Duke University
Department of Civil and Environmental Engineering
CEE 421L. Matrix Structural Analysis
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Review of Strain Energy Methods and Introduction to Stiffness Matrix Methods of Structural Analysis

1 Strain Energy

Strain energy is stored within an elastic solid when the solid is deformed under load. In the absence of energy losses, such as from friction, damping or yielding, the strain energy is equal to the work done on the solid by external loads. Strain energy is a type of potential energy.

Consider the work done on an elastic solid by a single point force F. When the elastic solid carries the load, F, it deforms with strains (ϵ and γ) and the material is stressed (σ and τ).

D is a displacement in the same location and in the same direction as a point force, F. D and F are *colocated*. The work done by the force F on the elastic solid is the area under the force vs. displacement curve.

$$W = \int F \, dD \tag{1}$$

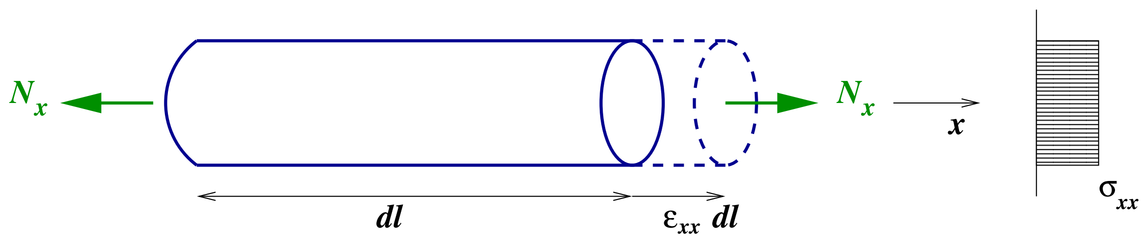
This work is stored as *strain energy* U within the elastic solid.

$$U = \frac{1}{2} \int_V (\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}) \, dV. \tag{2}$$

This is a very general expression for the strain energy, U, and is not very practical for structural elements like bars, beams, trusses, or frames.

1.1 Bars

For a *bar* in tension or compression, we have internal axial force, N, only,



so $\sigma_{yy} = 0$, $\sigma_{zz} = 0$, $\tau_{xy} = 0$, $\tau_{xz} = 0$, and $\tau_{yz} = 0$, and

$$U = \frac{1}{2} \int_V \sigma_{xx} \epsilon_{xx} \; dV \; ,$$

where $\sigma_{xx} = N/A$ and $\epsilon_{xx} = N/EA$. Substituting $dV = A\; dx$ we get

$$U = \frac{1}{2} \int_L \frac{N(x)^2}{E(x)\; A(x)} \; dx \; , \qquad (3)$$

and if N, E, and A are constant

$$U = \frac{1}{2} \frac{N^2\; L}{E\; A} \; .$$

Alternatively, we may express the strain as a function of the displacements along the bar $u_x(x)$, $\epsilon_{xx} = \partial u_x(x)/\partial x$, and $\sigma_{xx} = E\; \partial u_x(x)/\partial x$. Again substituting $dV = A\; dx$,

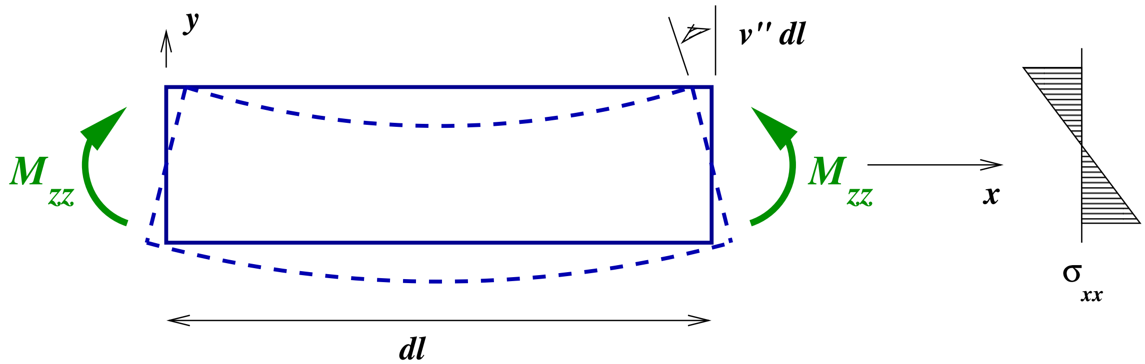
$$U = \frac{1}{2} \int_L E(x)\; A(x) \left(\frac{\partial u_x(x)}{\partial x} \right)^2 \; dx \; , \qquad (4)$$

and if E, A and $\partial u_x/\partial x = (u_2 - u_1)/L$ are constants,

$$U = \frac{1}{2} \frac{EA}{L} \; (u_2 - u_1)^2$$

1.2 Beams

For a *beam* in bending we have internal bending moments, M, and internal shear forces, V. For slender beams the effects of shear deformation are usually neglected.



As in the axially loaded bar, $\sigma_{yy} = 0$, $\sigma_{zz} = 0$, $\tau_{xy} = 0$, $\tau_{xz} = 0$, and $\tau_{yz} = 0$, and

$$U = \frac{1}{2} \int_V \sigma_{xx} \epsilon_{xx} \; dV \; .$$

For bending, $\sigma_{xx} = My/I$ and $\epsilon_{xx} = My/EI$. Substituting $dV = dA\; dx$,

$$U = \frac{1}{2} \int_L \int_A \frac{M(x)^2 \; y^2}{E(x) \; I(x)^2} \; dA \; dx \; ,$$

where $\int_A y^2 dA = I$, so

$$U = \frac{1}{2} \int_L \frac{M(x)^2}{E(x) \; I(x)} \; dx \; . \tag{5}$$

Alternatively, we may express the moment in terms of the curvature of the beam, $\phi \approx \partial^2 u_y / \partial x^2$,

$$M(x) = E(x) \; I(x) \; \frac{\partial^2 u_y(x)}{\partial x^2} \; ,$$

from which $\sigma_{xx} = E \; (\partial^2 u_y / \partial x^2) \; y$ and $\epsilon_{xx} = (\partial^2 u_y / \partial x^2) \; y$, so that

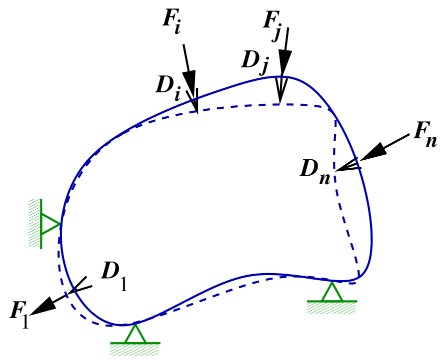
$$U = \frac{1}{2} \int_L \int_A E(x) \left(\frac{\partial^2 u_y(x)}{\partial x^2} \right)^2 \; y^2 \; dA \; dx$$

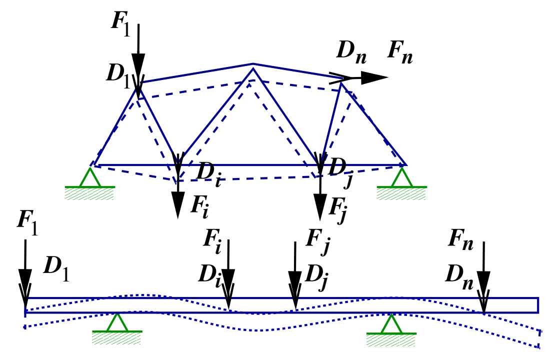
where, again, $\int_A y^2 dA = I$, so

$$U = \frac{1}{2} \int_L E(x) \; I(x) \left(\frac{\partial^2 u_y(x)}{\partial x^2} \right)^2 \; dx \; . \tag{6}$$

1.3 Summary

External work is done by a set of forces, F_i, on a linear elastic solid, producing a set of displacements, D_i, in the same locations and directions.





The work done by these forces is

$$W = \frac{1}{2}F_1 D_1 + \frac{1}{2}F_2 D_2 + \frac{1}{2}F_3 D_3 + \cdots$$

The external forces are resisted by internal moments, M, and axial forces, N. The total strain energy stored within the solid is

$$U = \frac{1}{2}\int_L \frac{M^2}{E\ I}\ dx\ + \frac{1}{2}\sum_j \frac{N_j^2\ L_j}{E_j\ A_j} \qquad (7)$$

where the first term is the integral over all lengths of all the beams and the second term is the sum over all the bars. If torsion and shear are included, then two additional terms are

$$\frac{1}{2}\int_L \frac{T^2}{G\ J}\ dx \qquad \text{and} \qquad \frac{1}{2}\int_L \frac{V^2}{G\ A/\alpha}\ dx\ .$$

Alternatively, we can think of external forces producing curvatures ($\partial^2 u_y/\partial x^2$) by bending, and axial stretches ($\partial u_x/\partial x$). In this case

$$U = \frac{1}{2}\int_L E\ I \left(\frac{\partial^2 u_y}{\partial x^2}\right)^2\ dx\ + \frac{1}{2}\sum_j \frac{E_j\ A_j}{L_j}\ (u_{2j} - u_{1j})^2 \qquad (8)$$

If torsion and shear are included, then two additional terms are

$$\frac{1}{2}\int_L G\ J \left(\frac{\partial u_{x\theta}}{\partial x}\right)^2\ dx\ , \qquad \text{and} \qquad \frac{1}{2}\int_L G\ A/\alpha \left(\frac{\partial u_y}{\partial x}\right)^2\ dx\ ,$$

where $u_{x\theta}$ is the torsional rotation about the x-axis, $\partial u_{x\theta}/\partial x$ is the torsional shear strain, $\gamma_{x\theta}$, (on the face perpendicular to the x-axis and in the θ-direction) and $\partial u_y/\partial x$ is the shear strain, γ_{xy}, (on the face perpendicular to the x-axis and in the y-direction).

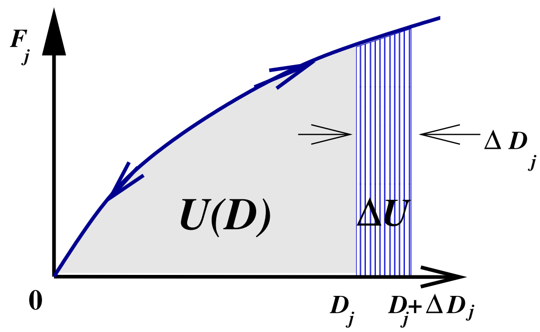
Analyses using expressions of the form of equations (3), (5), or (7) are called *force method* or *flexibility method* analyses.

Analyses using expressions of the form of equations (4), (6), or (8) are called *displacement method* or *stiffness method* analyses.

2 Castigliano's Theorems

2.1 Castigliano's Theorem - Part I

$$U = \int F \; dD \qquad \ldots \qquad \text{strain energy}$$

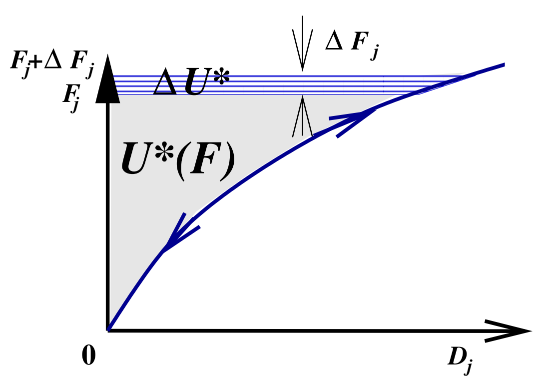


$$F_i = \frac{\Delta U}{\Delta D_i} = \frac{\partial U}{\partial D_i}$$

A force, F_i, on an elastic solid is equal to the derivative of the strain energy with respect to the displacement, D_i, in the direction and location of the force, F_i.

2.2 Castigliano's Theorem - Part II

$$U^* = \int D \; dF \qquad \ldots \qquad \text{complementary strain energy}$$



$$D_i = \frac{\Delta U^*}{\Delta F_i} = \frac{\partial U^*}{\partial F_i}$$

A displacement, D_i, on an elastic solid is equal to the derivative of the *complementary* strain energy with respect to the force, F_i, in the direction and location of the displacement, D_i.

If the solid is linear elastic, then $U^* = U$.

3 Superposition

Superposition is an extremely powerful idea that helps us solve problems that are statically indeterminate. To use the principle of superposition, the system must behave in a *linear* elastic fashion.

The principle of superposition states:

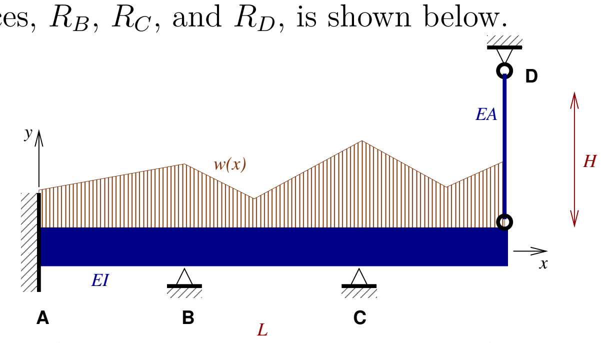
Any response of a system to multiple inputs can be represented as the sum of the responses to the inputs taken individually.

By "response" we can mean a strain, a stress, a deflection, an internal force, a rotation, an internal moment, etc.

By "input" we can mean an externally applied load, a temperature change, a support settlement, etc.

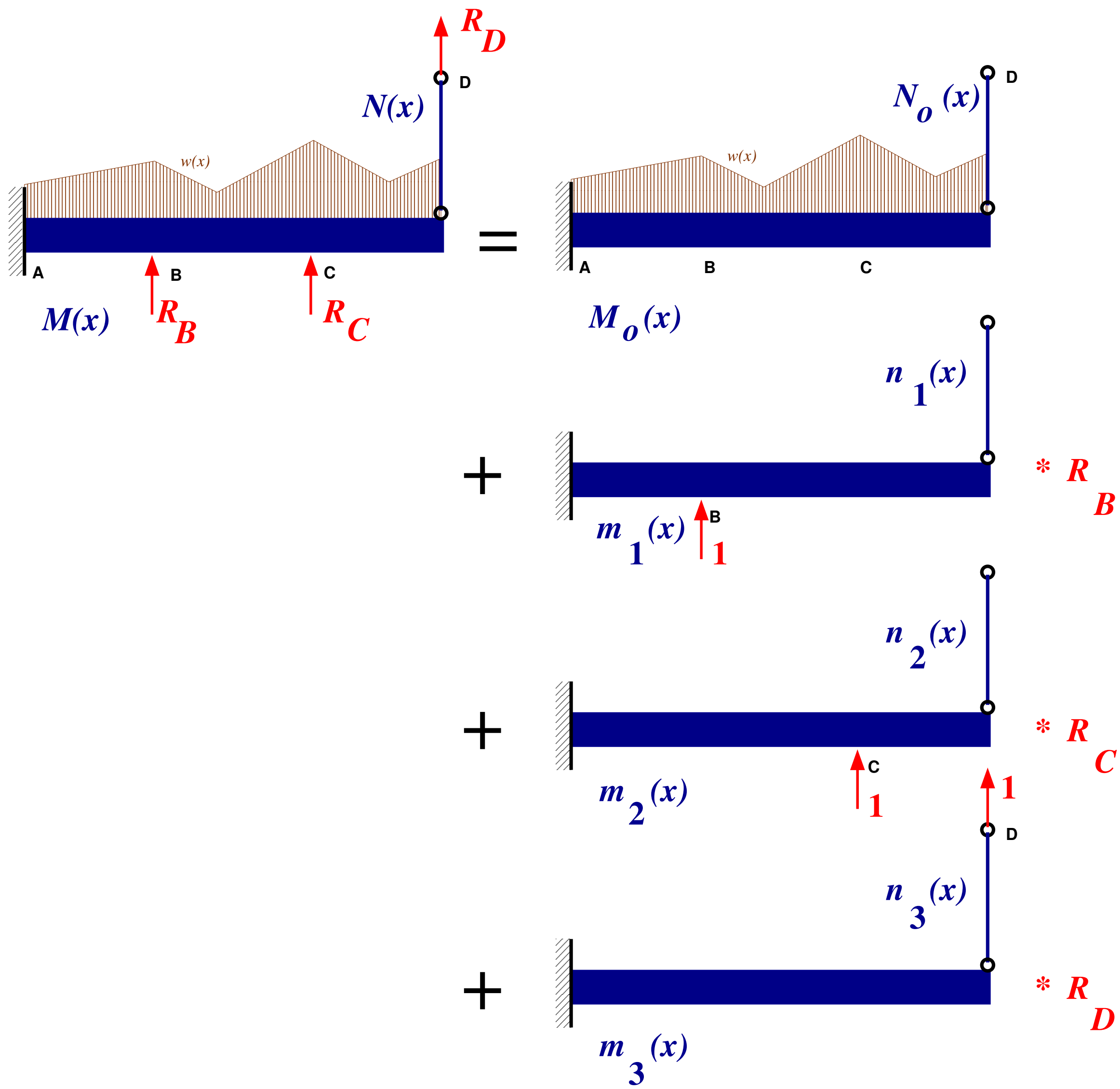
4 Detailed Example of Castigliano's Theorem and Superposition

An example of a statically indeterminate system with external loads $w(x)$ and three redundant reaction forces, R_B, R_C, and R_D, is shown below.



In general, the displacements at the locations of the unknown reaction forces are known, and, in this example these displacements will be taken as zero: $D_B = 0$, $D_C = 0$, $D_D = 0$.

Invoking the principle of superposition, we may apply the external loads, $(w(x))$ and the unknown reactions (R_B, R_C, and R_D) individually, and then sum-up the responses to each individual load. Further, we may represent the response to a reaction force, (e.g., R_B) as the response to a unit force co-located with the reaction force, times the value of the reaction force. Note that all four systems to the right of the equal sign in the following figure are statically determinate. Expressions for $Mo(x)$, $m_1(x)$, $m_2(x)$, $m_3(x)$, $N_o(x)$, $n_1(x)$, $n_2(x)$, and $n_3(x)$ may be found from static equilibrium alone.



In equation form, the principle of superposition says:

$$M(x) = M_o(x) + m_1(x)R_B + m_2(x)R_C + m_3(x)R_D \quad (9)$$

$$N = N_o + n_1 R_B + n_2 R_C + n_3 R_D \quad (10)$$

(Note that in this particular example, $N_o(x) = 0$, $n_1 = 0$, $n_2 = 0$, $n_3 = 1$, $m_1(x) = 0$ for $x > x_B$, and $m_2(x) = 0$ for $x > x_C$.)

The total strain energy, U, in systems with bending strain energy and axial strain energy is,

$$U = \frac{1}{2}\int_0^L \frac{M(x)^2}{EI}\,dx \quad + \quad \frac{1}{2}\sum \frac{N^2\,H}{EA} \tag{11}$$

We are told that the displacements at points B, C, and D are all zero and we need to assume the structure behaves linear elastically in order to invoke superposition in the first place. Therefore, from Castigliano's Second Theorem,

$$D_i = \frac{\partial U^*}{\partial F_i} = \frac{\partial U}{\partial F_i}\ ,$$

we obtain three expressions for the facts that $D_B = 0$, $D_C = 0$, and $D_D = 0$.

$$D_B = 0 = \frac{\partial U}{\partial R_B}$$

$$D_C = 0 = \frac{\partial U}{\partial R_C}$$

$$D_D = 0 = \frac{\partial U}{\partial R_D}$$

Inserting equation (11) into the three expressions for zero displacement at the fixed reactions, noting that EI and EA are constants in this problem, and noting that the strain energy, U, depends on the reactions R, only through the internal forces, M and N, we obtain

$$D_B = 0 = \frac{1}{EI}\int_0^L M(x)\frac{\partial M(x)}{\partial R_B}dx \quad + \quad \frac{H}{EA}N\frac{\partial N}{\partial R_B}$$

$$D_C = 0 = \frac{1}{EI}\int_0^L M(x)\frac{\partial M(x)}{\partial R_C}dx \quad + \quad \frac{H}{EA}N\frac{\partial N}{\partial R_C}$$

$$D_D = 0 = \frac{1}{EI}\int_0^L M(x)\frac{\partial M(x)}{\partial R_D}dx \quad + \quad \frac{H}{EA}N\frac{\partial N}{\partial R_D}$$

Now, from the superposition equations (9) and (10), $\partial M(x)/\partial R_B = m_1(x)$, $\partial M(x)/\partial R_C = m_2(x)$, $\partial M(x)/\partial R_D = m_3(x)$, $\partial N(x)/\partial R_B = n_1$, $\partial N(x)/\partial R_C = n_2$, and $\partial N(x)/\partial R_D = n_3$. Inserting these expressions and the superposition equations (9) and (10) into the above equations for D_B, D_C, and D_D,

$$D_B = 0 = \frac{1}{EI}\int_0^L [M_o + m_1 R_B + m_2 R_C + m_3 R_D]\ m_1\ dx + \frac{H}{EA}[N_o + n_1 R_B + n_2 R_C + n_3 R_D]\ n_1$$

$$D_C = 0 = \frac{1}{EI}\int_0^L [M_o + m_1 R_B + m_2 R_C + m_3 R_D]\ m_2\ dx + \frac{H}{EA}[N_o + n_1 R_B + n_2 R_C + n_3 R_D]\ n_2$$

$$D_D = 0 = \frac{1}{EI}\int_0^L [M_o + m_1 R_B + m_2 R_C + m_3 R_D]\ m_3\ dx + \frac{H}{EA}[N_o + n_1 R_B + n_2 R_C + n_3 R_D]\ n_3$$

These three expressions contain the three unknown reactions R_B, R_C, and R_D. Everything else in these equations ($m_1(x)$, $m_2(x)$ ... n_3) can be found without knowing the unknown reactions. By taking the unknown reactions out of the integrals (they are constants), we can write these three equations in matrix form.

$$\begin{bmatrix} \int_0^L \frac{m_1 m_1}{EI} dx + \frac{n_1 n_1 H}{EA} & \int_0^L \frac{m_1 m_2}{EI} dx + \frac{n_1 n_2 H}{EA} & \int_0^L \frac{m_1 m_3}{EI} dx + \frac{n_1 n_3 H}{EA} \\ \int_0^L \frac{m_2 m_1}{EI} dx + \frac{n_2 n_1 H}{EA} & \int_0^L \frac{m_2 m_2}{EI} dx + \frac{n_2 n_2 H}{EA} & \int_0^L \frac{m_2 m_3}{EI} dx + \frac{n_2 n_3 H}{EA} \\ \int_0^L \frac{m_3 m_1}{EI} dx + \frac{n_3 n_1 H}{EA} & \int_0^L \frac{m_3 m_2}{EI} dx + \frac{n_3 n_2 H}{EA} & \int_0^L \frac{m_3 m_3}{EI} dx + \frac{n_3 n_3 H}{EA} \end{bmatrix} \begin{bmatrix} R_B \\ R_C \\ R_D \end{bmatrix} = - \begin{bmatrix} \int_0^L \frac{M_o m_1}{EI} dx + \frac{N_o n_1 H}{EA} \\ \int_0^L \frac{M_o m_2}{EI} dx + \frac{N_o n_2 H}{EA} \\ \int_0^L \frac{M_o m_3}{EI} dx + \frac{N_o n_3 H}{EA} \end{bmatrix} \quad (12)$$

This 3-by-3 matrix is called a *flexibility matrix*, **F**. The values of the terms in the flexibility matrix depend only on the responses of the structure to unit loads placed at various points in the structure. The flexibility matrix is therefore a property of the structure alone, and does not depend upon the loads on the structure[1]. The vector on the right-hand-side depends on the loads on the structure. Recall that this matrix looks a lot like the matrix from the three-moment equation. All flexibility matrices share several properties:

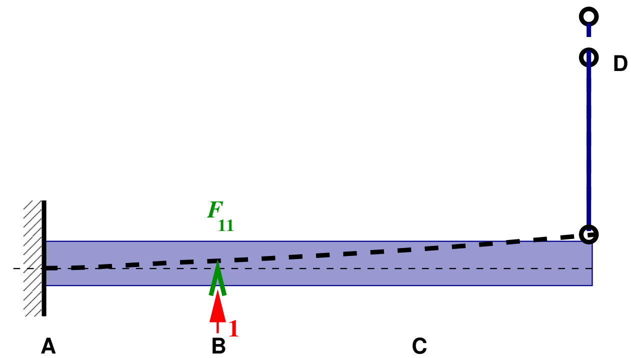
- All flexibility matrices are symmetric.
- No diagonal terms are negative.
- Flexibility matrices for structures which can not move or rotate without deforming are *positive definite*. This means that all of the eigenvalues of a flexibility matrix describing a fixed structure are positive.
- The unknowns in a flexibility matrix equation are forces (or moments).
- The number of equations (rows of the flexibility matrix) equals the number of unknown forces (or moments).

[1] There are some fascinating cases in which the behavior does depend upon the loads, but that is a story for another day!

It is instructive to now examine the meaning of the terms in the matrix, **F**

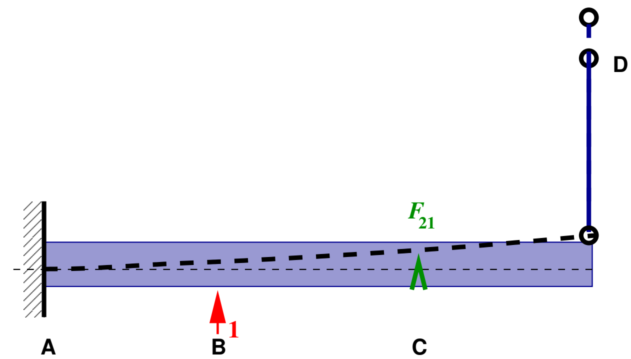
$$F_{11} = \int_0^L \frac{m_1 m_1}{EI} dx + \frac{n_1 n_1 H}{EA} = \delta_{11}$$

displacement at "1" due to unit force at "1"



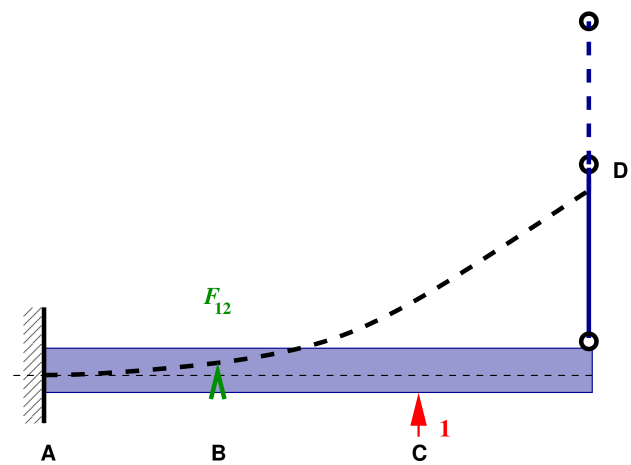
$$F_{21} = \int_0^L \frac{m_2 m_1}{EI} dx + \frac{n_2 n_1 H}{EA} = \delta_{21}$$

displacement at "2" due to unit force at "1"



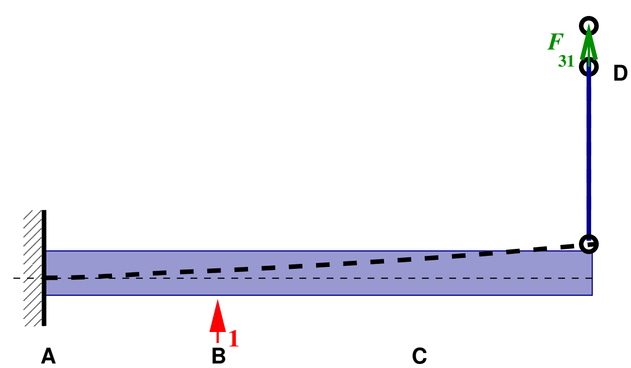
$$F_{12} = \int_0^L \frac{m_1 m_2}{EI} dx + \frac{n_1 n_2 H}{EA} = \delta_{12}$$

displacement at "1" due to unit force at "2"



$$F_{31} = \int_0^L \frac{m_3 m_1}{EI} dx + \frac{n_3 n_1 H}{EA} = \delta_{31}$$

displacement at "3" due to unit force at "1"



The fact that $F_{12} = F_{21}$ is called *Maxwell's Reciprocity Theorem.*

5 Introductory Example of the Stiffness Matrix Method

In this simple example, elements are springs with stiffness k. A spring with stiffness $k > 0$ connecting point i to point j, will have a force $f = k(d_j - d_i)$ where d_i is the displacement of point i and d_j is the displacement of point j. (Tension is positive so d_i points "into" the spring and d_j points "away" from the spring.)

The stiffness matrix for this structure can be found using equilibrium and force-deflection relationships ($f = kd$) for the springs.

#1: $\sum F_x = 0 : \quad f_1 - k_1 d_1 + k_2(d_2 - d_1) = 0$

#2: $\sum F_x = 0 : \quad f_2 - k_2(d_2 - d_1) - k_4 d_2 + k_3(d_3 - d_2) = 0$

#3: $\sum F_x = 0 : \quad f_3 - k_3(d_3 - d_2) - k_5 d_3 = 0$

In matrix form these three equations may be written:

$$\begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 + k_4 & -k_3 \\ 0 & -k_3 & k_3 + k_5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

The displacements are found by solving the stiffness matrix equation for $\mathbf{d}$, $\mathbf{d} = \mathbf{K}^{-1}\ \mathbf{f}$.

- The matrix $\mathbf{K}$ is called a *stiffness matrix.*
- All stiffness matrices are symmetric.

- All diagonal terms of all stiffness matrices are positive.
- Stiffness matrices are *diagonally dominant*. This means that the diagonal terms are usually larger than the off-diagonal term.
- If the structure is not free to translate or rotate without deforming, then the stiffness matrix is *positive definite*. This mathematical property guarantees that the stiffness matrix is invertible, and a unique set of displacements, $\mathbf{d}$, can be found by solving $\mathbf{K}\ \mathbf{d} = \mathbf{f}$.
- The total potential energy, U, in this system of springs is

$$U = \frac{1}{2}k_1 d_1^2 + \frac{1}{2}k_2(d_2 - d_1)^2 + \frac{1}{2}k_3(d_3 - d_2)^2 + \frac{1}{2}k_4 d_2^2 + \frac{1}{2}k_5 d_3^2 \ .$$

You should be able to confirm that this is equal to

$$U = \frac{1}{2}\ \mathbf{d}^T\ \mathbf{K}\ \mathbf{d}$$

Also, note that no matter what the values of the displacements, $\mathbf{d}$, may be, the energy U is always positive. The statement $\frac{1}{2}\mathbf{d}^T\mathbf{K}\mathbf{d} > 0\ \forall \mathbf{d} \neq 0$ is another way of saying that $\mathbf{K}$ is *positive-definite.*

- The set of forces required to deflect coordinate "i" by a deflection of "1 unit" equals the "i-th" column of the stiffness matrix. For example consider the case in which $d_1 = 1$, $d_2 = 0$, and $d_3 = 0$,

$$\left[\begin{array}{ccc} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3+k_4 & -k_3 \\ 0 & -k_3 & k_3+k_5 \end{array}\right] \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} k_1+k_2 \\ -k_2 \\ 0 \end{array}\right] \ ,$$

which is equal to the first column of the stiffness matrix.

This fact may be used to derive the stiffness matrix:

$d_1 = 1,\ d_2 = 0,\ d_3 = 0$

$f_1 = k_1 + k_2\ ,\quad f_2 = -k_2\ ,\quad f_3 = 0$ ... 1st column

$d_1 = 0,\ d_2 = 1,\ d_3 = 0$

$f_1 = -k_2\ ,\quad f_2 = k_2 + k_3 + k_4\ ,\quad f_3 = -k_3$ ... 2nd column

$d_1 = 0,\ d_2 = 0,\ d_3 = 1$

$f_1 = 0\ ,\quad f_2 = -k_3\ ,\quad f_3 = k_3 + k_5$ ... 3rd column

6 Basic Concepts of the Stiffness Matrix Method

The previous example illustrates some of the basic concepts needed to apply the stiffness matrix method to structures made out of bars and beams. There are, however, a few additional complications. Displacements in structures can be vertical, horizontal, or rotational, and structural bars and beams have a more complicated force-displacement relationships than those of simple springs.

In applying the matrix stiffness method of structural analysis, structures are described in terms of *elements* that connect *nodes* which can move in certain *coordinate* directions.

6.1 Elements

In the stiffness matrix method, structures are modeled as assemblies of elements such as bars, beams, cables, shafts, plates, and walls. Elements connect the nodes of the structural model. Like the simple springs in the previous example, structural elements have clearly defined, albeit more complicated, force-displacement relationships. The stiffness properties of structural elements can be determined from equilibrium equations, Castigliano's Theorems, the principle of minimum potential energy, and/or the principle of virtual work. Structural elements can be mathematically assembled with one another (like making a structural system using a set of tinker-toys), into a system of equations for the entire structure.

6.2 Nodes

The force-displacement relationship of a structural element is defined in terms of the forces and displacements at the nodes of the element. Nodes define the points where elements meet. The nodes in the model of a truss are at the joints between the truss bars. The nodes in the model of a beam or a frame are at the reaction locations, at locations at which elements connect to each other, and possibly at other intermediate locations.

6.3 Coordinates

Coordinates describe the location and direction at which forces and displacements act on an element or on a structure. Trusses are loaded with vertical and horizontal forces at the joints. The joints of a 2D truss can move vertically and horizontally; so there are two coordinates per node in a 2D truss. Beams and frames carry vertical and horizontal loads as well as bending moments. The nodes of a 2D frame can move vertically, horizontally, and can rotate; so there are three coordinates per node in a 2D frame.

Structural coordinates can be classified into two sets. *Displacement coordinates* have unknown displacements but know forces. *Reaction coordinates* have unknown forces but known displacements (usually zero).

6.4 Elements, Nodes and Coordinates

Planar (2D) truss bar elements have two nodes and four coordinates, two at each end.

Space (3D) truss bar elements have two nodes and six coordinates, three at each end.

Planar (2D) frame elements have two nodes and six coordinates, three at each end.

Space (3D) frame elements have two nodes and twelve coordinates, six at each end.

6.5 Structural Nodes and Coordinates

Planar (2D) truss nodes and coordinates

Planar (2D) frame nodes and coordinates

7 Relate the Flexibility Matrix to the Stiffness Matrix

- Column "j" of the stiffness matrix: The set of forces at all coordinates required to produce a unit displacement at coordinate "j"

- Column "j" of the flexibility matrix: The set of displacements at all coordinates resulting from a unit force at coordinate "j"

- Stiffness matrix equation:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2n} \\ K_{31} & K_{32} & K_{33} & \cdots & K_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & K_{n3} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix}$$

- Flexibility matrix equation:

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & \cdots & F_{1n} \\ F_{21} & F_{22} & F_{23} & \cdots & F_{2n} \\ F_{31} & F_{32} & F_{33} & \cdots & F_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{n1} & F_{n2} & F_{n3} & \cdots & F_{nn} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix}$$

- $\mathbf{K} = \mathbf{F}^{-1}$ and $\mathbf{K}^{-1} = \mathbf{F}$

- A useful fact for 2-by-2 matrices ...

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

... you should be able to prove this fact to yourself.