^2 dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) M_1 + \left(\int_0^L \frac{\frac{x}{L}\left(\frac{x}{L}-1\right) dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) M_2
\end{aligned}$$

and

$$\begin{aligned}
\theta_2 &= \frac{\partial U}{\partial M_2} \\
&= \int_0^L \frac{M(x)\frac{\partial M(x)}{\partial M_2}}{EI}\,dx + \int_0^L \frac{V(x)\frac{\partial V(x)}{\partial M_2}}{G(A/\alpha)}\,dx \\
&= \left(\int_0^L \frac{\frac{x}{L}\left(\frac{x}{L}-1\right) dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) M_1 + \left(\int_0^L \frac{\left(\frac{x}{L}\right)^2 dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) M_2
\end{aligned}$$

or, in matrix form,

$$\left\{ \begin{array}{c} \theta_1 \\ \\ \theta_2 \end{array} \right\} = \left[\begin{array}{cc} \left(\int_0^L \frac{\left(\frac{x}{L}-1\right)^2 dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) & \left(\int_0^L \frac{\frac{x}{L}\left(\frac{x}{L}-1\right) dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) \\ \\ \left(\int_0^L \frac{\frac{x}{L}\left(\frac{x}{L}-1\right) dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) & \left(\int_0^L \frac{\left(\frac{x}{L}\right)^2 dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) \end{array} \right] \left\{ \begin{array}{c} M_1 \\ \\ M_2 \end{array} \right\}$$

For prismatic beams, E, A, and I are constant along the length, and the flexibility relationship is

$$\left\{ \begin{array}{c} \theta_1 \\ \\ \theta_2 \end{array} \right\} = \left[\begin{array}{cc} \frac{L}{3EI} + \frac{1}{G(A/\alpha)L} & -\frac{L}{6EI} + \frac{1}{G(A/\alpha)L} \\ \\ -\frac{L}{6EI} + \frac{1}{G(A/\alpha)L} & \frac{L}{3EI} + \frac{1}{G(A/\alpha)L} \end{array} \right] \left\{ \begin{array}{c} M_1 \\ \\ M_2 \end{array} \right\}$$

To neglect shear deformation, set $\alpha = 0$.

The stiffness relationship is the inverse of the flexibility relationship, and for prismatic members,

$$\left\{ \begin{array}{c} M_1 \\ \\ M_2 \end{array} \right\} = \left[\begin{array}{cc} \frac{(4+\Phi)EI}{(1+\Phi)L} & \frac{(2-\Phi)EI}{(1+\Phi)L} \\ \\ \frac{(2-\Phi)EI}{(1+\Phi)L} & \frac{(4+\Phi)EI}{(1+\Phi)L} \end{array} \right] \left\{ \begin{array}{c} \theta_1 \\ \\ \theta_2 \end{array} \right\}$$

where

$$\Phi = \frac{12EI}{G(A/\alpha)L^2} = 24\alpha(1+\nu)\left(\frac{r}{L}\right)^2$$

and r is the "radius of gyration" of the cross section, $r = \sqrt{I/A}$.
To neglect shear deformation, set $\Phi = 0$:

$$\left\{ \begin{array}{c} M_1 \\ \\ M_2 \end{array} \right\} = \left[\begin{array}{cc} \frac{4EI}{L} & \frac{2EI}{L} \\ \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array} \right] \left\{ \begin{array}{c} \theta_1 \\ \\ \theta_2 \end{array} \right\} \tag{1}$$

2 Beam Element Stiffness Matrix in Local Coordinates, k

The beam element stiffness matrix **k** relates the shear forces and bending moments at the end of the beam $\{V_1, M_1, V_2, M_2\}$ to the deflections and rotations at the end of the beam $\{\Delta_1, \theta_1, \Delta_2, \theta_2\}$.

$$\left\{ \begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \right\} = \left[\begin{array}{cccc} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{array} \right] \left\{ \begin{array}{c} \Delta_1 \\ \theta_1 \\ \Delta_2 \\ \theta_2 \end{array} \right\}$$

The elements of this four-by-four stiffness matrix may be derived from equation (1) using arguments of equilibrium and symmetry.

The second column of the stiffness matrix is the set of forces and moments corresponding to the following set of displacements and rotations:

$$\{\Delta_1 = 0, \theta_1 = 1, \Delta_2 = 0, \theta_2 = 0\}$$

From equation (1) we know

$$M_1 = 4EI/L = k_{22}$$

and

$$M_2 = 2EI/L = k_{42}.$$

From equilibrium we know

$$V_1 = (M_1 + M_2)/L = 6EI/L^2 = k_{12}$$

and

$$V_2 = -(M_1 + M_2)/L = -6EI/L^2 = k_{32}.$$

This completes the second column of $\mathbf{k}$.

Similarly, the fourth column of $\mathbf{k}$ is the set of forces and moments corresponding to

$$\{\Delta_1 = 0, \theta_1 = 0, \Delta_2 = 0, \theta_2 = 1\}$$

These forces and moments are

$$M_1 = 2EI/L = k_{24} \ ,$$

$$M_2 = 4EI/L = k_{44} \ ,$$

$$V_1 = (M_1 + M_2)/L = 6EI/L^2 = k_{14} \ ,$$

and

$$V_2 = -(M_1 + M_2)/L = -6EI/L^2 = k_{34} \ .$$

Now the first column of **k** is the set of forces and moments corresponding to

$$\{\Delta_1 = 1, \theta_1 = 0, \Delta_2 = 0, \theta_2 = 0\}$$

From arguments of symmetry (of the element stiffness matrix) we know

$$M_1 = k_{21} = k_{12} = 6EI/L^2$$

and

$$M_2 = k_{41} = k_{14} = 6EI/L^2 \ .$$

And from equilibrium,

$$V_1 = (M_1 + M_2)/L = 12EI/L^3 = k_{11} \ ,$$

and

$$V_2 = -(M_1 + M_2)/L = -12EI/L^3 = k_{31} \ .$$

Finally, the third column of **k** is the set of forces and moments corresponding to

$$\{\Delta_1 = 0, \theta_1 = 0, \Delta_2 = 1, \theta_2 = 0\}$$

From arguments of symmetry (of the element stiffness matrix) we know

$$M_1 = k_{23} = k_{32} = -6EI/L^2$$

and

$$M_2 = k_{43} = k_{34} = -6EI/L^2 \ .$$

And from equilibrium,

$$V_1 = (M_1 + M_2)/L = -12EI/L^3 = k_{13} \ ,$$

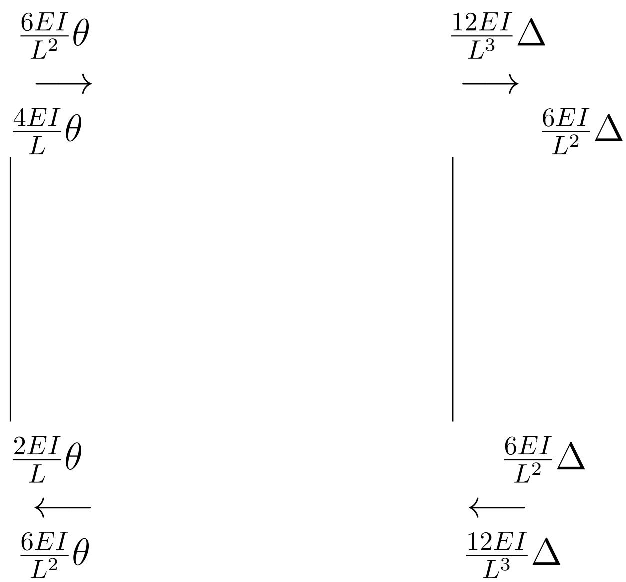
and

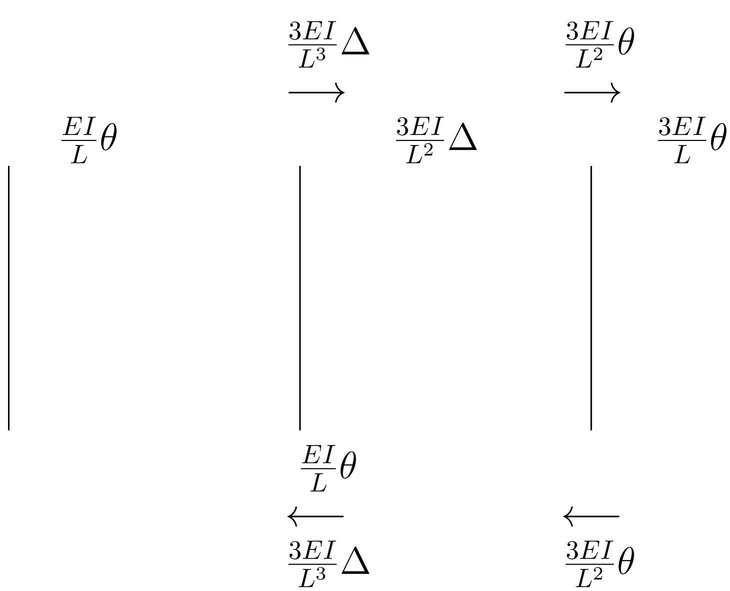
$$V_2 = -(M_1 + M_2)/L = 12EI/L^3 = k_{33} \ .$$

This analysis provides the sixteen terms of the beam element stiffness matrix.

$$\left\{\begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \end{array}\right\} = (EI)\left[\begin{array}{rrrr} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & 4/L & -6/L^2 & 2/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ 6/L^2 & 2/L & -6/L^2 & 4/L \end{array}\right]\left\{\begin{array}{c} \Delta_1 \\ \theta_1 \\ \Delta_2 \\ \theta_2 \end{array}\right\}$$

The images below summarize the stiffness coefficients for the standard fixed-fixed beam element as well as for the fixed-pinned beam element.





3 Notation

$\mathbf{u}$ = Element deflection vector $[\Delta_1, \theta_1, \Delta_2, \theta_2]$

$\mathbf{q}$ = Element force vector in the $[V_1, M_1, V_2, M_2]$

$\mathbf{k}$ = Element stiffness matrix in the Local coordinate system
... $\mathbf{q} = \mathbf{k}\ \mathbf{u}$

$\mathbf{d}$ = Structural deflection vector

$\mathbf{p}$ = Structural load vector

$\mathbf{K_s}$ = Structural stiffness matrix
... $\mathbf{p} = \mathbf{K_s}\ \mathbf{d}$

Element Deflection	$\mathbf{u}$
Element Force	$\mathbf{q}$
Element Stiffness	$\mathbf{k}$
Structural Deflection	$\mathbf{d}$
Structural Loads	$\mathbf{p}$
Structural Stiffness	$\mathbf{K_s}$

Example 1

```
>>  E = 30000;                                  % modulus of elasticity
>>  I1 = 1000; I2 = 500; I3 = 250;              % moments of inertia
>>  L1 = 150; L2 = 120; L3 = 100;               % element lengths

% enter the stiffness matrix

>>  Ks = [ (4*E*I1/L1 + 4*E*I2/L2)   2*E*I2/L2             ;
            2*E*I2/L2                4*E*I2/L2 + 4*E*I3/L3 ]

Ks =

  1300000   250000
   250000   800000

>>  w = 0.1;               % uniform distributed load on member number 1
>>  p = [ w*L1^2/12 0 ]'    % fixed end forces from member 1 applied to DoF 1

p =
                            % external load vector
  187.50000
    0.00000

>>  d = inv(Ks) * p           % compute the displacement Vector

d =
    1.5345e-04
   -4.7954e-05

>>  M1 = 4*E*I1/L1 * d(1)
M1 = 122.76                 % Moment in member 1 at DoF 1 due to rotation D1

>>  M2 = 4*E*I2/L2 * d(1) + 2*E*I2/L2 * d(2)

M2 = 64.738                 % Moment in member 2 at DoF 2 due to rotation D2


>>  M1 - 187.5              % Moment in member 1 at DoF 1  MINUS  the

ans = -64.738               % fixed end moment equals the Moment M2 ... YAY!
```