Mathematical Properties of Stiffness Matrices

CEE 421L. Matrix Structural Analysis
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Fall, 2012

These notes describe some of the mathematical properties of element stiffness matrices and structural stiffness matrices.

A stiffness matrix, $[K]$, relates point forces, $\{p\}$, applied at a set of coordiantes on the structure , to the displacements, $\{d\}$, at the same set of coordinates.

$$[K]\{d\} = \{p\} \tag{1}$$

The locations and directions of the point forces and displacements are called the *coordinates* of the structural model. Force coordinates and displacement coordinates are *co-located*.

1 Structural stiffness matrices

Here is a structure with two coordinates:

The structural stiffness matrix for these two coordinates may be written

$$[K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{2}$$

This stiffness matrix represents a set of two equations with two unknowns.

$$K_{11}d_1 + K_{12}d_2 = p_1 \tag{3}$$
$$K_{21}d_1 + K_{22}d_2 = p_2 \tag{4}$$

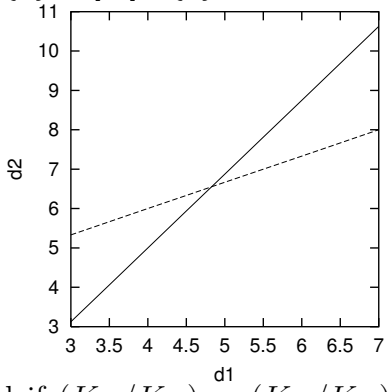
All stiffness matrices are *symmetric*; $[K] = [K]^\mathsf{T}$ and $K_{ij} = K_{ji}$. This is a statement of Maxwell's Reciprocity Theorem, which says that the deflection d_i (at coordinate i) due to a unit force p_j (at coordinate j) is equal to the deflection d_j (at coordinate j) due to a unit force p_i (at coordinate i).

Solving equations (3) and (4) for d_2 in terms of d_1 gives the two equations

$$d_2 = -(K_{11}/K_{12})d_1 + (1/K_{12})p_1 \quad (5)$$
$$d_2 = -(K_{21}/K_{22})d_1 + (1/K_{22})p_2 \quad (6)$$

Given numerical values for $\{p\}$ and $[K]$, these two equations may be plotted. The intersection of these two lines is the solution, $\{d\} = [K]^{-1}\{p\}$. For example,

$$[K] = \begin{bmatrix} 1.5 & -0.8 \\ -0.8 & 1.2 \end{bmatrix} \qquad \{p\} = \begin{Bmatrix} 2 \\ 4 \end{Bmatrix}$$



$$\{d\} = \begin{Bmatrix} 4.8276 \\ 6.5517 \end{Bmatrix}$$

Note that the lines are parallel if $(K_{11}/K_{12}) = (K_{21}/K_{22})$. There is no unique point at which parallel lines cross, and there is therefore no unique solution $\{d\} = [K]^{-1}\{p\}$. In other words the matrix $[K]$ can not be inverted if $(K_{11}/K_{12}) = (K_{21}/K_{22})$.

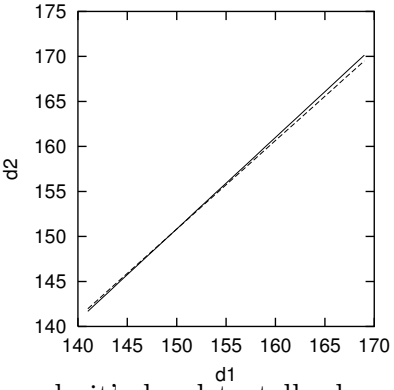
The following statements are equivalent:

$$K_{11}/K_{12} = K_{21}/K_{22} \quad \Leftrightarrow \quad K_{11}K_{22} - K_{12}K_{21} = 0 \quad \Leftrightarrow \quad \det([K]) = 0. \quad (7)$$

A stiffness matrix $[K]$ is called *ill-conditioned* or *nearly singular* if its determinant, $\det([K])$, is "close to zero". In computer applications, "close to zero" means close to the smallest number a computer can represent.

One way in which this can happen in our 2-by-2 example, is if the diagonal elements of $[K]$ are equal to one another and the off diagonal elements are the negatives of the diagonal elements. Consider an ill-conditioned 2-by-2 stiffness matrix. The d_2 vs. d_1 lines for the deflections are almost parallel.

$$[K] = \begin{bmatrix} 1.22 & -1.2 \\ -1.2 & 1.22 \end{bmatrix} \qquad \{p\} = \begin{Bmatrix} 2 \\ 4 \end{Bmatrix}$$

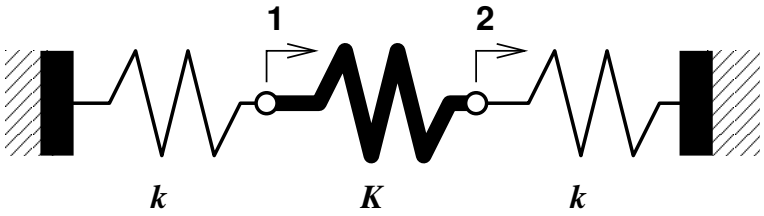


$$\{d\} = \begin{Bmatrix} 149.59 \\ 150.41 \end{Bmatrix}$$

In contrast to the previous example it's hard to tell where exactly these two lines cross. There are large ranges for the values of d_1 and d_2 for which the lines are nearly touching. Numerical

computation involving the inverse of ill-conditioned matrices can lose precision because there is a range of values in the solution $\{d\}$ that can satsify $[K]\{d\} = \{p\}$.

How can this arise in the stiffness matrix of a structural system? Consider the three-spring system shown below:



The stiffness matrix for this system is

$$\begin{bmatrix} K+k & -K \\ -K & K+k \end{bmatrix}, \tag{8}$$

which (for $K \gg k$) is very close to

$$\begin{bmatrix} K & -K \\ -K & K \end{bmatrix}. \tag{9}$$

If $K \gg k$ the determinant of this stiffness matrix is close to zero. How big does K/k need to be before the solution loses accuracy? It depends on the specified precision of the computations.

	Single Precision	Double Precision
significant figures	8	15
K/k	10^n	10^n
remaining accuracy	$8-n$	$15-n$

As a rule of thumb, we would like to have at least 4 or 5 significant figures of accuracy in our results, so in single precision computations if we want to make an element "very stiff" as compared to the other elements, we should make it no stiffer than 1000 times the stiffness of the other elements; for double precision, K should be less than 10-billion times k.

2 Eigenvalues of stiffness matrices

The mathematical meaning of the *eigenvalues* and *eigenvectors* of a symmetric stiffness matrix $[K]$ can be interpreted geometrically. The stiffness matrix $[K]$ maps a displacement vector $\{d\}$ to a force vector $\{p\}$. If the vectors $\{x\}$ and $[K]\{x\}$ point in the same direction, then the vector $\{x\}$ is called the eigenvector of $[K]$. The ratio of the length of vector $[K]\{x\}$ to the length of vector $\{x\}$ is called the eigenvalue, λ, of $[K]$ associated with the eigenvector $\{x\}$. So, the standard eigenvalue problem can be written

$$[K]\{x\} = \lambda\{x\} \ . \tag{10}$$

All eigenvalues of symmetric matrices (e.g., stiffness matrices) are real-valued.

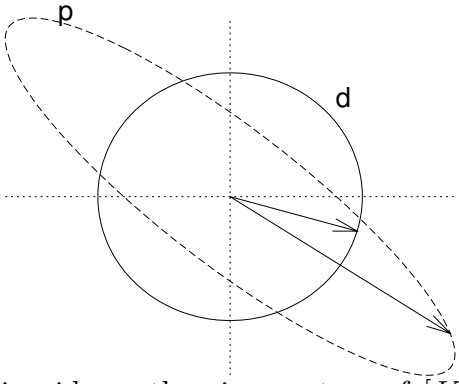
Now consider a set of displacement vectors consistent with the reactions (or constraints) of the structure. For a given stiffness matrix, the equation $\{p\} = [K]\{d\}$ will produce a corresponding set of force vectors (in equilibrium). For each displacement vector, there is one, and only one, force vector. Now, consider a "spheroidal" set of displacement vectors. In other words, for each displacement vector $\{d\}$ in our set,

$$d_1^2 + d_2^2 + d_3^2 + \cdots + d_N^2 = 1. \tag{11}$$

If $N = 2$ then the tips of the $\{d\}$ vectors trace out a unit circle. A "spheroidal" set of $\{d\}$ vectors produces an "ellipsoidal" set of $\{p\}$ vectors. This ellipsoid has principle axes with principle lengths. If $N = 2$ then the tips of the $\{p\}$ vectors trace out an ellipse. For example,

$$[K] = \begin{bmatrix} 1.5 & -0.8 \\ -0.8 & 1.2 \end{bmatrix}$$

$$\lambda_1 = 0.536 \qquad \lambda_2 = 2.164$$



The principle axis directions of the force-ellipsoid are the eigenvectors of $[K]$, and the lengths of these principle axes are the eigenvalues of $[K]$. These principle directions are perpendicular (*orthogonal*) to one another. In the example above, if the radius of the circle 'd' is 1, then the principle axes of the ellipse 'f' have lengths equal to λ_1 and λ_2. If $\{x\}$ is an eigen-vector of $[K]$, then

$$[K]\{x\} - \lambda\{x\} = 0 \quad \Leftrightarrow \quad [[K] - \lambda[I]_N]\{x\} = 0 \tag{12}$$

where $[I]_N$ is the N-by-N identity matrix. In other words, the matirx $[[K] - \lambda[I]_N]$ has linearly-dependent columns. In other words, the matrix $[[K] - \lambda[I]_N]$ is not invertible. In other words, the determinant of the matrix $[[K] - \lambda[I]_N]$ is zero. So, one way to compute the eigenvalues is to find the roots of

$$\det([K] - \lambda\,[I]_N) = 0, \tag{13}$$

which is called the *characteristic polynomial* of $[K]$.

If a structure is *stable* (internally and externally), then its stiffness matrix is invertible. Otherwise, the structure is free to move or deflect without deforming. If a structure is free to move in this way, then applied forces can produce infinite or undetermined displacements.

As we saw earlier, a structure has an invertible stiffness matrix if and only if $\det([K]) \neq 0$. The determinant of a matrix is the product of its eigenvalues,

$$\det([K]) = (\lambda_1)(\lambda_2)\cdots(\lambda_N), \tag{14}$$

therefore, no eigenvalue of an invertible matrix can be zero. All of the eigenvalues of a *positive definite matrix* are positive numbers. Hence, the determinant of a positive definite stiffness matrix is also a positive number and a positive definite stiffness matrix is invertible. If $[K]$ is positive definite then any set of displacements will result in positive strain energy,

$$U = \frac{1}{2}\{d\}^{\mathsf{T}}[K]\{d\} > 0 \qquad \forall \;\; \{d\} \neq 0 \quad \Leftrightarrow \quad \lambda_i > 0 \quad \forall \;\; i. \tag{15}$$

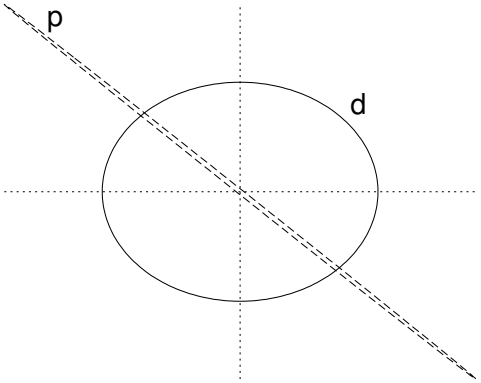
Prove this fact to yourself by substituting $[K]\{d\} = \lambda\{d\}$ into (15).

If a structure is *not stable* (internally or externally), then its stiffness matrix will have one or more eigenvalue equal to zero. Unstable structures can be moved to a displaced condition without applying any forces, i.e., $[K]\{d\} = \{0\}$. In this case, the stiffness matrix is said to be *singular.*

We can make use of the geometric illustration of $[K]\{d\} = \{p\}$ to interpret what a zero eigenvalue means. If a stiffness matrix $[K]$ has one eigenvalue equal to zero, then the force ellipsoid will have a principle axis of zero length; it will have one less dimension than the displacement spheroid. In other words, for $N = 2$, the force ellipse will become a line. In this case, two displacement vectors map to the same point on the force line (which is actually a degenerate ellipse). So, inversely, for a given force vector and a singular stiffness matrix, there is more than one displacement vector, there is not a unique displacement for a given force, and $[K]$ can not be inverted.

$$[K] = \begin{bmatrix} 1.22 & -1.2 \\ -1.2 & 1.22 \end{bmatrix}$$

$\lambda_1 = 0.02 \qquad\qquad \lambda_2 = 2.42$



A matrix is called *stiff* if the ratio of the largest to smallest eigenvalue, $\lambda_{\max}/\lambda_{\min}$ is much greater than one. (The eigenvalues of the stiffness matrix in equation (8) are k and $2K + k$.)

3 Element stiffness matrices

Element stiffness matrices $[k]$ relate the forces and displacements of *all* of the coordinates of the element, regardless of whether or not any of the coordinates are reaction coordinates in the assembled structure. An assembled structural stiffness matrix relating forces and displacements at *all* of the structural coordinates (displacement coordinates *and* reaction coordinates) can be viewed as a kind of super-element stiffness matrix, which can be useful for large structures assembled from a set of repeated sub-structures (e.g., a "Brown Truss").

The following five statements are equivalent:

- All element stiffness matrices are singular.
- Element stiffness matrices can not be inverted.
- For element stiffness matrices, there is no unique solution to $\{q\} = [k]\{u\}$.
- For element stiffness matrices, there is at least one non-trivial (non-zero) vector $\{u\}$ for which $[k]\{u\} = \{0\}$.
- Element stiffness matrices have at least one eigenvalue equal to zero.

Consider the element stiffness matrix matrix for a truss bar.

$$\left\{ \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \right\} = \left[\begin{array}{cccc} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \right\} \tag{16}$$

It is not hard to come up with element displacements for which the truss bar does not stretch, and for which $\{q\} = \{0\}$. In fact, any set of displacements for which $u_1 = u_3$ satisfies $[k]\{u\} = \{0\}$. Any vector $\{u\}$ for which $u_1 = u_3$ is said to be in the *null space* of the truss bar local element stiffness matrix $[k]$. The determinant of $[k]$ is seen to be equal to $(EA/L) \times 0 - (EA/L) \times 0 = 0$. The eigenvalues of $[k]$ are the roots of $\det([k - \lambda I])$ where

$$\det([k - \lambda I]) = \left(\frac{EA}{L} - \lambda\right)(-\lambda)\left(\frac{EA}{L} - \lambda\right)(-\lambda) + \left(-\frac{EA}{L}\right)(-\lambda)\left(\frac{EA}{L} - \lambda\right)(-\lambda) \tag{17}$$

$$= \lambda^2\left(\frac{EA}{L} - \lambda\right)\left(\frac{EA}{L} - \lambda\right) + \lambda^2\left(\frac{EA}{L}\right)\left(\lambda - \frac{EA}{L}\right) \tag{18}$$

The element stiffness matrix $[k]$ has three eigenvalues that are zero, corresponding to two rigid translations and one rigid rotation. These displacements are called *rigid body modes*. The displacement vectors of these three rigid translations and rotations are in the *null-space* of $[k]$, which is three-dimensional for a 2D truss element stiffness matrix.

Coordinate transformation ($[K] = [T]^{\mathsf{T}}[k][T]$) does not affect the eigenvalues of a stiffness matrix. Consider the global element stiffness matrix equation for a truss bar.

$$\left\{ \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \right\} = \frac{EA}{L} \left[\begin{array}{cccc} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{array} \right] \left\{ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \right\} \tag{19}$$

Any set of displacements that do not result in tension in the truss bar are in the null space of $[K]$. Knowing the equation for the tension in a truss bar, it can be seen easily that the truss bar tension is zero (and $\{f\} = 0$). if $v_1 = v_3$, or $v_2 = v_4$, or if $v_2 = -v_4$. These three conditions ($v_1 = v_3$, $v_2 = v_4$, $v_2 = -v_4$) are linearly independent, which implies that the null space of $[K]$ is three-dimensional, and that $[K]$ has three eigenvalues equal to zero.

The beam element stiffness matrix equation,

$$\left\{ \begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \right\} = (EI) \left[\begin{array}{cccc} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & 4/L & -6/L^2 & 2/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ 6/L^2 & 2/L & -6/L^2 & 4/L \end{array} \right] \left\{ \begin{array}{c} \Delta_1 \\ \theta_1 \\ \Delta_2 \\ \theta_2 \end{array} \right\} \tag{20}$$

has a two-dimensional null space. This null space corresponds to the set of displacements and rotations that will not deform the beam in shear or in bending. It's not too hard to see that the two (linearly independent) vectors

$$\{u\} = \left\{ \begin{array}{l} \Delta_1 = \delta \\ \theta_1 = 0 \\ \Delta_2 = \delta \\ \theta_2 = 0 \end{array} \right\} \quad \text{and} \quad \{u\} = \left\{ \begin{array}{l} \Delta_1 = -\delta \\ \theta_1 = 2\delta/L \\ \Delta_2 = \delta \\ \theta_2 = 2\delta/L \end{array} \right\} \tag{21}$$

both satisfy $[k]\{u\} = \{0\}$. These two displacement vectors span the null space of the beam element stiffness matrix.

4 Solving $[K]\{d\} = \{p\}$ for $\{d\}$ without finding $[K]^{-1}$ ... LDL^T decomposition

An efficient, simple, and accurate way to solve $[K]\{d\} = \{p\}$ for $\{d\}$ (where $[K]$ is symmetric and invertible) is by a method called LDL^T *decomposition*.[1] In this method, the stiffness matrix is represented by the product of three matrices, $[L]$, $[D]$, and $[L]^\mathsf{T}$.

$$[K] = [L][D][L]^\mathsf{T}, \tag{22}$$

where $[L]$ is *lower triangular* and $[D]$ is *diagonal*.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix} \begin{bmatrix} D_{11} & 0 & 0 & 0 \\ 0 & D_{22} & 0 & 0 \\ 0 & 0 & D_{33} & 0 \\ 0 & 0 & 0 & D_{44} \end{bmatrix} \begin{bmatrix} 1 & L_{21} & L_{31} & L_{41} \\ 0 & 1 & L_{32} & L_{42} \\ 0 & 0 & 1 & L_{43} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Here, the term *decomposition* means the separation of the matrix $[K]$ into the product of three simple matrix factors. Once the factors $[D]$ and $[L]$ have been found, the vector $([D][L]^\mathsf{T}\{d\})$ is computed via back-substitution starting with the first equation.

$$\{p\} = [L][D][L]^\mathsf{T}\{d\} = [L]([D][L]^\mathsf{T}\{d\}) = [L]\{y\} \tag{24}$$

$$f_1 = 1 \cdot y_1 \quad \ldots \quad \text{solve for } y_1 \tag{25}$$

$$f_2 = L_{21} \cdot y_1 + 1 \cdot y_2 \quad \ldots \quad \text{solve for } y_2 \tag{26}$$

etc.

Once $\{y\}$ is computed, it is just as easy to find $\{d\}$ from $\{y\} = [D][L]^\mathsf{T}\{d\}$, by starting with the last equation. It is an interesting and useful fact that the number of positive values of $[D]$ equals the number of positive eigenvalues of $[K]$, and that the number of negative values of $[D]$ equals the number of negative eigenvalues of $[K]$. Since the LDL^T decomposition of a matrix is much faster than its eigenvalue decomposition, LDL^T decomposition is an easy way to check if a matrix is positive definite or negative definite.

[1] A. Kassimali, *Matrix Analysis of Structures, 2nd ed.* Brooks/Cole 2012, section 9.9.

5 Solving $[K]\{d\} = \{p\}$ for $\{d\}$ without finding $[K]^{-1}$ ... singular value decomposition

Singular value decomposition (SVD) is another way to solve for the deflections of a system, and is a generalization of LDL^{T} decomposition. In general, the singular value decomposition of a matrix $[A]$ is the product of three matrices, $[U]$, $[\Sigma]$, and $[V]^{\mathsf{T}}$.

$$[A]_{(m\times n)} = [U]_{(m\times m)}\ [\Sigma]_{(m\times n)}\ [V]^{\mathsf{T}}_{(n\times n)} \tag{27}$$

$$= \begin{bmatrix} | & | & | & & | \\ u_1 & u_2 & u_3 & \cdots & u_m \\ | & | & | & & | \end{bmatrix}_{(m\times m)} \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & 0 & \\ & & \sigma_3 & & \\ & 0 & & \ddots & \\ & & & & \sigma_n \\ & & 0 & & \end{bmatrix}_{(m\times n)} \begin{bmatrix} -v_1^{\mathsf{T}}- \\ -v_2^{\mathsf{T}}- \\ -v_3^{\mathsf{T}}- \\ \vdots \\ -v_n^{\mathsf{T}}- \end{bmatrix}_{(n\times n)} \tag{28}$$

The matrix $[\Sigma]$ is a diagonal matrix of the *singular values* sorted in decreasing numerical order, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_h \geq 0$ The vectors $\{u_i\}$ and $\{v_i\}$ are the i^{th} columns of the matrices $[U]$ and $[V]$. The matrices $[U]$ and $[V]$ are ortho-normal, $[U]^{\mathsf{T}} = [U]^{-1}$ and $[V]^{\mathsf{T}} = [V]^{-1}$, or

$$[U]^{\mathsf{T}}[U] = [I]_m\ , \qquad [V]^{\mathsf{T}}[V] = [I]_n\ , \qquad ||u_i|| = 1\ , \qquad ||v_i|| = 1 \tag{29}$$

The SVD of a matrix $[A]$ can be written as a sum of rank-1 matrices.

$$[A] = \sum_{i=1}^{n} \sigma_i \left[\begin{Bmatrix} | \\ u_i \\ | \end{Bmatrix}_{(m\times 1)} \begin{Bmatrix} -v_i^{\mathsf{T}}- \end{Bmatrix}_{(1\times n)} \right]_{(m\times n)} \tag{30}$$

Each rank-1 matrix is the outer-product of unit-length vectors, so all the rank-1 matrices in the series have a norm of 1. The singular values are like scale factors for each of these matrices. So $[A]$ is mostly $\sigma_1[u_1 v_1^{\mathsf{T}}]$; $\sigma_n[u_n v_n^{\mathsf{T}}]$ contributes much less to $[A]$. The matrix equation

$$[A]\{x\} = \{b\} \tag{31}$$

can be solved for $\{x\}$ using the SVD of $[A]$, by inverting only the positive singular values in $[\Sigma]$.

$$\{x\}_{(n\times 1)} = \sum_{i=1}^{n} \frac{1}{\sigma_i} \left[\begin{Bmatrix} | \\ v_i \\ | \end{Bmatrix}_{(n\times 1)} \begin{Bmatrix} -u_i^{\mathsf{T}}- \end{Bmatrix}_{(1\times m)} \right]_{(n\times m)} \begin{Bmatrix} | \\ b \\ | \end{Bmatrix}_{(m\times 1)} \tag{32}$$

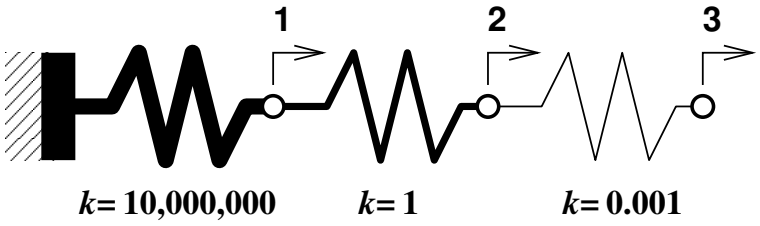
$$= [V][\Sigma]^{-1}[U]^{\mathsf{T}}\{b\}, \tag{33}$$

So the smallest singular values of $[A]$ contribute the most to the solution $\{x\}$.

When applied to square, symmetric, and positive definite stiffness matrices, $[K]_{(n\times n)}$, the singular values, σ_i, are equal to the squares of the eigenvalues of $[K]$, and $[U] = [V]$. The matrices $[U]$, $[\Sigma]$ and $[V]$ are all $(n \times n)$. If some eigenvalues of $[K]$ are nearly zero, the corresponding singular values will also be very small, and the stiffness matrix is said to be *nearly singular* or *ill-conditioned* or *stiff*.

Numerical Example[2]

Consider the SVD of an ill-conditioned stiffness matrix for the system shown below.



$$\begin{bmatrix} 10^7+1 & -1 & 0 \\ -1 & 1.001 & -0.001 \\ 0 & -0.001 & 0.001 \end{bmatrix} = \begin{bmatrix} -1 & -10^{-7} & 10^{-10} \\ 10^{-7} & -1 & 10^{-3} \\ 0 & 10^{-3} & 1 \end{bmatrix} \begin{bmatrix} 10^7+1 & & \\ & 1.001 & \\ & & 9.99\times10^{-4} \end{bmatrix} \begin{bmatrix} -1 & 10^{-7} & 0 \\ -10^{-7} & -1 & 10^{-3} \\ 10^{-10} & 10^{-3} & 1 \end{bmatrix} \tag{34}$$

This stiffness matrix is ill-conditioned because the ratio of the first-to-last (largest-to-smallest) singular values is very large (10^{10}). The stiffest aspects of this system are related to the largest singular value, but will deform very little. The most flexible aspects of this system are related to the smallest singular value, and will deform a lot. When the forces at all the coordinates are about the same, the computed displacements are always dominated by the *smallest* singular values of $[K]$. For example for loading $p_1 = 1$, $p_2 = 1$, and $p_3 = 1$, the displacements are $d_1 = 3\times10^{-7} \approx 0$, $d_2 = 2$, and $d_3 = 1002$. The deformation of the stiff spring does not affect the displacements of coordinates 2 and 3 in any significant way, so coordinate 1 could be removed from the model. Alternatively, coordinate 1 could be retained but the displacements could be computed from the two smaller singular values,

$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \frac{1}{1.001} \left[\begin{Bmatrix} -10^{-7} \\ -1 \\ 10^{-3} \end{Bmatrix} \begin{Bmatrix} -10^{-7} & -1 & 10^{-3} \end{Bmatrix} \right] \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} + \frac{1}{9.99\times10^{-4}} \left[\begin{Bmatrix} 10^{-10} \\ 10^{-3} \\ 1 \end{Bmatrix} \begin{Bmatrix} 10^{-10} & 10^{-3} & 1 \end{Bmatrix} \right] \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2\times10^{-7} \approx 0 \\ 2 \\ 1002 \end{Bmatrix} \tag{35}$$

The SVD is useful in determining what aspects of a model contribute negligibly to the responses. The SVD is used in *model reduction* methods to simplify models by eliminating their least-significant aspects.

[2] contributed by: Yrd.Doç.Dr. Gürsoy Turan, Insaat Mühendisligi Bölümü, Izmir Yüksek Teknoloji Enstitüsü, Gülbahçe-Urla