Geometric Stiffness Effects in 2D and 3D Frames

CEE 421L. Matrix Structural Analysis
Department of Civil and Environmental Engineering
Duke University

Henri Gavin

Fall, 2012

In situations in which deformations are not *infinitesimally*[1] small, linear elastic analyses may not capture the true structural response. For large-strain problems, results of *finite*[2] deformation analysis is significantly more accurate than linear, infinitesimal deformation analysis. Incorporating the details of finite deformation, the analysis may also be applied to a buckling analysis of the structural system.

In the derivation of the linear elastic stiffness matrix for frame elements, the potential energy function includes strain energy due to bending, axial and shear deformation effects. Axial effects are decoupled from shear and bending effects in the resulting linear elastic stiffness matrices.[3] In finite deformation analysis, on the other hand, the potential energy function includes additional terms, which accounts for the interaction between the axial load effects on the frame element and the lateral deformation of the frame element. These effects are often called "$P - \Delta$ effects".

We will separate the potential energy function U into an elastic part U_{E} (which contains the infinitesimal strain energy) and a geometric part, U_{G} (which includes the interaction of lateral deformations and axial loads). The linear elastic strain energy results in the same frame element stiffness matrices $\mathbf{k}_{\mathrm{E}}$ that we have found previously. So, this document focuses only on the geometric component of the potential energy function. From this geometric part of the potential energy, we will derive the geometric stiffness matrix $\mathbf{k}_{\mathrm{G}}$.

As in the finite deformation analysis of trusses, we need to know the deformation of the structure in order to find the internal axial loads, but we need to know the internal axial loads to determine the geometric stiffness matrix and the deformations. This "chicken-and-egg" problem can be solved with the same type of Newton-Raphson iteration approach which we used previously for finite deformation analysis of trusses.

---

[1] **infinitesimal** means arbitrarily close to zero, as in infinitesimal calculus.

[2] **finite** means neither infinite nor infinitesimal, as in a finite distance.

[3] The rows and columns corresponding to axial effects (1st and 4th) have non-zero elements only in the 1st and 4th columns and rows. Also, the rows and columns corresponding to bending and shear effects have non-zero elements only in the 2nd, 3rd, 5th, and 6th columns and rows.

1 Deformed shape of a frame element in bending

To start with, we need to introduce the deformed shape of a frame element. The deformed shape of a frame element, $h(x)$, subjected to end-forces, $\mathbf{q}$, is a cubic polynomial. A cubic polynomial may be written in a power-polynomial form:

$$h(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3. \tag{1}$$

Likewise, the slope of the beam, $h'(x)$ may be expressed

$$h'(x) = a_1 + 2a_2 \cdot x + 3a_3 \cdot x^2. \tag{2}$$

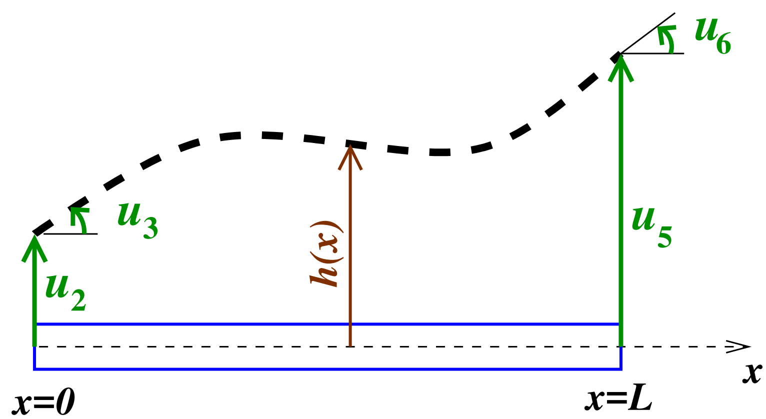


Figure 1. The deformed shape of a beam, $h(x)$ is assumed to be a cubic function of x.

The four polynomial coefficients, $a_0, \cdots, a_3$, are found by matching $h(0)$, $h'(0)$, $h(L)$ and $h'(L)$ to the specified end displacements and rotations of the beam. Assuming small rotations, $\tan\theta \approx \theta$, and neglecting shear deformation effects,

$$\begin{aligned} h(0) = u_2 &\;\rightarrow\; a_0 = u_2 \\ h'(0) = u_3 &\;\rightarrow\; a_1 = u_3 \\ h(L) = u_5 &\;\rightarrow\; a_0 + a_1 L + a_2 L^2 + a_3 L^3 = u_5 \\ h'(L) = u_6 &\;\rightarrow\; a_1 + 2a_2 L + 3a_3 L^2 = u_6 \end{aligned}$$

These four equations with four unknowns $(a_0, \cdots, a_3)$ have the solution

$$a_0 = u_2 \tag{3}$$

$$a_1 = u_3 \tag{4}$$

$$a_2 = 3(u_5 - u_2)/L^2 - (2u_3 + u_6)/L \tag{5}$$

$$a_3 = -2(u_5 - u_2)/L^3 + (u_3 + u_6)/L^2. \tag{6}$$

You should be able to confirm this solution for the polynomial coefficients. Note that the cubic deformation function $h(x)$ may also be written as a weighted sum of cubic polynomials.

$$h(x) = u_2 \cdot b_2(x) + u_3 \cdot b_3(x) + u_5 \cdot b_5(x) + u_6 \cdot b_6(x), \tag{7}$$

The "weights" u_i are simply the set of local element displacements and the functions $b_i(x)$ are each cubic functions in x. These cubic *shape functions* represent beam deformations due to a unit applied displacement in the corresponding coordinate (only). Neglecting shear deformation effects, the frame element shape functions are the Hermite polynomials,

$$\begin{array}{lllll}
b_2(x) = 1 - 3\,(x/L)^2 + 2\,(x/L)^3 & b_2(0) = 1 & b_2'(0) = 0 & b_2(L) = 0 & b_2'(L) = 0 \\
b_3(x) = x\,(1 - x/L)^2 & b_3(0) = 0 & b_3'(0) = 1 & b_3(L) = 0 & b_3'(L) = 0 \\
b_5(x) = 3\,(x/L)^2 - 2\,(x/L)^3 & b_5(0) = 0 & b_5'(0) = 0 & b_5(L) = 1 & b_5'(L) = 0 \\
b_6(x) = (x/L)^2\,(x/L - 1) & b_6(0) = 0 & b_6'(0) = 0 & b_6(L) = 0 & b_6'(L) = 1
\end{array}$$

Equations (1) and (7) are two different ways of expressing *exactly* the same equation, $h(x)$. The finite element method makes use of polynomials in the form of equation (7). To complete the picture, for axial deformations, (which contribute to transverse deformations only through the axial load in the geometric stiffness matrix),

$$\begin{aligned}
b_1(x) &= (1 - x/L) \\
b_4(x) &= (x/L)\,.
\end{aligned}$$

You should confirm that with the given definitions of $b_i(x)$, and the coefficients a_i, that equations (1) and (7) are equivalent.

We now relate this polynomial for the assumed deformation shape to a potential energy function. Recall the internal elastic strain energy of a beam due to bending effects,

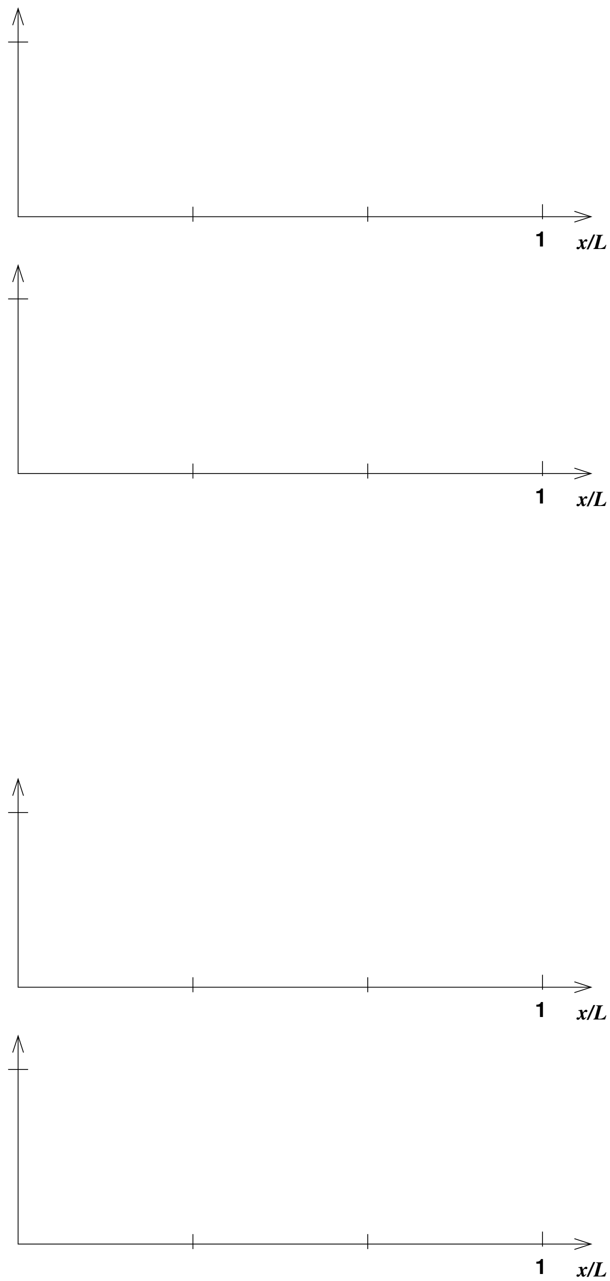
$$U_{\mathrm{B}} = \frac{1}{2}\int_0^L \frac{M^2(x)}{EI}\;dx. \tag{8}$$

Since the curvature of the beam is $M(x)/(EI)$ and assuming infinitesimal deformation, $h''(x)$ is practically the same as the curvature, and

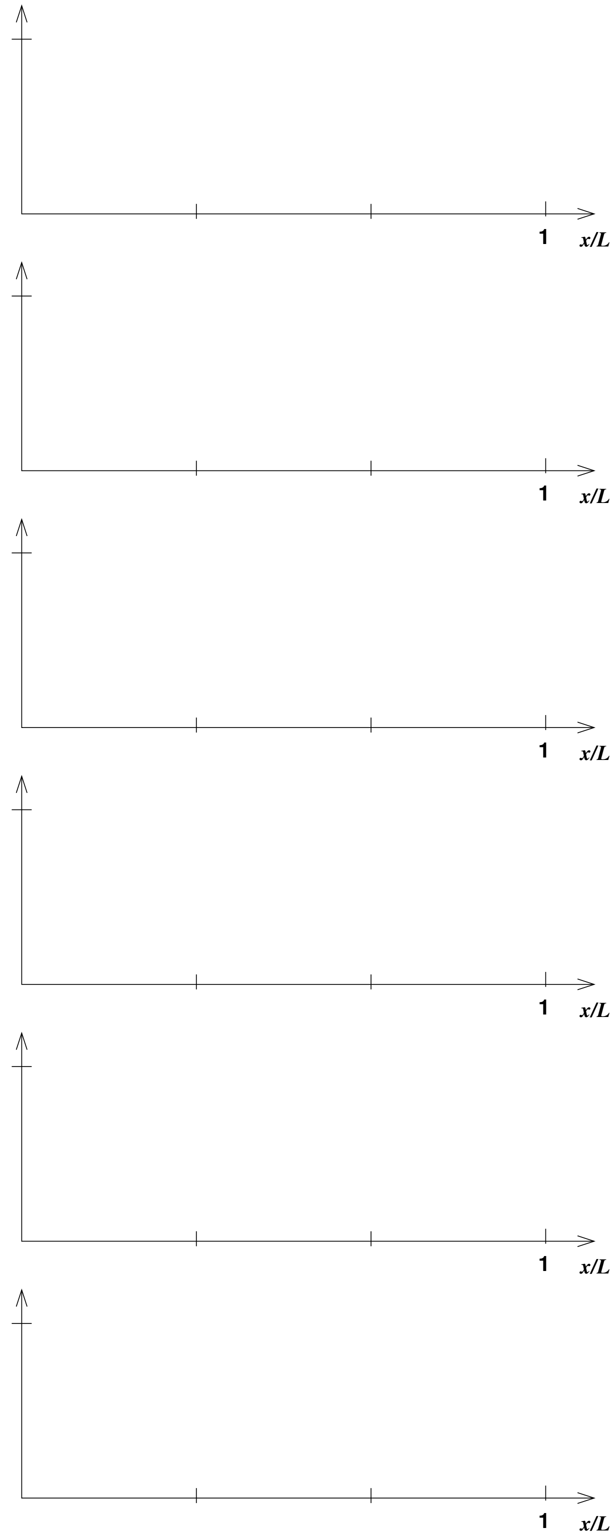
$$\begin{aligned}
U_{\mathrm{B}} &= \frac{1}{2}\int_0^L M(x) h''(x)\;dx && (9)\\
&= \frac{1}{2}\;EI\int_0^L h''(x) \cdot h''(x)\;dx\;. && (10)
\end{aligned}$$

Equation (8) is an expression of the strain energy in terms of the internal bending moment; equation (9) is an expression of the strain energy in terms of the internal bending moment and the assumed cubic deformation function; and equation (10) is an expression of the strain energy in terms of the assumed cubic deformation function only. In the finite element method it is common to express the potential energy using forms like equation (10). If the internal bending moment $M(x)$ does indeed generate the assumed deformation function $h(x)$ then all three forms of the elastic strain energy are exactly equivalent and completely interchangeable.

Power polynomial basis



Hermite polynomial basis



2 Finite deformation effects in transversely-displaced frame elements

Frame elements carrying large axial loads or undergoing large displacements have nonlinear behavior arising from the internal moments that are the product of the axial loads P and the displacements transverse to the loads, Δ. This kind of nonlinearity is sometimes called the "$P\Delta$" effect.

For frame elements with transverse rotation $h'(x)$, but without displacement along the local axial coordinate, x, the frame element undergoes axial deformation $u'(x)$. Figure 2 shows how transverse rotation $h'(x)$ relate to axial deformation $u'(x)$ when there is no displacement along the local element axial coordinate. These are related using the Pythagorean theorem:

$$\begin{aligned}
(dx)^2 &= (dx - du)^2 + (dh)^2 \\
(dx)^2 &= (dx)^2 - 2(du)(dx) + (du)^2 + (dh)^2 \\
2(du)(dx) &= (du)^2 + (dh)^2 \\
\frac{du}{dx} &= \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dh}{dx}\right)^2
\end{aligned}$$

For frame elements in which axial strain is much less than transverse rotations, which is always the case in elastic frame elements made from common structural materials, $(u')^2 \ll (h')^2$ and $u'(x) \approx (1/2)(h'(x))^2$. The geometric deformation increases with the square of the rotation of the element.

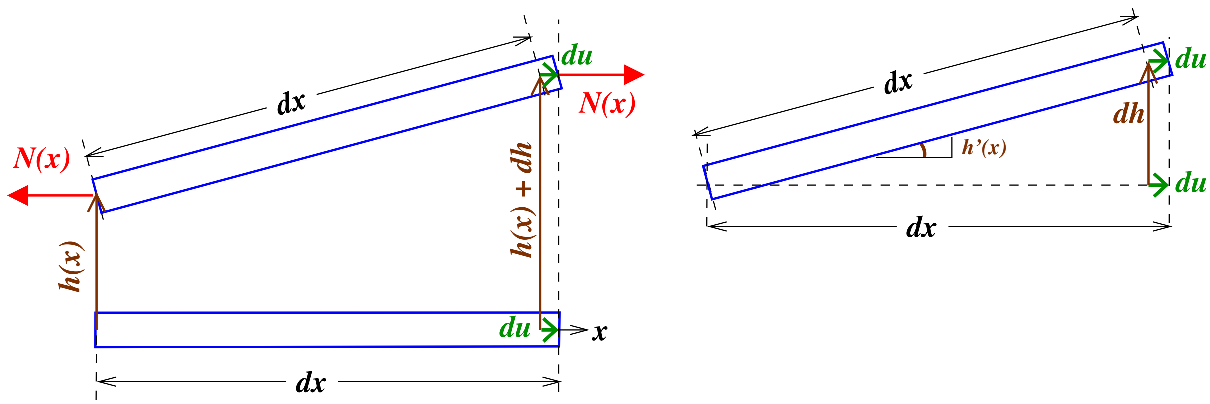


Figure 2. Axial end displacements due to transverse displacements, without axial deformation.

To determine the strain energy associated with axial forces $N(x)$ and geometric deformation $u'(x) = (1/2)(h'(x))^2$, recall that the work of a constant force F moving through a displacement d, is simply Fd. Since $N(x)$ does not increase linearly with $h(x)$, the associated internal strain energy of a force $N(x)$ distributed along the length of a frame element and geometric deformation $u'(x) = (1/2)(h'(x))^2$ associated with transverse displacements $h(x)$

$$U_G = \int_0^L N(x)\frac{du(x)}{dx}\, dx = \frac{1}{2}\int_0^L N(x)\left(\frac{dh(x)}{dx}\right)^2\, dx. \tag{11}$$

3 Geometric stiffness of frame elements

The previous section shows that the potential energy due to axial loads, $N(x)$ and transverse displacements, $h(x)$, is

$$U_{\mathrm{G}} = \frac{1}{2}\int_0^L N(x) \left(\frac{dh}{dx}\right)^2 \, dx \; . \tag{12}$$

If the axial load $N(x)$ is constant over the length of the beam, $N(x) = T$, and

$$U_{\mathrm{G}} = \frac{1}{2} \, T \int_0^L (h'(x))^2 \, dx \; . \tag{13}$$

Substituting the beam shape functions, equation (7), and carrying out the integral leads to the potential energy function in terms of transverse end displacements, u_2 and u_5, and end rotations, u_3 and u_6,

$$\begin{aligned} U_{\mathrm{G}} = \tfrac{T}{30L} \;\; & (-Lu_3u_6 - 3u_5Lu_3 - 3u_5Lu_6 + 3u_2Lu_3 + 3u_2Lu_6 \\ & +18u_5^2 - 36u_5u_2 + 18u_2^2 + 2L^2u_3^2 + 2L^2u_6^2) \; . \end{aligned} \tag{14}$$

Invoking Castigliano's theorem, the partial derivative of the potential energy function with respect to a displacement coordinate is the force in the direction of that displacement coordinate. The end forces due to geometric stiffness effects can then be found as follows:

$$\begin{aligned}
q_2 &= \frac{\partial U_{\mathrm{G}}}{\partial u_2} = \frac{T}{30L}(36u_2 + 3Lu_3 - 36u_5 + 3Lu_6) \\
q_3 &= \frac{\partial U_{\mathrm{G}}}{\partial u_3} = \frac{T}{30L}(3Lu_2 + 4L^2u_3 - 3Lu_5 - L^2u_6) \\
q_5 &= \frac{\partial U_{\mathrm{G}}}{\partial u_5} = \frac{T}{30L}(-36u_2 - 3Lu_3 + 36u_5 - 3Lu_6) \\
q_6 &= \frac{\partial U_{\mathrm{G}}}{\partial u_6} = \frac{T}{30L}(3Lu_2 - L^2u_3 - 3Lu_5 + 4L^2u_6).
\end{aligned}$$

Writing these expressions in matrix form, we arrive at the geometric stiffness matrix for a frame element:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \frac{T}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2L^2}{15} & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{L}{10} & 0 & \frac{6}{5} & -\frac{L}{10} \\ 0 & \frac{L}{10} & -\frac{L^2}{30} & 0 & -\frac{L}{10} & -\frac{2L^2}{15} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}, \tag{15}$$

where the tension in the beam is given by $T = EA(u_4 - u_1)/L$. The geometric stiffness matrix for a 2D (planar) frame element in local coordinates is:

$$\mathbf{k}_{\mathrm{G}} = \frac{T}{L} \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & \frac{-6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2L^2}{15} & 0 & \frac{-L}{10} & \frac{-L^2}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-6}{5} & \frac{-L}{10} & 0 & \frac{6}{5} & \frac{-L}{10} \\ 0 & \frac{L}{10} & \frac{-L^2}{30} & 0 & \frac{-L}{10} & \frac{2L^2}{15} \end{array} \right] . \tag{16}$$

The coordinate transformation process is identical to the process carried out before for the elastic element stiffness matrix. The coordinate transformation matrix, $\mathbf{T}$, is

$$\mathbf{T} = \left[\begin{array}{cccccc} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] , \tag{17}$$

where s and c are the sine and cosine of the counter-clockwise angle from global element coordinate number 1 to the frame element. Here we are making the approximation that the deformed inclination of the frame element is approximately the same as the original inclination of the frame element. The element stiffness matrix in global coordinates is found by applying the coordinate transformation matrix.

$$\mathbf{K}_{\mathrm{G}} = \mathbf{T}^{\mathsf{T}}\ \mathbf{k}_{\mathrm{G}}\ \mathbf{T} = \frac{T}{L} \left[\begin{array}{cccccc} \frac{6}{5}s^2 & \frac{-6}{5}sc & \frac{-L}{10}s & \frac{-6}{5}s^2 & \frac{6}{5}sc & \frac{-L}{10}s \\ \frac{-6}{5}sc & \frac{6}{5}c^2 & \frac{L}{10}c & \frac{6}{5}sc & \frac{-6}{5}c^2 & \frac{L}{10}c \\ \frac{-L}{10}s & \frac{L}{10}c & \frac{2L^2}{15} & \frac{L}{10}s & \frac{-L}{10}c & \frac{-L^2}{30} \\ \frac{-6}{5}s^2 & \frac{6}{5}sc & \frac{L}{10}s & \frac{6}{5}s^2 & \frac{-6}{5}sc & \frac{L}{10}s \\ \frac{6}{5}sc & \frac{-6}{5}c^2 & \frac{-L}{10}c & \frac{-6}{5}sc & \frac{6}{5}c^2 & \frac{-L}{10}c \\ \frac{-L}{10}s & \frac{L}{10}c & \frac{-L^2}{30} & \frac{L}{10}s & \frac{-L}{10}c & \frac{2L^2}{15} \end{array} \right] \tag{18}$$

It is not hard to confirm this expression for $\mathbf{K}_{\mathrm{G}}$, and you should feel encouraged to do so. The assembly of the structural stiffness matrix $\mathbf{K_s}$ with elastic and geometric effects proceeds exactly as with the elastic stiffness matrix.

4 Derivation of stiffness coefficients directly from the strain energy function

It is common to derive the coefficients of a stiffness matrix directly from the strain energy function. Note that the i, j component of the stiffness matrix is

$$k_{ij} = \frac{\partial}{\partial u_j} q_i = k_{ji} = \frac{\partial}{\partial u_i} q_j \ ,$$

and that the i^{th} component of the end force, q_i, is

$$q_i = \frac{\partial}{\partial u_i} U \ .$$

Therefore, the stiffness coefficients may be written

$$k_{ij} = \frac{\partial^2 \, U}{\partial u_i \, \partial u_j} \ .$$

If the stiffness matrix to be determined is for bending effects only, then, as seen before,

$$U = U_{\mathrm{B}} = \frac{1}{2} \ EI \int_0^L h''(x) \cdot h''(x) \ dx \ .$$

Now, since integration and differentiation are both linear operations, it does not matter which is done first, integration or differentiation. Therefore, the stiffness coefficient may be written,

$$k_{\mathrm{E}_{ij}} = \frac{1}{2} \ EI \int_0^L \frac{\partial h''(x)}{\partial u_i} \cdot \frac{\partial h''(x)}{\partial u_j} \ dx \ .$$

The elastic stiffness matrix incorporating bending effects only may be determined directly from this expression. Likewise, the geometric stiffness matrix may be determined directly from

$$k_{\mathrm{G}_{ij}} = \frac{1}{2} \ T \int_0^L \frac{\partial h'(x)}{\partial u_i} \cdot \frac{\partial h'(x)}{\partial u_j} \ dx \ .$$

The coefficients for the elastic stiffness matrix and the geometric stiffness matrix for frame elements in three dimensions are derived using this method in the remaining sections of this document.

5 Cubic shape functions for beams including shear deformations

Consider the twelve local coordinates of a three dimensional frame element. The transverse

Figure 3. The twelve local coordinates of a three-dimensional frame element.

displacements in the local $x-y$ plane, $h_y(x)$, of an elastic beam may be separated into a shear-related component, $h_s(x)$ and a bending-related component $h_b(x)$,

$$h_y(x) = h_s(x) + h_b(x). \tag{19}$$

The shear force at the end of the beam in the local y direction, q_2 may be found in terms of the beam end-displacements in the local y direction and the end-rotations about the local z axis,

$$q_2 = k_{22}\ u_2 + k_{26}\ u_6 + k_{28}\ u_8 + k_{2\ 12}\ u_{12}\ .$$

The effective shear strain is simply

$$h'_s(x) = -\frac{q_2}{GA_{sy}} = -\frac{1}{GA_{sy}}(k_{22}\ u_2 + k_{26}\ u_6 + k_{28}\ u_8 + k_{2\ 12}\ u_{12})\ . \tag{20}$$

The internal bending moment, $M_z(x)$, due to the effects of the end displacements and end rotations is simply

$$M_z(x) = q_2\ x - q_6,$$

and the curvature is approximately

$$\begin{aligned} h''_b(x) \ &= \tfrac{1}{EI_z} \ \ [\ (k_{22}\ u_2 + k_{26}\ u_6 + k_{28}\ u_8 + k_{2\ 12}\ u_{12})\ x \\ &\qquad -(k_{62}\ u_2 + k_{66}\ u_6 + k_{68}\ u_8 + k_{6\ 12}\ u_{12})\ ]. \end{aligned} \tag{21}$$

in which small angles are assumed. By computing the potential energy function for shear and bending deformations,

$$U = \frac{1}{2}EI_z\int_0^L (h_b''(x))^2 \ dx + \frac{1}{2}GA_{sy}\int_0^L (h_s'(x))^2 \ dx,$$

and taking the partial derivatives of the potential energy function with respect to the displacements coordinates u_2, u_6, u_8 and u_{12}, rows 2, 6, 8, and 12 of the elastic stiffness matrix may be computed.

The shape of the deformed beam may therefore by found by integrating equations (20) and (21) to obtain $h_s(x)$ and $h_b(x)$ and by solving for the constants of integration using the end conditions. So doing,

$$h_y'(x) = h_s'(x) + h_b'(x) = -\frac{1}{GA_{sy}}q_2 + \frac{1}{EI_z}\left(\frac{1}{2}q_2x^2 - q_6x + C_1\right).$$

Inserting the end condition $h_y'(0) = u_6$ the constant of integration, C_1, is EI_zu_6. Integrating again,

$$h_y(x) = h_s(x) + h_b(x) = -\frac{1}{GA_{sy}}q_2x + \frac{1}{EI_z}\left(\frac{1}{6}q_2x^3 - \frac{1}{2}q_6x^2\right) + u_6x + C_2.$$

Inserting the boundary condition $h(0) = u_2$, and noting that $1/(GA_{sy}) = \Phi_yL^2/(12EI_z)$, the deformed shape of the beam becomes

$$h_y(x) = \frac{1}{EI_z}\left(\frac{1}{6}q_2 \ x^3 - \frac{1}{2}q_6 \ x^2 - \frac{1}{12}\Phi_yL^2q_2 \ x\right) + u_6 \ x + u_2 \ .$$

The transverse deformation shape function for a beam with bending and shear deformation in the $x - y$ plane.

$$\begin{aligned} h_y(x) \ = \tfrac{1}{L^3}\tfrac{1}{1+\Phi_y} \ \{ \ & [2x^3 - 3Lx^2 - \Phi_yL^2x + L^3(1+\Phi_y)] \ u_2 + \\ & [Lx^3 - L^2(2+\Phi_y/2)x^2 + L^3(1+\Phi_y/2)x] \ u_6 + \\ & [-2x^3 + 3Lx^2 + \Phi_yL^2x] \ u_8 + \\ & [Lx^3 - L^2(1-\Phi_y/2)x^2 - \Phi_yL^3x/2] \ u_{12} \ \} \ . \end{aligned} \tag{22}$$

For bending and shear deformations in the $x-z$ plane, the shape function may be found using an analogous method, while respecting the right-hand coordinate system. The transverse deflection, $h_z(x)$ will consist of shear and bending components,

$$h_z(x) = h_s(x) + h_b(x) \ .$$

The end-shear force and the end-bending moment arise for end displacements in the local z direction and end moments about the local y axis,

$$q_3 = k_{33} \ u_3 + k_{35} \ u_5 + k_{39} \ u_9 + k_{3\ 11} \ u_{11} \ ,$$

and

$$q_5 = k_{53}\ u_3 + k_{55}\ u_5 + k_{59}\ u_9 + k_{5\ 11}\ u_{11}\ .$$

The effective shear strain is

$$h_s'(x) = -\frac{q_3}{GA_{sz}} = -\frac{1}{GA_{sz}}(\ k_{33}\ u_3 + k_{35}\ u_5 + k_{39}\ u_9 + k_{3\ 11}\ u_{11}\ ),$$

and the bending curvature is approximately

$$\begin{aligned} h_b''(x) &= \tfrac{1}{EI_y} \quad (q_3\ x + q_5) \\ &= \tfrac{1}{EI_y} \quad [\ (k_{33}\ u_3 + k_{35}\ u_5 + k_{39}\ u_9 + k_{3\ 11}\ u_{11})\ x \\ & \qquad\quad + (k_{53}\ u_3 + k_{55}\ u_5 + k_{59}\ u_9 + k_{5\ 11}\ u_{11})\ ], \end{aligned}$$

where small angles are again assumed. Integrating $h_b''(x)$ and combining with $h_s'(x)$,

$$h_z'(x) = -\frac{1}{GA_{sz}}q_3 + \frac{1}{EI_y}\left(\frac{1}{2}q_3\ x^2 + q_5\ x + C_1\right)\ .$$

Inserting the end condition, $h_z'(0) = -u_5$ gives $C_1 = -EI_y u_5$. Integrating again,

$$h_z(x) = -\frac{1}{GA_{sz}}q_3\ x + \frac{1}{EI_y}\left(\frac{1}{6}q_3\ x^3 + \frac{1}{2}q_5\ x^2\right) - u_5\ x + C_2\ .$$

Now inserting the end condition $h_z(0) = u_3$ gives $C_2 = u_3$ and noting that $1/(GA_{sz}) = \Phi_z L^2/(12EI_y)$ the deformed shape may be written

$$h_z(x) = \frac{1}{EI_y}\left(\frac{1}{6}q_3\ x^3 + \frac{1}{2}q_5\ x^2 - \frac{1}{12}\Phi_z L^2 q_3\ x\right) - u_5\ x + u_3\ .$$

Finally, the shape function for a frame element bending and shear in the local $x - z$ plane is

$$\begin{aligned} h_z(x) = \tfrac{1}{L^3}\tfrac{1}{1+\Phi_z}\ \{ \quad & [2x^3 - 3Lx^2 - \Phi_z L^2 x + L^3(1+\Phi_z)]\ u_3 + \\ & [-Lx^3 + L^2(2+\Phi_z/2)x^2 - L^3(1+\Phi_z/2)x]\ u_5 + \\ & [-2x^3 + 3Lx^2 + \Phi_z L^2 x]\ u_9 + \\ & [-Lx^3 + L^2(1-\Phi_z/2)x^2 + \Phi_z L^3 x/2]\ u_{11}\ \ \}\ . \end{aligned} \tag{23}$$

Note that this expression is equivalent to equation (22) except for the fact that Φ_z replaces Φ_y and that the signs of the u_5 and u_6 shape functions are reversed, as are the signs of the u_{11} and u_{12} shape functions. This is consistent with the right-hand coordinate system.

For axial displacements, the shape function is the same as for a truss,

$$h_x(x) = \left(1 - \frac{x}{L}\right)u_1 + \frac{x}{L}u_7\ . \tag{24}$$

Likewise, for torsional displacements, the shape function is analogous to the axial displacement shape function

$$h_{\theta x}(x) = \left(1 - \frac{x}{L}\right)u_4 + \frac{x}{L}u_{10}\ . \tag{25}$$

6 The 3D elastic stiffness matrix for frame elements including shear and bending effects

Differentiating and integrating the shape functions derived above, the three-dimensional elastic stiffness matrix for frame elements in local coordinates including bending and shear deformation effects is:

$$
\mathbf{k}_{\mathrm{E}} = \left[\begin{array}{cccccccccccc}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & \frac{-12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{6EI_z}{L^2(1+\Phi_y)} \\
0 & 0 & \frac{12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & 0 & 0 & \frac{-12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{L(1+\Phi_z)} & 0 & 0 & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(2-\Phi_z)EI_y}{L(1+\Phi_z)} & 0 \\
0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)} & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} \\
\frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & \frac{12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} \\
0 & 0 & \frac{-12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & 0 & 0 & \frac{12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 \\
0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(2-\Phi_z)EI_y}{L(1+\Phi_z)} & 0 & 0 & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{L(1+\Phi_z)} & 0 \\
0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)}
\end{array}\right],
$$

where

$$
\Phi_y = \frac{12EI_z}{GA_{sy}L^2}\ , \qquad \text{and} \qquad \Phi_z = \frac{12EI_y}{GA_{sz}L^2}\ .
$$

7 Formulation of the geometric stiffness matrix from the cubic shape functions

For the elements of the geometric stiffness matrix in rows 2,6,8, and 12, the potential energy function is

$$U_{\mathrm{G}y} = \frac{1}{2} \, T \int_0^L h_y'(x) \; h_y'(x) \; dx \; ,$$

where $h_y(x)$ is given by equation (22). Likewise, for rows 3,5,9, and 11, the potential energy function is

$$U_{\mathrm{G}z} = \frac{1}{2} \, T \int_0^L h_z'(x) \; h_z'(x) \; dx \; ,$$

where $h_z(x)$ is given by equation (23). For rows 1 and 7, the potential energy function is

$$U_{\mathrm{G}x} = \frac{1}{2} \, T \int_0^L h_x'(x) \; h_x'(x) \; dx \; ,$$

where $h_x(x)$ is given by equation (24).

8 Torsion

In the torsion of non-circular sections, torsional displacements result in axial deformation (warping) of the cross-section. In such cases, the work of the axial tension, T, moving through the warping displacements provides the potential energy function for rows 4 and 10,

$$U_{\mathrm{G}\theta} = \frac{1}{2} \, T \frac{J_x}{A_x} \int_0^L h_{\theta x}'(x) \; h_{\theta x}'(x) \; dx \; ,$$

where $h_{\theta x}(x)$ is given by equation (25) and J_x is the torsional moment of inertia.

The geometric stiffness coefficients may then be found by forming the Hessian of the appropriate potential energy function,

$$k_{\mathrm{G}_{ij}} = \frac{\partial^2 \; U_{\mathrm{G}}}{\partial u_i \; \partial u_j} .$$

9 The 3D geometric stiffness matrix for frame elements including shear and bending effects

Differentiating and integrating the shape functions as described above, the three-dimensional geometric stiffness matrix for frame elements in local coordinates including axial, bending, shear and torsional warping effects is:

$$
\mathbf{k}_{\mathrm{G}} = \frac{T}{L} \left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{6/5+2\Phi_y+\Phi_y{}^2}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{L/10}{(1+\Phi_y)^2} \\
0 & 0 & \frac{6/5+2\Phi_z+\Phi_z{}^2}{(1+\Phi_z)^2} & 0 & \frac{-L/10}{(1+\Phi_z)^2} & 0 \\
0 & 0 & 0 & \frac{J_x}{A_x} & 0 & 0 \\
0 & 0 & \frac{-L/10}{(1+\Phi_z)^2} & 0 & \frac{2L^2/15+L^2\Phi_z/6+L^2\Phi_z{}^2/12}{(1+\Phi_z)^2} & 0 \\
0 & \frac{L/10}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{2L^2/15+L^2\Phi_y/6+L^2\Phi_y{}^2/12}{(1+\Phi_y)^2} \\
-0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-6/5-2\Phi_y-\Phi_y{}^2}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{-L/10}{(1+\Phi_y)^2} \\
0 & 0 & \frac{-6/5-2\Phi_z-\Phi_z{}^2}{(1+\Phi_z)^2} & 0 & \frac{L/10}{(1+\Phi_z)^2} & 0 \\
0 & 0 & 0 & -\frac{J_x}{A_x} & 0 & 0 \\
0 & 0 & \frac{-L/10}{(1+\Phi_z)^2} & 0 & \frac{-L^2/30-L^2\Phi_z/6-L^2\Phi_z{}^2/12}{(1+\Phi_z)^2} & 0 \\
0 & \frac{L/10}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{-L^2/30-L^2\Phi_y/6-L^2\Phi_y{}^2/12}{(1+\Phi_y)^2}
\end{array} \right.
$$

$$
\left. \begin{array}{cccccc}
-0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-6/5-2\Phi_y-\Phi_y{}^2}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{L/10}{(1+\Phi_y)^2} \\
0 & 0 & \frac{-6/5-2\Phi_z-\Phi_z{}^2}{(1+\Phi_z)^2} & 0 & \frac{-L/10}{(1+\Phi_z)^2} & 0 \\
0 & 0 & 0 & -\frac{J_x}{A_x} & 0 & 0 \\
0 & 0 & \frac{L/10}{(1+\Phi_z)^2} & 0 & \frac{-L^2/30-L^2\Phi_z/6-L^2\Phi_z{}^2/12}{(1+\Phi_z)^2} & 0 \\
0 & \frac{-L/10}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{-L^2/30-L^2\Phi_y/6-L^2\Phi_y{}^2/12}{(1+\Phi_y)^2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{6/5+2\Phi_y+\Phi_y{}^2}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{-L/10}{(1+\Phi_y)^2} \\
0 & 0 & \frac{6/5+2\Phi_z+\Phi_z{}^2}{(1+\Phi_z)^2} & 0 & \frac{L/10}{(1+\Phi_z)^2} & 0 \\
0 & 0 & 0 & \frac{J_x}{A_x} & 0 & 0 \\
0 & 0 & \frac{L/10}{(1+\Phi_z)^2} & 0 & \frac{2L^2/15+L^2\Phi_z/6+L^2\Phi_z{}^2/12}{(1+\Phi_z)^2} & 0 \\
0 & \frac{-L/10}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{2L^2/15+L^2\Phi_y/6+L^2\Phi_y{}^2/12}{(1+\Phi_y)^2}
\end{array} \right] ,
$$

where

$$T = EA(u_7 - u_1)/L \ ,$$

$$\Phi_y = \frac{12EI_z}{GA_{sy}L^2} \qquad \text{and} \qquad \Phi_z = \frac{12EI_y}{GA_{sz}L^2} \ .$$