Introduction to The Principles of Virtual Work

CEE 421L. Matrix Structural Analysis
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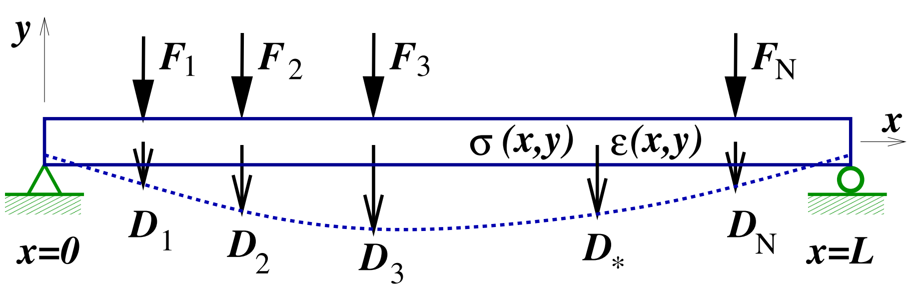
Forces F_i, Displacements D_i and D_*

Consider a linear elastic beam with N (*"real"*) forces, F_i, applied at points 1,2, ..., N.



All of these forces together produce deflections D_1, D_2, ... D_N at the corresponding loading points. There are also deflections at other points on the beam, for example the deflection D_* at point *. The sum of these forces produce stresses σ and strains ϵ throughout the volume of the beam. With this system of loading the external work and internal strain energy are:

$$W = \frac{1}{2}\sum_{i=1}^{N} F_i D_i = \frac{1}{2}\{F\}^{\mathsf{T}}\{D\} \tag{1}$$

and

$$U = \frac{1}{2}\int_V \{\sigma\}^{\mathsf{T}}\{\epsilon\}\, dV \tag{2}$$

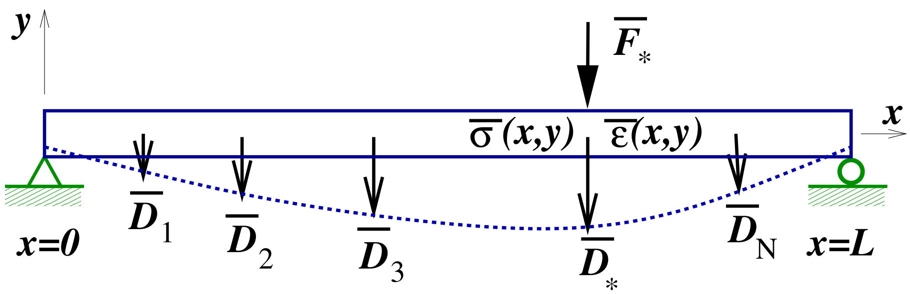
where the vectors of stresses $\{\sigma\}$ and strains $\{\epsilon\}$ are written

$$\{\sigma\}^{\mathsf{T}} = \left\{ \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \tau_{xy} \quad \tau_{xz} \quad \tau_{yz} \right\} \tag{3}$$

$$\{\epsilon\}^{\mathsf{T}} = \left\{ \epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{zz} \quad \gamma_{xy} \quad \gamma_{xz} \quad \gamma_{yz} \right\} \tag{4}$$

Force $\bar{F}_*$, Displacements $\bar{D}_i$ and $\bar{D}_*$

Next, consider the same beam with a single point load, $\bar{F}_*$, (This force could be at one of the loading points, $(1, \cdots, N)$ but it doesn't have to be.)



The force $\bar{F}_*$ by itself produces a collocated deflection $\bar{D}_*$ at point $*$, as well as deflections $\bar{D}_1$, $\bar{D}_2$, ..., $\bar{D}_N$, at the other locations on the beam.

The force $\bar{F}_*$ by itself creates stresses $\bar{\sigma}$ and strains $\bar{\epsilon}$ throughout the beam.

The external work of the force $\bar{F}_*$ passing through the displacement $\bar{D}_*$ is

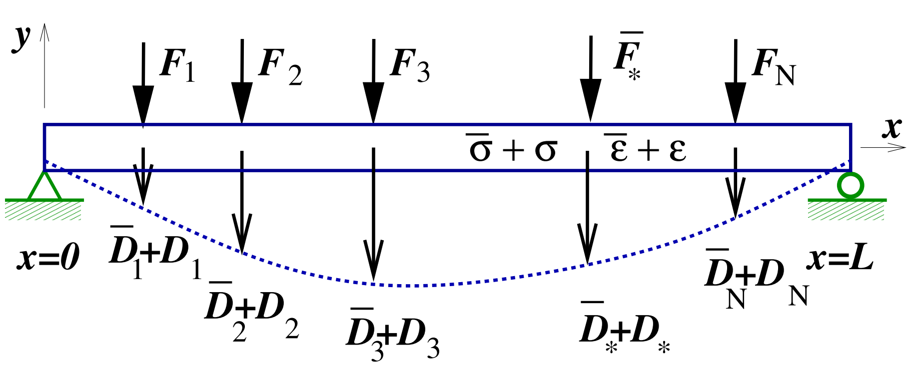
$$W = \frac{1}{2}\bar{F}_*\bar{D}_*, \tag{5}$$

and the total internal work associated with this single point load is

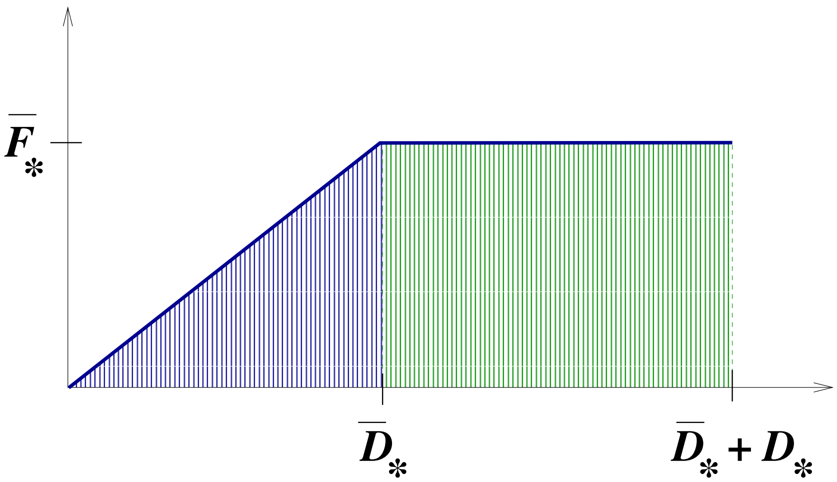
$$U = \frac{1}{2}\int_V \{\bar{\sigma}\}^{\mathsf{T}}\{\bar{\epsilon}\}\ dV \tag{6}$$

Forces $\bar{F}_* + F_i$, Displacements $\bar{D}_i + D_i$ and $\bar{D}_* + D_*$

Finally, consider what happens when we apply $\bar{F}_*$ first, then apply all the other (*"real"*) loads F_1, F_2, ..., F_N, while holding $\bar{F}_*$ constant.



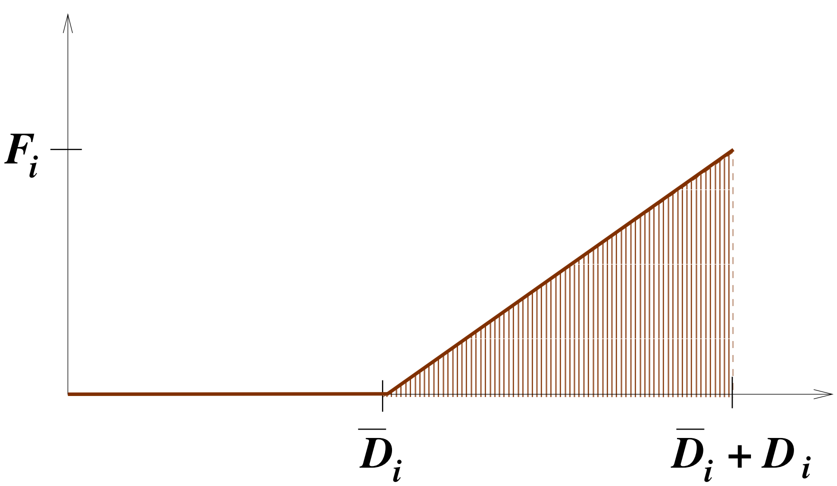
For the force $\bar{F}_*$:



and the external work for force $\bar{F}_*$ is

$$W = \frac{1}{2}\bar{F}_*\bar{D}_* + \bar{F}_* D_*. \tag{7}$$

Note that $\bar{F}_*$ is held constant as the other loads, F_i, and deflections, D_i, increase. For the forces at the other locations, F_i, $i = 1, \ldots, N$:

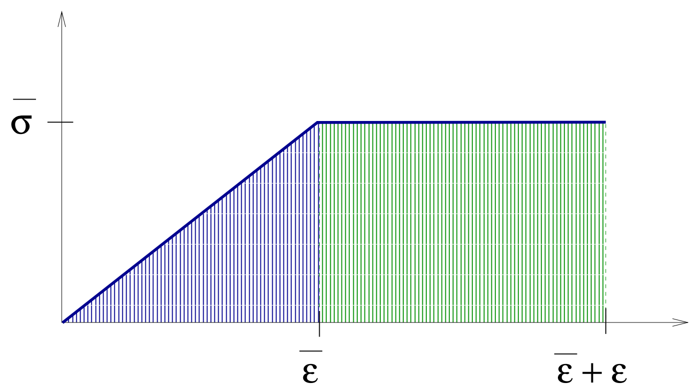


$$W = \frac{1}{2}\sum_{i=1}^{N} F_i D_i \tag{8}$$

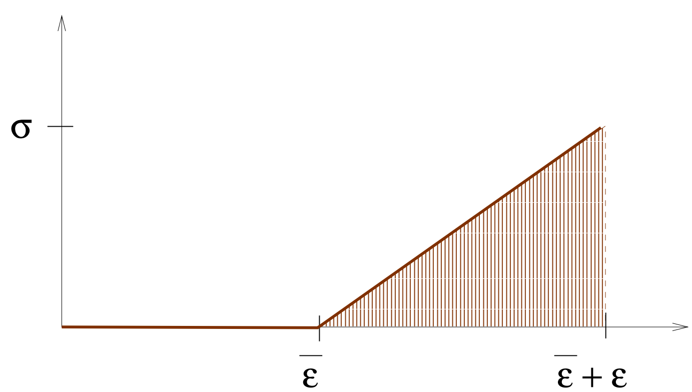
The total external work is

$$W = \frac{1}{2}\bar{F}_*\bar{D}_* + \bar{F}_* D_* + \frac{1}{2}\sum_{i=1}^{N} F_i D_i. \tag{9}$$

Now, let's find the internal work for the combination of the load $\bar{F}_*$ with all the other loads F_i, $i = 1, \ldots, N$. As before, we will say that the stresses and strains caused by the force $\bar{F}_*$ are $\bar{\sigma}$ and $\bar{\epsilon}$ and that the stresses and strains caused by the set of forces F_i, $i = 1, \ldots, N$ are σ and ϵ. First we will look at the stresses due to the force $\bar{F}_*$. The stress $\bar{\sigma}$ increases linearly until the strain $\bar{\epsilon}$ is attained. After the force $\bar{F}_*$ is applied the stress $\bar{\sigma}$ remains constant as the strains from the forces F_i, $i = 1, \ldots, N$ are applied.



Next we will look at the stresses σ due to the forces F_1, F_2, ..., F_N. As the load $\bar{F}_*$ is applied, (i.e., before the forces F_i are applied), the stresses σ are all zero, but the strains increase from 0 to $\bar{\epsilon}$. Once the force $\bar{F}_*$ has been applied, the forces F_i are applied, and the strains increase linearly with the stress.



The total internal work due to the combined actions of all the loads, applied sequentially, is

$$U = \frac{1}{2} \int_V \{\bar{\sigma}\}^{\mathsf{T}} \{\bar{\epsilon}\} \, dV + \int_V \{\bar{\sigma}\}^{\mathsf{T}} \{\epsilon\} \, dV + \frac{1}{2} \int_V \{\sigma\}^{\mathsf{T}} \{\epsilon\} \, dV. \qquad (10)$$

Equating the external and internal work (equations (9) and (10)), noting from page 1 that

$$\frac{1}{2}\bar{F}_*\bar{D}_* = \frac{1}{2}\int_V \{\bar{\sigma}\}^{\mathsf{T}}\{\bar{\epsilon}\}\ dV, \tag{11}$$

and from page 2 that

$$\frac{1}{2}\sum_{i=1}^{N} F_i D_i = \frac{1}{2}\int_V \{\sigma\}^{\mathsf{T}}\{\epsilon\}\ dV, \tag{12}$$

we obtain *the principle of virtual work*:

$$\bar{F}_* D_* = \int_V \{\bar{\sigma}\}^{\mathsf{T}}\{\epsilon\}\ dV. \tag{13}$$

This expression holds for beams, bars, trusses, frames, plates, shells, bricks, etc. etc. It is customary to call the left hand side of this expression the *external virtual work*

$$\bar{W} = \bar{F}_* D_*, \tag{14}$$

and the right hand side the *internal virtual work.*

$$\bar{U} = \int_V \{\bar{\sigma}\}^{\mathsf{T}}\{\epsilon\}\ dV. \tag{15}$$

Recall the definitions of the various terms in the principle of virtual work (13).

- D_* is the *real deflection* at some point in the structure (point *) caused by the *real forces* $F_1, F_2, ..., F_N$. Often the displacement D_* is the unknown response to be found.
- $\bar{F}_*$ is a *virtual force* in the direction and in the location of D_*. Often we set this force equal to 1 unit of force.
- ϵ are the *real strains* associated with the system *real forces* F_i. These strains are found by analyzing the structure with loads F_i.
- $\bar{\sigma}$ are the *virtual stresses* caused by the *virtual force* $\bar{F}_*$. These stresses are found by analyzing the structure with load $\bar{F}_*$.

In words, the external virtual work of a virtual force ($\bar{F}_*$) moving through a real displacement (D_*) equals the product of the virtual stresses ($\{\bar{\sigma}\}$) corresponding to the virtual force ($\bar{F}_*$) with the real strains ($\{\epsilon\}$) corresponding to the real system of forces (F_i), integrated over the volume of the solid.

Equation (13) is sometimes called the *principle of virtual forces*, since the forces and stresses correspond to the "*virtual*" loading system and the displacements and strains correspond to the real system of loading. In a completely analogous way we can write the *the principle of virtual displacements*:

$$\sum_{i=1}^{N} F_i \bar{D}_i = \int_V \{\sigma\}^{\mathsf{T}} \{\bar{\epsilon}\} \; dV. \tag{16}$$

in which,

- F_i are a system of *real forces* at a set of coordinates $(i = 1, \cdots, N)$.
- $\bar{D}_i$ are the *virtual deflections* collocated with the system of forces F_i, arising from some arbitrary system of *virtual forces*. Note that since these virtual deflections are found from a system of virtual forces, the virtual displacements are consistent with all the constraints (reaction points) on the structure. In fact, the only requirement for the virtual displacements is that they adhere to the constraints of the structure.
- $\bar{\epsilon}$ are the *virtual strains* associated with the *virtual displacements* $\bar{D}_i$.
- σ are the *real stresses* caused by the system of *real forces* F_i.

In words, the external virtual work of a real forces (F_i) moving through collocated virtual displacements ($\bar{D}_i$) equals the product of real stresses ($\{\sigma\}$) associated with forces F_i and the virtual strains ($\{\bar{\epsilon}\}$) associated with virtual displacements $\bar{D}_i$, integrated over the volume of the solid.

The principle of virtual work applies to linear elastic and non-linear elastic structures also.

There are many ways in which the principle of virtual work is applied to problems in many fields of engineering and applied mathematics, including solid mechanics, fluid mechanics, and electro-statics, to name a few. The principle of virtual work is fundamental to the finite element method, which is used to solve problems described by systems of partial differential equations in many disciplines. Here is a typical example of how we can apply the principle of virtual work to find the deflections at some point in an elastic solid. Consider a beam carrying some loads

The problem is to find the displacement D^* at some point (*) on this structure. Note that the principle of real work can not be applied to this problem.

The figure above shows the real displacements of the structure. This system is the one we are "really" interested in and it has internal strains that are "really" there.

To apply the principle of virtual work to this problem, we remove all of the externally applied loads in the figure above and apply a unit force in the direction and location of the unknown displacement D^*. This unit *virtual force* $(\bar{F}_*)$ will cause bending and shear in this particular structure, which will have associated *real stresses*. Knowing how to:
— relate the real external loads to internal bending moments $M(x)$ and shear forces $V(x)$;
— Relate these internal moments and shears to strains, ϵ
— Relate the unit virtual load to internal virtual bending moments $\bar{M}(x)$ and shear forces $\bar{V}(x)$; and
— relate those internal virtual moments and shears to virtual stresses, $\bar{\sigma}$,
equation (15) becomes

$$1 \cdot D^* = \int_V \{\bar{\sigma}\}^\mathsf{T} \{\epsilon\} \, dV. \tag{17}$$

The relationships between internal moments, shears, torques, and axial loads to stresses and strains may be simplified by considering those cases individually. Doing so simplifies the integral in equation (17) for these special cases.