Frame Element Stiffness Matrices

CEE 421L. Matrix Structural Analysis
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Truss elements carry axial forces only. Beam elements carry shear forces and bending moments. Frame elements carry shear forces, bending moments, and axial forces. This document picks up with the previously-derived truss and beam element stiffness matrices in local element coordinates and proceeds through frame element stiffness matrices in global coordinates.

1 Frame Element Stiffness Matrix in Local Coordinates, $\mathbf{k}$

A frame element is a combination of a truss element and a beam element. The forces and displacements in the local axial direction are independent of the shear forces and bending moments.

$$
\left\{ \begin{array}{c} N_1 \\ V_1 \\ M_1 \\ N_2 \\ V_2 \\ M_2 \end{array} \right\} = \left\{ \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{array} \right\} = \left[\begin{array}{cccccc} \frac{EA}{L} & & & & & \\ 0 & \frac{12EI}{L^3} & & & \text{SYM} & \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & & & \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & & \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \right\}
$$

2 Relationships between Local Coordinates and Global Coordinates: $\mathbf{T}$

The geometric relationship between local displacements, $\mathbf{u}$, and global displacements, $\mathbf{v}$, is

$$u_1 = v_1 \cos\theta + v_2 \sin\theta \qquad u_2 = -v_1 \sin\theta + v_2 \cos\theta \qquad u_3 = v_3$$

or, $\mathbf{u} = \mathbf{T}\ \mathbf{v}$.

The equilibrium relationship between local forces, $\mathbf{q}$, and global forces, $\mathbf{f}$, is

$$q_1 = f_1 \cos\theta + f_2 \sin\theta \qquad q_2 = -f_1 \sin\theta + f_2 \cos\theta \qquad q_3 = f_3$$

or, $\mathbf{q} = \mathbf{T}\ \mathbf{f}$, where, in both cases,

$$\mathbf{T} = \left[\begin{array}{cccccc} c & s & 0 & & & \\ -s & c & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ & & & c & s & 0 \\ & 0 & & -s & c & 0 \\ & & & 0 & 0 & 1 \end{array} \right] \qquad c = \cos\theta = \frac{x_2 - x_1}{L} \qquad s = \sin\theta = \frac{y_2 - y_1}{L}$$

The coordinate transformation matrix, $\mathbf{T}$, is orthogonal, $\mathbf{T}^{-1} = \mathbf{T}^{\mathsf{T}}$.

3 Frame Element Stiffness Matrix in Global Coordinates: K

Combining the coordinate transformation relationships,

$$\begin{aligned}\mathbf{q} &= \mathbf{k}\,\mathbf{u}\\ \mathbf{T}\,\mathbf{f} &= \mathbf{k}\,\mathbf{T}\,\mathbf{v}\\ \mathbf{f} &= \mathbf{T}^{\mathsf{T}}\,\mathbf{k}\,\mathbf{T}\,\mathbf{v}\\ \mathbf{f} &= \mathbf{K}\,\mathbf{v}\end{aligned}$$

which provides the force-deflection relationships in global coordinates. The stiffness matrix in global coordinates is $\mathbf{K} = \mathbf{T}^{\mathsf{T}}\,\mathbf{k}\,\mathbf{T}$

$$\mathbf{K} = \begin{bmatrix} \frac{EA}{L}c^2 + \frac{12EI}{L^3}s^2 & \frac{EA}{L}cs - \frac{12EI}{L^3}cs & -\frac{6EI}{L^2}s & -\frac{EA}{L}c^2 - \frac{12EI}{L^3}s^2 & -\frac{EA}{L}cs + \frac{12EI}{L^3}cs & -\frac{6EI}{L^2}s \\ & \frac{EA}{L}s^2 + \frac{12EI}{L^3}c^2 & \frac{6EI}{L^2}c & -\frac{EA}{L}cs + \frac{12EI}{L^3}cs & -\frac{EA}{L}s^2 - \frac{12EI}{L^3}c^2 & \frac{6EI}{L^2}c \\ & & \frac{4EI}{L} & \frac{6EI}{L^2}s & -\frac{6EI}{L^2}c & \frac{2EI}{L} \\ & & & \frac{EA}{L}c^2 + \frac{12EI}{L^3}s^2 & \frac{EA}{L}cs - \frac{12EI}{L^3}cs & \frac{6EI}{L^2}s \\ & \text{SYM} & & & \frac{EA}{L}s^2 + \frac{12EI}{L^3}c^2 & -\frac{6EI}{L^2}c \\ & & & & & \frac{4EI}{L} \end{bmatrix}$$

4 Frame Element Stiffness Matrices for Elements with End-Releases

Some elements in a frame may not be fixed at both ends. For example, an element may be fixed at one end and pinned at the other. Or, the element may be guided on one end so that the element shear forces at that end are zero. Or, the frame element may be pinned at both ends, so that it acts like a truss element. Such modifications to the frame element naturally affect the elements stiffness matrix.

Consider a frame element in which a set of end-coordinates r are released, and the goal is to find a stiffness matrix relation for the primary p retained coordinates. The element end forces at the released coordinates, $\mathbf{q}_r$ are all zero. One can partition the element stiffness matrix equation as follows

$$\left[\begin{array}{c} \mathbf{q}_p \\ \mathbf{q}_r \end{array}\right] \left[\begin{array}{cc} \mathbf{k}_{pp} & \mathbf{k}_{pr} \\ \mathbf{k}_{rp} & \mathbf{k}_{rr} \end{array}\right] \left[\begin{array}{c} \mathbf{u}_p \\ \mathbf{u}_r \end{array}\right]$$

The element displacement coordinates at the released coordinates do not equal the structural displacements at the collocated structural coordinates, since the coordinates r are released. Since the element end forces at the released coordinates are all zero ($\mathbf{q}_r = 0$), the element end displacements at the released coordinates must be related to the displacements at the primary (retained) coordinates as:

$$\mathbf{u}_r = -\mathbf{k}_{rr}^{-1}\mathbf{k}_{rp}\mathbf{u}_p$$

The element stiffness matrix equation relating $\mathbf{q}_p$ and $\mathbf{u}_p$ is

$$\mathbf{q}_p = \left[\mathbf{k}_{pp} - \mathbf{k}_{pr}\mathbf{k}_{rr}^{-1}\mathbf{k}_{rp}\right]\mathbf{u}_p$$

The rows and columns of the released element stiffness matrix corresponding to the released coordinates, r, are set to zero. The rows and columns of the released element stiffness matrix corresponding to the retained coordinates, p, are $[\mathbf{k}_{pp} - \mathbf{k}_{pr}\mathbf{k}_{rr}^{-1}\mathbf{k}_{rp}]$. This is the element stiffness matrix that should assemble into the structural coordinates collocated with the primary (retained) coordinates p. The following sections give examples for pinned-fixed and fixed-pinned frame elements. Element stiffness matrices for many other end-release cases can be easily computed.

4.1 Pinned-Fixed Frame Element in Local Coordinates, $\mathbf{k}\ldots(r=3)$

$$
\mathbf{k} = \left[\begin{array}{cccccc}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
 & \frac{3EI}{L^3} & 0 & 0 & -\frac{3EI}{L^3} & \frac{3EI}{L^2} \\
 & & 0 & 0 & 0 & 0 \\
 & & & \frac{EA}{L} & 0 & 0 \\
 & \text{SYM} & & & \frac{3EI}{L^3} & -\frac{3EI}{L^2} \\
 & & & & & \frac{3EI}{L}
\end{array} \right]
$$

4.2 Pinned-Fixed Frame Element in Global Coordinates, $\mathbf{K} = \mathbf{T}^{\mathsf{T}}\mathbf{k}\mathbf{T}$

$$
\mathbf{K} = \left[\begin{array}{cccccc}
\begin{array}{c} \frac{EA}{L}c^2 \\ +\frac{3EI}{L^3}s^2 \end{array} & \begin{array}{c} \frac{EA}{L}cs \\ -\frac{3EI}{L^3}cs \end{array} & 0 & \begin{array}{c} -\frac{EA}{L}c^2 \\ -\frac{3EI}{L^3}s^2 \end{array} & \begin{array}{c} -\frac{EA}{L}cs \\ +\frac{3EI}{L^3}cs \end{array} & -\frac{3EI}{L^2}s \\
 & \begin{array}{c} \frac{EA}{L}s^2 \\ +\frac{3EI}{L^3}c^2 \end{array} & 0 & \begin{array}{c} -\frac{EA}{L}cs \\ +\frac{3EI}{L^3}cs \end{array} & \begin{array}{c} -\frac{EA}{L}s^2 \\ -\frac{3EI}{L^3}c^2 \end{array} & \frac{3EI}{L^2}c \\
 & & 0 & 0 & 0 & 0 \\
 & & & \begin{array}{c} \frac{EA}{L}c^2 \\ +\frac{3EI}{L^3}s^2 \end{array} & \begin{array}{c} \frac{EA}{L}cs \\ -\frac{3EI}{L^3}cs \end{array} & \frac{EI}{L^2}s \\
 & \text{SYM} & & & \begin{array}{c} \frac{EA}{L}s^2 \\ \frac{3EI}{L^3}c^2 \end{array} & -\frac{3EI}{L^2}c \\
 & & & & & \frac{3EI}{L}
\end{array} \right]
$$

4.3 Fixed-Pinned Frame Element in Local Coordinates, $\mathbf{k}\ldots(r=6)$

$$
\mathbf{k} = \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
 & \frac{3EI}{L^3} & \frac{3EI}{L^2} & 0 & -\frac{3EI}{L^3} & 0 \\
 & & \frac{3EI}{L} & 0 & -\frac{3EI}{L^2} & 0 \\
 & & & \frac{EA}{L} & 0 & 0 \\
 & \text{SYM} & & & \frac{3EI}{L^3} & 0 \\
 & & & & & 0
\end{bmatrix}
$$

4.4 Fixed-Pinned Frame Element in Global Coordinates, $\mathbf{K} = \mathbf{T}^{\mathsf{T}}\mathbf{k}\mathbf{T}$

$$
\mathbf{K} = \begin{bmatrix}
\frac{EA}{L}c^2 + \frac{3EI}{L^3}s^2 & \frac{EA}{L}cs - \frac{3EI}{L^3}cs & -\frac{3EI}{L^2}s & -\frac{EA}{L}c^2 - \frac{3EI}{L^3}s^2 & -\frac{EA}{L}cs + \frac{3EI}{L^3}cs & 0 \\
 & \frac{EA}{L}s^2 + \frac{3EI}{L^3}c^2 & \frac{3EI}{L^2}c & -\frac{EA}{L}cs + \frac{3EI}{L^3}cs & -\frac{EA}{L}s^2 - \frac{3EI}{L^3}c^2 & 0 \\
 & & \frac{3EI}{L} & \frac{3EI}{L^2}s & -\frac{3EI}{L^2}c & 0 \\
 & & & \frac{EA}{L}c^2 + \frac{3EI}{L^3}s^2 & \frac{EA}{L}cs - \frac{3EI}{L^3}cs & 0 \\
 & \text{SYM} & & & \frac{EA}{L}s^2 + \frac{3EI}{L^3}c^2 & 0 \\
 & & & & & 0
\end{bmatrix}
$$

5 Notation

$\mathbf{u}$ = Element deflection vector in the Local coordinate system
$\mathbf{q}$ = Element force vector in the Local coordinate system
$\mathbf{k}$ = Element stiffness matrix in the Local coordinate system
... $\mathbf{q} = \mathbf{k}\ \mathbf{u}$
$\mathbf{T}$ = Coordinate Transformation Matrix
... $\mathbf{T}^{-1} = \mathbf{T}^\mathsf{T}$
$\mathbf{v}$ = Element deflection vector in the Global coordinate system
... $\mathbf{u} = \mathbf{T}\ \mathbf{v}$
$\mathbf{f}$ = Element force vector in the Global coordinate system
... $\mathbf{q} = \mathbf{T}\ \mathbf{f}$
$\mathbf{K}$ = Element stiffness matrix in the Global coordinate system
... $\mathbf{K} = \mathbf{T}^\mathsf{T}\ \mathbf{k}\ \mathbf{T}$
$\mathbf{d}$ = Structural deflection vector in the Global coordinate system
$\mathbf{p}$ = Structural load vector in the Global coordinate system
$\mathbf{K_s}$ = Structural stiffness matrix in the Global coordinate system
... $\mathbf{p} = \mathbf{K_s}\ \mathbf{d}$

	Local	**Global**
Element Deflection	$\mathbf{u}$	$\mathbf{v}$
Element Force	$\mathbf{q}$	$\mathbf{f}$
Element Stiffness	$\mathbf{k}$	$\mathbf{K}$
Structural Deflection	-	$\mathbf{d}$
Structural Loads	-	$\mathbf{p}$
Structural Stiffness	-	$\mathbf{K_s}$

For frame element stiffness matrices including shear deformations, see:
J.S. Przemieniecki, *Theory of Matrix Structural Analysis,* Dover Press, 1985.
(... a steal at $12.95)