# Geometric Stiffness Effects in 2D Trusses

**CEE 421L. Matrix Structural Analysis**
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In structural analysis if deformations are negligibly small (or *infinitesimal*) analyzing equilibrium in the undeformed configuration provides sufficiently accurate results. If deformations are not negligible, (i.e., if they are *finite*) equilibrium should be analyzed in the deformed configuration.

Finite deformation analysis is always more accurate than small deformation analysis, and when strains are large ($> 0.05\%$) the increased accuracy may be significant. Further, finite deformation analysis may be used to analyze buckling potential of structures.

It will be shown that in finite deformation analysis the stiffness depends on the bar tensions; one needs to know the bar tensions in order to compute the stiffness but one needs the stiffness to compute the tensions. The solution to such problems is to proceed incrementally by increasing deformations in steps until equilibrium is achieved for the specified loads, in the deformed configuration, and with the desired precision. Two algorithms for step-by-step deformation are described later in this document.

In section 1 we will find that the element stiffness matrix may be separated into an elastic part, $\mathbf{k}_\mathrm{E}$, plus a *geometric* part, $\mathbf{k}_\mathrm{G}$ that accounts for the effects of finite deformation. The elastic stiffness matrix for a bar element in local coordinates, as found earlier, is

$$\mathbf{k}_\mathrm{E} = \frac{EA}{L_o} \left[ \begin{array}{rrrr} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

We start by finding the analog of this stiffness matrix that accounts for finite deformation.

## 1 Finite Deformation Analysis of a Bar in Local Coordinates

Recall that in local coordinates the four deflections at the bar ends are $u_1$, $u_2$, $u_3$, and $u_4$; the four forces at the bar ends are $q_1$, $q_2$, $q_3$, and $q_4$, and the element stiffness matrix in local coordinates is $\mathbf{k}$. Define the angle $\alpha$ as the counter-clockwise inclination of the deformed bar with respect to local coordinate 1, i.e., the bar's original un-deformed orientation. Taking equilibrium at nodes 1 and 2,

$$\begin{aligned} q_1 &= -T\cos\alpha \\ q_2 &= -T\sin\alpha \\ q_3 &= \phantom{-}T\cos\alpha \\ q_4 &= \phantom{-}T\sin\alpha \end{aligned} \tag{1}$$

The angle $\alpha$ may be determined from the geometry of the deformed shape, i.e., the displacements $u_1$, $u_2$, $u_3$, and $u_4$.

$$\sin\alpha = \frac{1}{L}(u_4 - u_2),$$

and

$$\cos\alpha = \frac{1}{L}(L_o + u_3 - u_1) \approx 1,$$

where $L_o$ is the original length of the bar and $L$ is the deformed length of the bar. Substituting the expressions for $\sin\alpha$ and $\cos\alpha$ into the equilibrium equations (1) for $\mathbf{q}$, and writing the expressions in matrix form,

$$\left\{\begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \end{array}\right\} = \left\{\begin{array}{c} -T \\ 0 \\ T \\ 0 \end{array}\right\} + \frac{T}{L}\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]\left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\right\}. \tag{2}$$

Note that so far we require *one approximation* in this "exact" analysis: $\cos\alpha \approx 1$. If the bar deforms only along it's original direction, then $u_2 = u_4 = 0$; $u_3 - u_1 = \Delta$; and $T = \frac{EA}{L_o}\Delta = \frac{EA}{L_o}(u_3 - u_1) = -q_1 = q_3$. Therefore the vector $\{-T\ 0\ T\ 0\}^{\mathsf{T}}$, in equation (2) may be found from the elastic stiffness matrix, and

$$\left\{\begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \end{array}\right\} = \frac{EA}{L_o}\left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\right\} + \frac{T}{L_o}\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]\left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\right\}, \tag{3}$$

where the first matrix is the elastic stiffness matrix, which we have already seen, and the second matrix is the *geometric element stiffness matrix*, $\mathbf{k}_G$. The approximation $(T/L) \approx (T/L_o)$ in equation (3) (a *second approximation*) is sufficiently accurate in most applications.

## 2 Coordinate Transformation

The coordinate transformation process for finite strain is nearly identical to coordinate transformation for infinitesimal strain. The coordinate transformation matrix, $\mathbf{T}$, is

$$\mathbf{T} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix},$$

where $s$ and $c$ are now the sine and cosine of the counter-clockwise angle $\theta + \alpha$ from global element coordinate number 1 to the bar. The $s$ and $c$ terms in the coordinate transformation matrix can include the effects of the deformation as follows:

$$c = (x_2 + v_3 - x_1 - v_1)/L \qquad \text{and} \qquad s = (y_2 + v_4 - y_1 - v_2)/L \ ,$$

where $v_1, v_2, v_3, v_4$ are the element end displacements in the global directions, $(x_1, y_1)$ and $(x_2, y_2)$ are the node locations of the undeformed structure, and

$$L = \sqrt{(x_2 + v_3 - x_1 - v_1)^2 + (y_2 + v_4 - y_1 - v_2)^2} \ .$$

Alternatively, we may make a *third approximation*: that the deformed inclination of the bar is approximately the same as the original inclination of the bar, or $|\alpha| \ll 1$ in which case,

$$c = (x_2 - x_1)/L_o \qquad \text{and} \qquad s = (y_2 - y_1)/L_o \ .$$

$$L_o = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ .$$

In either case, the element stiffness matrix in global coordinates is found by applying the coordinate transformation operation.

$$\begin{aligned} \mathbf{K} &= \mathbf{T}^{\mathsf{T}}(\mathbf{k}_E + \mathbf{k}_G)\mathbf{T} \\ &= \mathbf{T}^{\mathsf{T}}\,\mathbf{k}_E\,\mathbf{T} + \mathbf{T}^{\mathsf{T}}\,\mathbf{k}_G\,\mathbf{T}. \end{aligned} \tag{4}$$

From these expressions,

$$\mathbf{K}_{\mathrm{E}} = \mathbf{T}^{\mathsf{T}}\ \mathbf{k}_{\mathrm{E}}\ \mathbf{T} \ = \ \frac{EA}{L_o}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \tag{5}$$

$$\mathbf{K}_{\mathrm{G}} = \mathbf{T}^{\mathsf{T}}\ \mathbf{k}_{\mathrm{G}}\ \mathbf{T} \ = \ \frac{T}{L_o}\begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix} \tag{6}$$

It is not hard to confirm these two expressions for $\mathbf{K}_{\mathrm{E}}$ and $\mathbf{K}_{\mathrm{G}}$, and you should feel encouraged to do so.

## 3 Matrix Assembly

The assembly of the structural stiffness matrix $\mathbf{K_s}$ proceeds exactly as with the elastic stiffness matrix.

## 4 Bar Forces

To compute the internal bar forces, finite deformation considerations must be included. For finite (logarithmic) strain, the bar tension is given exactly by $T = EA \log(L/L_o)$, where

$$L = \sqrt{(x_2 + v_3 - x_1 - v_1)^2 + (y_2 + v_4 - y_1 - v_2)^2}\ .$$

and

$$L_o = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

Alternatively, we may approximate the bar tension in terms of the four end displacements, $u_1$, $u_2$, $u_3$, and $u_4$. Let's look at the deformed bar in local coordinates again. The bar tension is approximated by $T \approx (EA/L_o)\Delta$, where the stretch in the bar, $\Delta$, is found from the four end displacements, $u_1$, $u_2$, $u_3$, and $u_4$.

$$\begin{aligned} (L_o + \Delta)^2 &= (L_o + u_3 - u_1)^2 + (u_4 - u_2)^2 \\ L_o^2 + 2L_o\Delta + \Delta^2 &= L_o^2 + u_3^2 + u_1^2 + 2L_o u_3 - 2L_o u_1 - 2u_3 u_1 + (u_4 - u_2)^2 \\ 2L_o\Delta + \Delta^2 &= 2L_o(u_3 - u_1) + (u_3 - u_1)^2 + (u_4 - u_2)^2 \end{aligned}$$

This is a quadratic equation in $\Delta$. Limiting strains to the elastic range of metals, $|\Delta/L_o| < 0.002$ so $|\Delta^2/(2L_o\Delta)| < 0.001$, and the $\Delta^2$ term can be neglected. This leads to the *fourth approximation* in this method

$$\Delta \approx (u_3 - u_1) + \frac{1}{2L_o}(u_3 - u_1)^2 + \frac{1}{2L_o}(u_4 - u_2)^2 \ , \tag{7}$$

which provides a relation for $\Delta$ that is not quite as complicated as the quadratic formula. The first term in parenthesis is the stretch considering infinitesimal deformations; the second and third terms are the contribution of the finite deformation effects. Optionally the $(u_3 - u_1)^2$ term can be neglected, since axial displacements are much smaller than transverse displacements. This approximation is typically accurate to within to within 0.0001% to 0.001% for strains up to 0.1%, which is on the order of the yield strain of most metals. The tension in a bar, including finite deformation effects, is

$$T \approx \frac{EA}{L_o}\left[(u_3 - u_1) + \frac{1}{2L_o}(u_4 - u_2)^2\right], \tag{8}$$

The bar tensions can be found from the bar displacements in the global coordinate directions, $v_1$, $v_2$, $v_3$, $v_4$.

$$T \approx \frac{EA}{L_o}\left[((v_3 - v_1)c + (v_4 - v_2)s) + \frac{1}{2L_o}\left((v_1 - v_3)s + (v_4 - v_2)c\right)^2\right] \tag{9}$$

Again, you should feel encouraged to confirm these equations.

In general, for trusses with metallic elements that are stressed to near their yield stress (axial strains $\approx$ 0.1%), geometric stiffness effects can affect node displacements by a fraction of a percent and can affect bar forces by a few percent. Trusses made of stronger or more flexible materials, such as plastics with elastic strains reaching 1% to 5%, geometric stiffness effects can be much more significant. Furthermore, geometric stiffness effects can be more significant in frames than in trusses.

Here is a summary of the approximations invoked in this finite strain analysis:

- $\cos\alpha \approx 1$ ...so that $\mathbf{k}_\mathrm{E}$ is independent of $T$ and $\mathbf{u}$ (error $\approx$ 0.1%);
- $(T/L) \approx (T/L_o)$ ...so that $\mathbf{k}_\mathrm{G}$ depends on $\mathbf{u}$ only through $T$ (error $\approx$ 0.0001% to 0.001%);

- $\alpha \ll 1$ ...so that $c$ and $s$ are independent of $\mathbf{v}$ (error $\approx 0.1\%$); and
- $T \approx \frac{EA}{L_o}\left[(u_3 - u_1) + \frac{1}{2L_o}(u_4 - u_2)^2\right]$ ...so that $T$ can be found without computing quadratic roots or logarithms (error $\approx 0.0001\%$ to $0.001\%$)

Only the first approximation is required for the stiffness matrix assembly process. The other approximations merely simplify the calculations. Given $\mathbf{x}$, $\mathbf{y}$, $\mathbf{v}$, $E$, and $A$ for an element, it is possible to compute $L_o$, $L$, $c$, $s$, and $T$ exactly. However, the effects of these approximations for strains less than 1% are truly minuscule.

## 5 Solving Nonlinear Problems using Newton-Raphson and Broyden Methods

Recall the truncated Taylor series expansion of a nonlinear function $\mathbf{f}(\mathbf{x})$,

$$\mathbf{f} - \mathbf{f_o} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right](\mathbf{x} - \mathbf{x_o}) \ + \ \text{h.o.t.} \ .$$

To solve a system of nonlinear equations $\mathbf{f}(\mathbf{x_o}) = \mathbf{f_o}$ for the vector of unknowns $\mathbf{x_o}$, we may proceed in an incremental fashion, by first evaluating $\mathbf{f^{(i)}} = \mathbf{f}(\mathbf{x^{(i)}})$ for a trial vector $\mathbf{x^{(i)}}$. If we know the *Jacobian matrix* $[\partial \mathbf{f}/\partial \mathbf{x}]$ evaluated at the vector $\mathbf{x} = \mathbf{x^{(i)}}$, then the next trial value of the unknown vector should be

$$\mathbf{x^{(i+1)}} = \mathbf{x^{(i)}} - \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]^{\mathbf{-1}}_{\mathbf{x}=\mathbf{x^{(i)}}} (\mathbf{f^{(i)}} - \mathbf{f_o})$$

This is the essence of the Newton-Raphson method for solving sets of nonlinear equations. In the problem of finite deformation analysis of structures, the *equilibrium condition* is represented by a system of non-linear algebraic equations

$$\mathbf{p}(\mathbf{d}) = \mathbf{K_s}(\mathbf{d}) \ \mathbf{d} \ ,$$

where the structural stiffness matrix $\mathbf{K_s}(\mathbf{d})$ includes geometric stiffness effects, which depend on the bar tensions, $T$, which, in turn depend on the displacements, $\mathbf{d}$. (See equation (9).) In other words, $\mathbf{K_s}(\mathbf{d})$ is a matrix that depends on the unknown displacements, $\mathbf{d}$. Note that the Jacobian matrix of this problem is $[\partial \mathbf{p}/\partial \mathbf{d}]$, which is not equal to $\mathbf{K_s}(\mathbf{d})$. An explicit expression for the Jacobian $[\partial \mathbf{p}/\partial \mathbf{d}]$ is extremely difficult to derive but we can approximate it using a numerical technique. There are many ways to approximate an unknown Jacobian, and two approaches will be described here.

In the first approach, we use the structural stiffness matrix, $\mathbf{K_s}$, as if it were the Jacobian matrix. In this approach, the iterations proceed as follows:

1. Initialize the displacements and bar tensions to be zero, $\mathbf{d}^{(0)} = \mathbf{0}$.
2. Assemble the structural stiffness matrix, for the initial configuration with zero displacements and zero bar tensions. Call this matrix $\mathbf{K_s}^{(0)} = \mathbf{K_s}(\mathbf{d}^{(0)})$.
3. Find the first approximation to the displacements, $\mathbf{d}^{(1)}$, by solving $\mathbf{p} = \mathbf{K_s}^{(0)}\mathbf{d}^{(1)}$.
4. With these displacements, find the bar forces, including finite deformation effects, using equation (9).
5. Re-compute the structural stiffness matrix using this set of bar forces for deflections $\mathbf{d} = \mathbf{d}^{(1)}$, $\mathbf{K_s}^{(1)} = \mathbf{K_s}(\mathbf{d}^{(1)})$.
6. Go back to step 3, and continue to iterate until $||\mathbf{p} - \mathbf{K_s}^{(i-1)}\mathbf{d}^{(i)}|| < \epsilon$ where $\epsilon$ is the convergence tolerance for the equilibrium error, or until you get tired of iterating.

In principle, this approach will converge to the correct solution for both stiffening systems and softening systems. However, this approach can be quite slow to converge, and may not converge at all if the problem has an inflection point.

In the second approach, the Jacobian matrix, which is also called the *tangent stiffness matrix*,[1] is approximated by a *secant stiffness matrix*, for which we will use the symbol $\bar{\mathbf{K}}_\mathbf{s}$. The secant stiffness matrix will be calculated using a technique attributed to C.G. Broyden.[2,3] In each iteration with the secant stiffness approach, we find incremental displacements, $\delta\mathbf{d}$, and add those displacements to the previously computed displacements. The procedure using Broyden's secant stiffness matrix is as follows:

1. Initialize the displacements and bar tensions to be zero, $\mathbf{d}^{(0)} = \mathbf{0}$.

[1] Can you guess why?

[2] Press, W.H., et. al, *Numerical Recipes in C*, Cambridge, 1992, section 9.7

[3] Broyden, C.G., "A class of methods for solving simultaneous nonlinear equations," *Mathematics of Computation,* vol. 19, no. 92 1965, pp. 577–593.

2. Assemble the structural stiffness matrix, for the initial configuration with zero displacements and zero bar tensions, call this matrix $\mathbf{K_s}^{(0)} = \mathbf{K_s}(\mathbf{d}^{(0)})$. This matrix will also be the initial secant stiffness matrix, $\bar{\mathbf{K}}_\mathbf{s}^{(0)} = \mathbf{K_s}^{(0)}$. [4]

3. Find the first incremental displacements, $\delta\mathbf{d}^{(0)}$, by solving $\mathbf{p} = \mathbf{K_s}^{(0)}\, \delta\mathbf{d}^{(0)}$.

4. Add these displacements to the initial displacements, $\mathbf{d}^{(0)}$, (which are zero), $\mathbf{d}^{(1)} = \mathbf{d}^{(0)} + \delta\mathbf{d}^{(0)} = \delta\mathbf{d}^{(0)}$.

5. With the displacements $\mathbf{d}^{(1)}$ find the bar forces, including finite deformation effects, using equation (9).

6. Compute the structural stiffness matrix, $\mathbf{K_s}^{(1)} = \mathbf{K_s}(\mathbf{d}^{(1)})$ using the set of bar forces for deflections $\mathbf{d}^{(1)}$, and compute the equilibrium error $\mathbf{p} - \mathbf{K_s}^{(1)}\, \mathbf{d}^{(1)}$

7. Update the secant stiffness matrix using Broyden's formula.[5]

$$\bar{\mathbf{K}}_\mathbf{s}^{(1)} = \bar{\mathbf{K}}_\mathbf{s}^{(0)} - \frac{(\mathbf{p} - \mathbf{K_s}^{(1)}\mathbf{d}^{(1)}) \times \delta\mathbf{d}^{(0)}}{\delta\mathbf{d}^{(0)} \cdot \delta\mathbf{d}^{(0)}}$$

   The *times* symbol $(\times)$ represents the vector outer product, which results in a matrix, and the *dot* $(\cdot)$ represents the vector inner product, which results in a scalar.

8. Using this secant stiffness matrix, find the next set of incremental displacements, $\delta\mathbf{d}^{(1)}$.

$$\bar{\mathbf{K}}_\mathbf{s}^{(1)}\, \delta\mathbf{d}^{(1)} = \mathbf{p} - \mathbf{K_s}^{(1)}\mathbf{d}^{(1)}$$

9. Add the incremental displacements, $\delta\mathbf{d}^{(1)}$ to the current displacements, $\mathbf{d}^{(1)}$,

$$\mathbf{d}^{(2)} = \mathbf{d}^{(1)} + \delta\mathbf{d}^{(1)}$$

10. Increment the iteration counter, $i = i+1$, return to step 6, and continue to iterate until the equilibrium error is sufficiently small, $||\mathbf{p} - \mathbf{K_s}^{(i)}\, \mathbf{d}^{(i)}|| < \epsilon$.

Note that the secant stiffness matrix as computed by Broyden's method is not symmetric. This could potentially lead to problems. The following diagram illustrates Newton-Raphson iterations using Broyden's secant stiffness approach. It is not hard to see how this approach can be substantially more efficient than the first approach.

[4]This is a good idea. Can you see why?
[5] Press, W.H., et. al, *Numerical Recipes in C*, Cambridge Univ. Press, 1992, section 9.7.

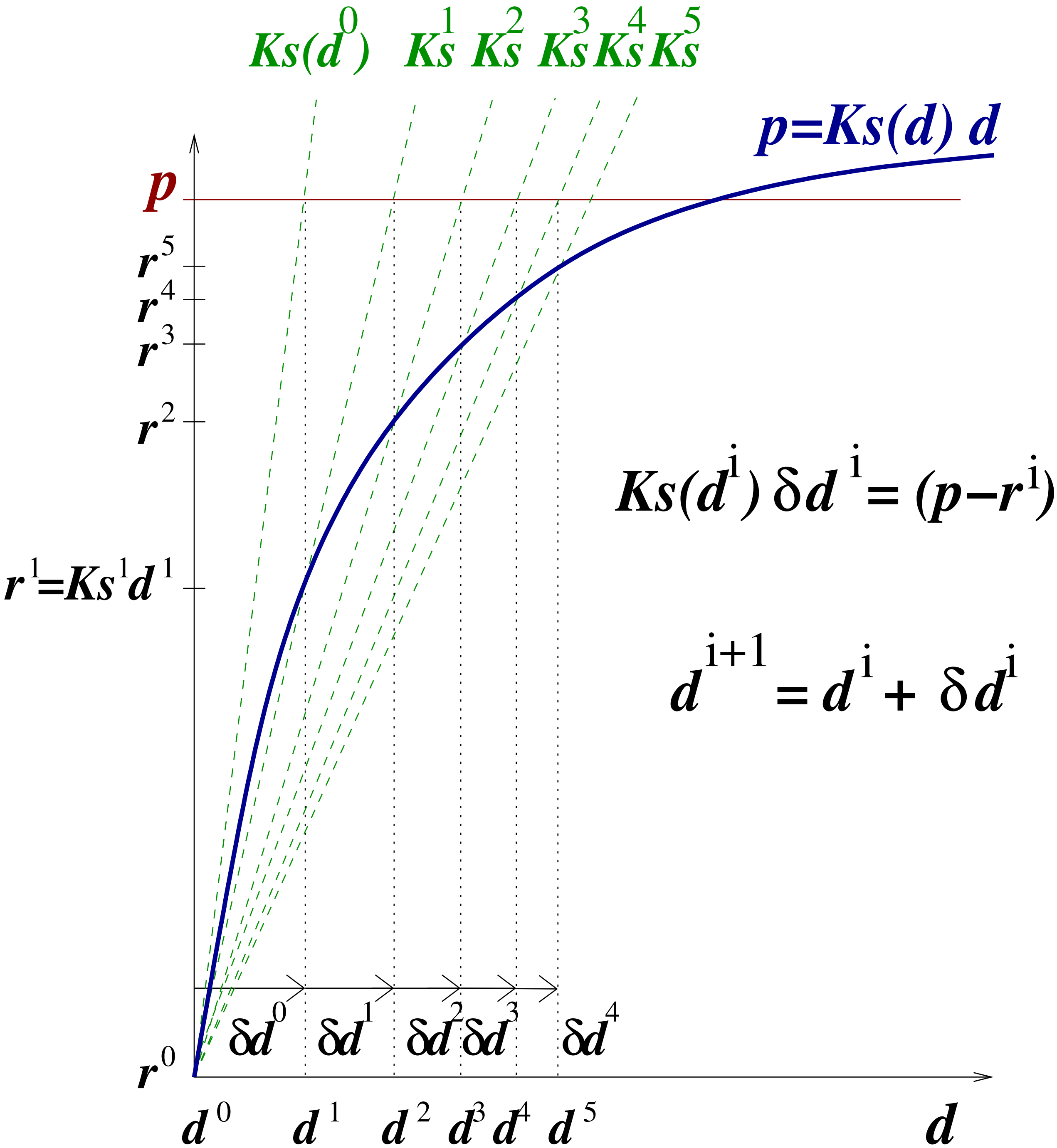


Figure 1. Modified Newton Raphson Method.

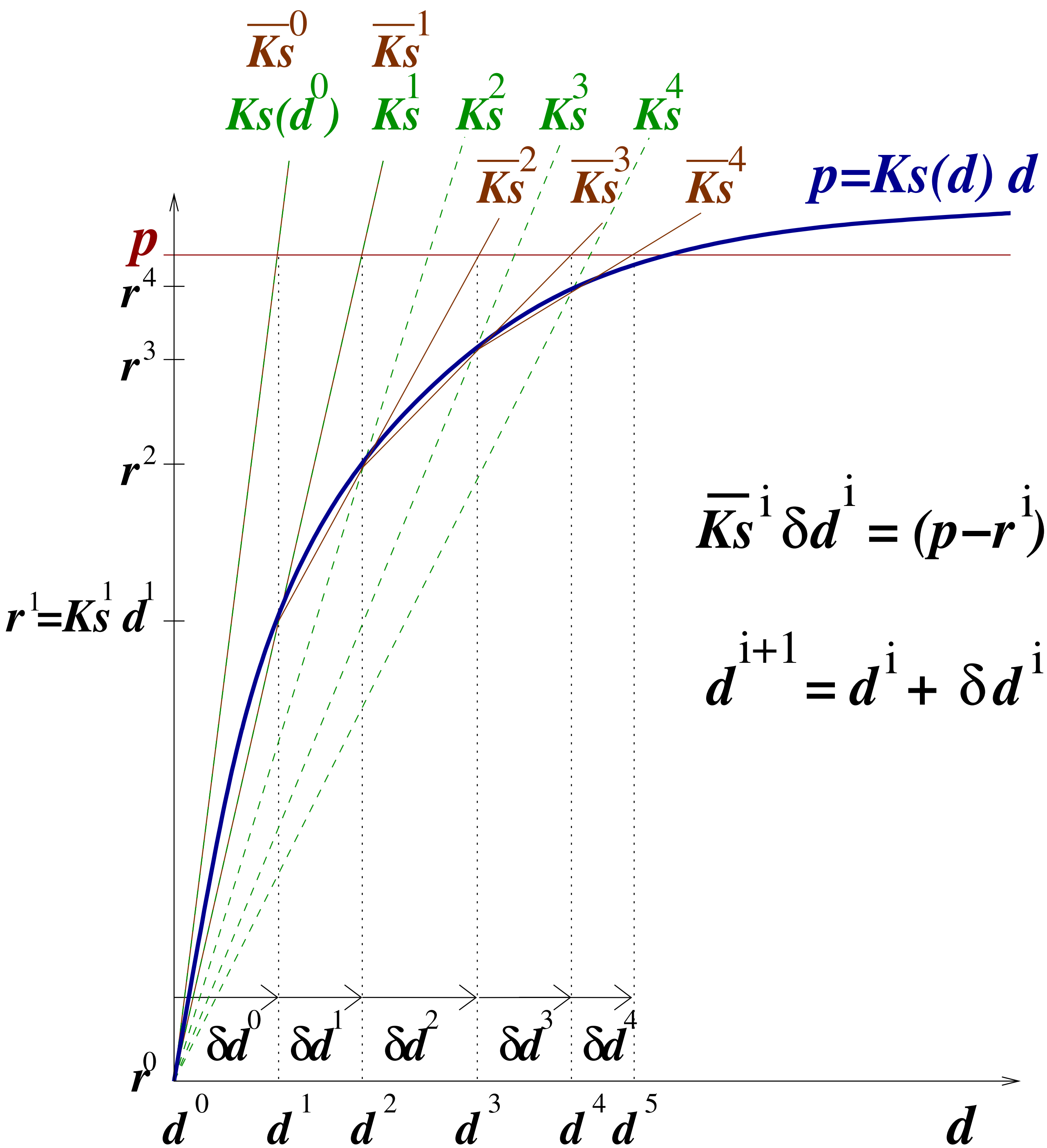


Figure 2. Modified Newton Raphson Method with Broyden secant stiffness updating.