Stress transformation:
tractors, linear algebra, and circles within circles.

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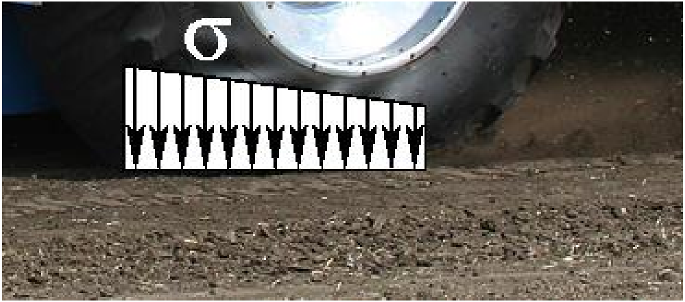
Spring 2017

traction is stress at a point on a plane

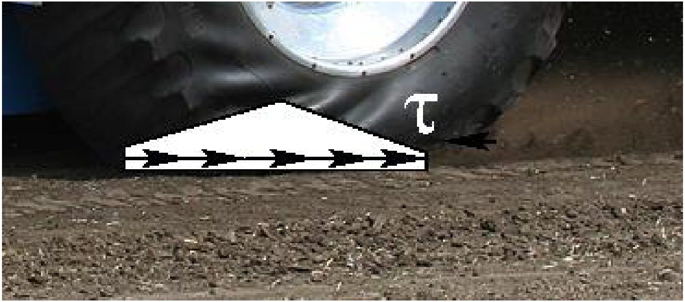
Sunday! SUNDAY!! **SUNDAY!!!**

traction forces in equilibrium

nonuniform distribution of traction stress components

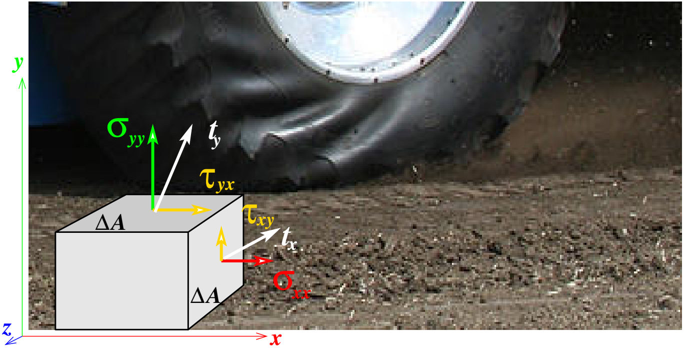


$\sigma(x)$



$\tau(x)$

a small volume under traction stress vectors, $\vec{t}_x$, $\vec{t}_y$



- Traction is a stress vector acting on a plane.
- The traction stress vector $\vec{t}_x$ acts on the *x*-normal plane and has components σ_{xx} and τ_{xy}.
- The traction stress vector $\vec{t}_y$ acts on the *y*-normal plane and has components τ_{yx} and σ_{yy}.

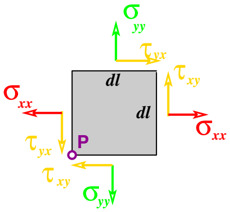
the state of stress is coordinate invariant

the state of stress in 2D

- The state of stress on an infinitesimal cubic volume is completely described by the traction vectors acting on the faces of the cube.
- In 2D (plane stress), the complete set of traction stresses is

$$S = \begin{bmatrix} \vec{t}_x \\ \vec{t}_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

- S is called the **stress tensor**
- $\tau_{xy} = \tau_{yx}$ ... S is a symmetric tensor



$$\sum M_P = 0 :$$

$$+\sigma_{xx}(\Delta A)(dl/2) - \sigma_{xx}(\Delta A)(dl/2)$$
$$+\sigma_{yy}(\Delta A)(dl/2) - \sigma_{yy}(\Delta A)(dl/2)$$
$$+\tau_{xy}(\Delta A)(dl) - \tau_{yx}(\Delta A)(dl) = 0$$

traction stress vectors on coordinate planes

- The traction stress vector $\vec{t}_x$ is the first row of S.

$$\vec{t}_x = [1\ 0] \cdot \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} = \vec{\imath} \cdot S$$

  Note: $\vec{\imath}$ is the unit normal to the x-plane.

- The traction stress vector $\vec{t}_y$ is the second row of S.

$$\vec{t}_y = [0\ 1] \cdot \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} = \vec{\jmath} \cdot S$$

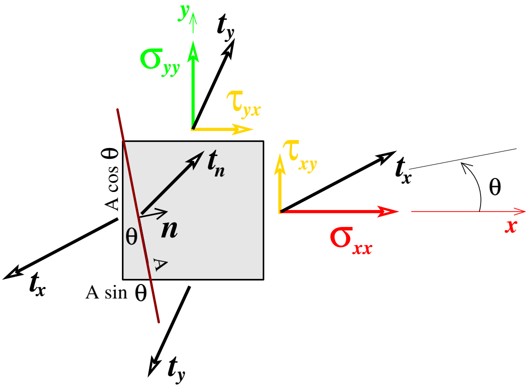
  Note: $\vec{\jmath}$ is the unit normal to the y-plane.

- Just as $\vec{t}_x = \vec{\imath} \cdot S$ and $\vec{t}_y = \vec{\jmath} \cdot S$, the stress vector on any other plane (with unit normal $\vec{n}$) is given by

$$\vec{t}_n = \vec{n} \cdot S$$

  This is what the stress tensor does.
  It defines tractions on any plane.

traction stress vectors on any plane: $\vec{t}_n = \vec{n} \cdot S$



equilibrium on the wedge:

$$A\vec{t}_n = (A\cos\theta)\vec{t}_x + (A\sin\theta)\vec{t}_y$$

$$\vec{t}_n = [\cos\theta \quad \sin\theta] \cdot \begin{bmatrix} \vec{t}_x \\ \vec{t}_y \end{bmatrix}$$

$$\vec{n} = [\cos\theta \quad \sin\theta]$$

$$S = \begin{bmatrix} \vec{t}_x \\ \vec{t}_y \end{bmatrix}$$

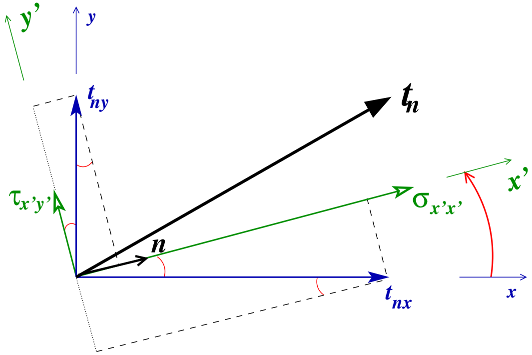
$$\vec{t}_n = \vec{n} \cdot S$$

The traction $\vec{t}_n = \vec{n} \cdot S$ is expressed in the *xy* coordinate system.

note: $(A\cos\theta) = An_x$ and $(A\sin\theta) = An_y$

traction stress vectors in other coordinates: $\vec{t}'_n = \vec{t}_n \cdot T$

- Let's express this traction $\vec{t}_n$ in a new coordinate system $x'y'$, where x' is along the unit normal $\vec{n}$ and y' is perpendicular to x'.
- These rotated components of $\vec{t}_n$ **are** the normal stress and shear stress on the plane with normal $\vec{n}$: $t'_n = [\sigma_{x'x'}\ ,\ \tau_{x'y'}]$



$$c = \cos\theta$$
$$s = \sin\theta$$

$$\sigma_{x'x'} = t_{nx}c + t_{ny}s$$
$$\tau_{x'y'} = -t_{nx}s + t_{ny}c$$

$$\left[\sigma_{x'x'}\ ,\ \tau_{x'y'}\right] = \left[t_{nx}\ ,\ t_{ny}\right]\begin{bmatrix} c & -s \\ s & c \end{bmatrix}; \qquad \vec{t}'_n = \vec{t}_n \cdot T\ ; \qquad T^{-1} = T^T$$

unit vectors and coordinate transformation in 2D

- The unit vectors along x and y are

$$\vec{\imath} = [1\ ,\ 0] \quad \text{and} \quad \vec{\jmath} = [0\ ,\ 1]$$

- The unit vectors along x' and y' ... rotated by θ (ccw) ... are

$$\vec{i}' = [\cos\theta\ ,\ \sin\theta] \quad \text{and} \quad \vec{j}' = [-\sin\theta\ ,\ \cos\theta]$$

- The coordinate transformation
  from a vector $\vec{t}$ (in xy) to the vector $\vec{t}'$ (in $x'y'$) is

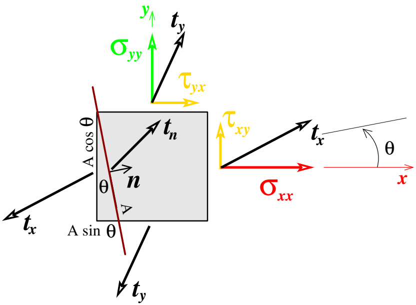
$$\left[\begin{array}{cc} t'_x & t'_y \end{array}\right] = \left[\begin{array}{cc} t_x & t_y \end{array}\right] \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right] = \left[\begin{array}{cc} t_x & t_y \end{array}\right] \left[\begin{array}{cc} \vec{\imath}\cdot\vec{i}' & \vec{\imath}\cdot\vec{j}' \\ \vec{\jmath}\cdot\vec{i}' & \vec{\jmath}\cdot\vec{j}' \end{array}\right]$$

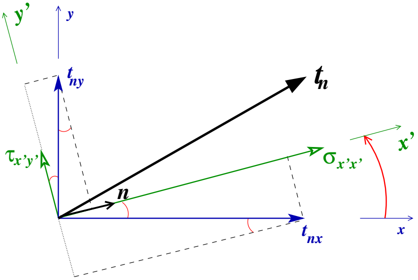
- The coordinate transformation
  from a vector $\vec{t}'$ (in $x'y'$) to the vector $\vec{t}$ (in xy) is

$$\left[\begin{array}{cc} t_x & t_y \end{array}\right] = \left[\begin{array}{cc} t'_x & t'_y \end{array}\right] \left[\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array}\right] = \left[\begin{array}{cc} t_x & t_y \end{array}\right] \left[\begin{array}{cc} \vec{i}'\cdot\vec{\imath} & \vec{i}'\cdot\vec{\jmath} \\ \vec{j}'\cdot\vec{\imath} & \vec{j}'\cdot\vec{\jmath} \end{array}\right]$$

- Tractions (stress vectors) can be transformed by coordinate rotation.

summary





- In the *xy* coordinate system, the traction on a plane with unit normal *n* is

$$\vec{t}_n = \vec{n} \cdot S$$
$$= [\cos\theta \,,\ \sin\theta] \cdot S$$

- The coordinate transformation from $\vec{t}_n$ (in *xy*) to $\vec{t}'_n$ (in $x'y'$) is

$$\vec{t}'_n = \vec{t}_n \cdot T$$

- Unit vectors can be similarly transformed, $\vec{n}' = \vec{n} \cdot T$ ... and ...

$$\vec{n} = \vec{n}' \cdot T^T$$
$$[\cos\theta \,,\ \sin\theta] = [1 \,,\ 0] \cdot T^T$$

putting it all together ... stress transformation

- Given:
  - stress in an *xy* coordinate system, S.
  - a unit vector $\vec{n}$ along x' in an $x'y'$ coordinate system.
- Find: The stress S' in the $x'y'$ coordinate system.

- Substitute ...

$$\begin{aligned}
\vec{t}'_n &= \vec{t}_n \cdot T \\
\vec{t}'_n &= \vec{n} \cdot S \cdot T \\
\vec{t}'_n &= \vec{n}' \cdot T^T \cdot S \cdot T \\
\vec{t}'_n &= \vec{n}' \cdot S' \\
S' &= T^T \cdot S \cdot T
\end{aligned}$$

In 2D, the stress transformation formula for a CCW rotation θ is:

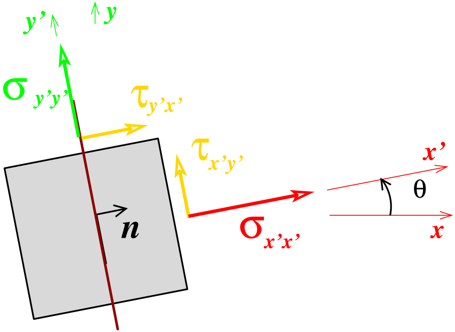
$$\begin{bmatrix} \sigma_{x'x'} & \tau_{x'y'} \\ \tau_{y'x'} & \sigma_{y'y'} \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

carry out the matrix product $S' = T^T \cdot S \cdot T$

$$\begin{bmatrix} \sigma_{x'x'} & \tau_{x'y'} \\ \tau_{y'x'} & \sigma_{y'y'} \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{xx}c^2 + \sigma_{yy}s^2 + 2\tau_{xy}cs & (\sigma_{yy} - \sigma_{xx})cs + \tau_{xy}(c^2 - s^2) \\ (\sigma_{yy} - \sigma_{xx})cs + \tau_{xy}(c^2 - s^2) & \sigma_{xx}s^2 + \sigma_{yy}c^2 - 2\tau_{xy}cs \end{bmatrix}$$

(simplified using symmetry ... $\tau_{xy} = \tau_{yx}$)



For what angle θ is $\tau_{x'y'} = 0$?

1. On what planes is the shear stress zero?
2. What is the transformation matrix T that diagonalizes S'?
3. What are the normal stresses on planes with no shear stress?

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$\begin{bmatrix} c & s \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} = \sigma_1 \begin{bmatrix} c & s \end{bmatrix}$$

$$\begin{bmatrix} -s & c \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} = \sigma_2 \begin{bmatrix} -s & c \end{bmatrix}$$

1. Planes with no shear stress are normal to eigenvectors of S.
2. The transformation matrix T of eigenvectors of S diagonalizes S.
3. The normal stresses on planes with no shear stress are the eigenvalues of S, σ_1 and σ_2. These are called **principle stresses**.

principle stresses are the eigenvalues of the stress tensor

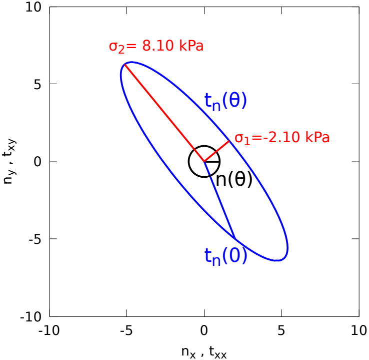
That sounds impressive.

But what does it mean?

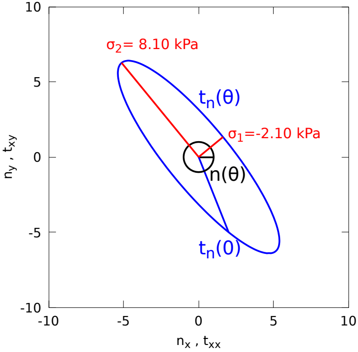
Let's look at an example in 2D.

$$S = \begin{bmatrix} \vec{t}_x \\ \vec{t}_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -5 & 4 \end{bmatrix} kPa$$

Here's a plot of $\vec{t}_n(\theta) = \vec{n}(\theta) \cdot S$ for all values of $\vec{n}(\theta) = [\cos\theta\ ,\ \sin\theta]$



principle stresses are the eigenvalues of the stress tensor



- At $\theta = 0$, $\vec{n}(\theta) = [1\ ,\ 0]$ and $\vec{t}_n = \vec{t}_x = [S_{11}, S_{12}] = [2, -5]$
- In general, the traction vector is not in line with the normal vector.
- There are two normal vectors for which the traction vectors align with the normal vector. These are the **eigenvectors** of S.
- For normal vectors aligned with the eigenvectors of S, the traction vector $\vec{t}_n$ is in line with the normal vector $\vec{n}$, and has a length equal to the **eigenvalues** of S. These are the principal stresses. (red lines).

Mohr’s Circle (for plane stress)

trigonometry and algebra lead to Mohr's circle

Substitute:

$$\cos^2\theta = (1+\cos 2\theta)/2$$

$$\sin^2\theta = (1-\cos 2\theta)/2$$

$$2\sin\theta\cos\theta = \sin 2\theta$$

into:

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{x'y'} = (\sigma_{yy}-\sigma_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

square equations, add them, and do some algebra to obtain:

$$\left(\sigma_{x'x'} - \frac{\sigma_{xx}+\sigma_{yy}}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2$$

and compare to:

$$(x-c)^2 + y^2 = r^2$$

to recognize the equation for a circle with center $(c, 0)$ and radius r!

Mohr's circle of stress

- The center of Mohr's circle is at $\left(\frac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right)$.
- The radius of Mohr's circle is the maximum shear stress, τ_{max}

$$\tau_{max} = \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2\right]^{\frac{1}{2}}$$

- There exists a plane on which the shear stresses are zero. This plane is inclined at an angle α from the *xy* axes. The normal stresses acting at this orientation are called the *principle stresses*, σ_1 and σ_2.
- From Mohr's circle:

$$\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \tau_{max} \qquad \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \tau_{max}$$

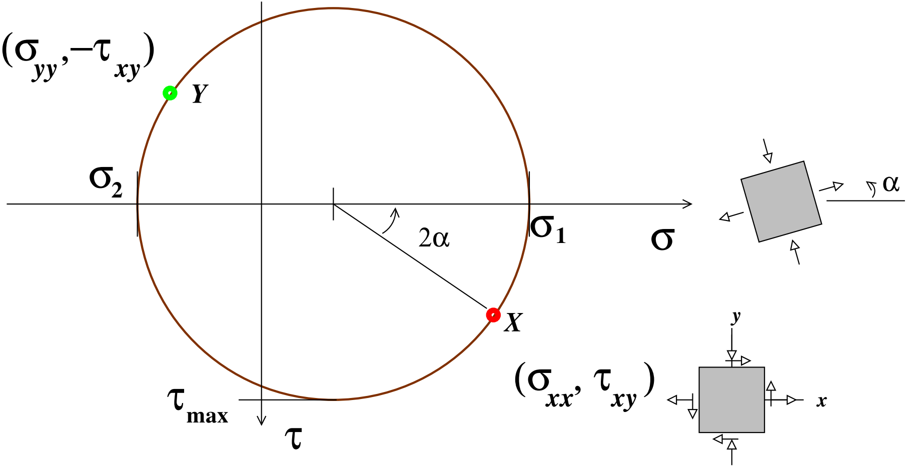
- The angle, α, is given by:

$$2\alpha = \arctan\left(\frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}\right)$$

Mohr's Circle Procedure - draw the circle

1. Draw the stress element aligned with the original x, y axes.
   Show the known stresses, σ_{xx}, σ_{yy}, and τ_{xy} on the stress element.
   Tension is positive.
   Compression is negative.
2. Draw the σ, τ axes for Mohr's circle with τ pointing down.
3. Locate the center C of the circle at $\left(\frac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right)$.
4. Locate the point X on the circle at (σ_{xx}, τ_{xy}).
5. Draw the circle passing through point X with center at point C.
   The circle will pass through point Y $(\sigma_{yy}, -\tau_{xy})$ 180 degrees from point X.

Mohr's Circle Procedure - draw the circle



Mohr's Circle Procedure - transform the stress

1. Draw new axes $x'y'$ rotated an angle θ counter-clockwise from the xy axes on the stress element from step 1a above.
2. Find numerical values for the principle stresses, the maximum shear stresses, and the angle, α, from the xy axes to the principle axes.
3. Separate the angle θ into two parts: $\theta = \alpha + \beta$.
4. Calculate the new stresses:

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \tau_{max}\cos(-2\beta)$$

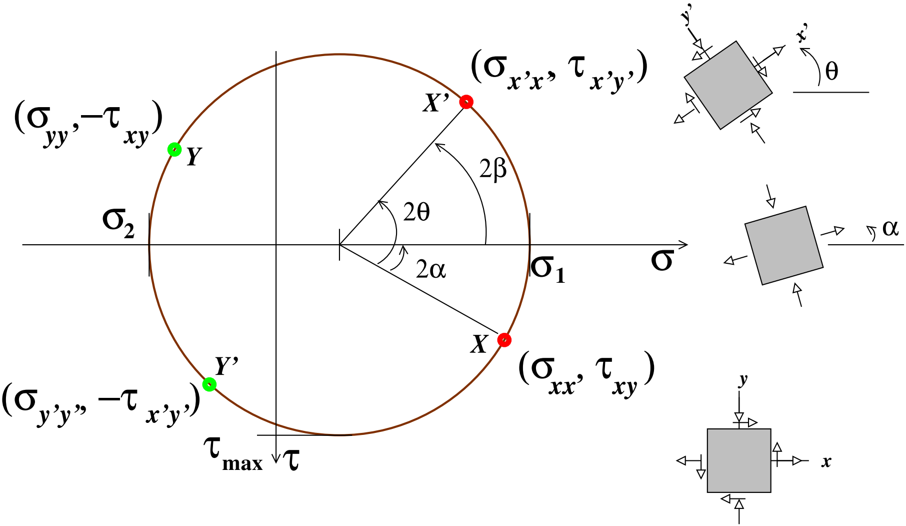
$$\sigma_{y'y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \tau_{max}\cos(-2\beta)$$

$$\tau_{x'y'} = \tau_{max}\sin(-2\beta)$$

5. Points X', $(\sigma_{x'x'}, \tau_{x'y'})$ and Y', $(\sigma_{y'y'}, -\tau_{x'y'})$ lie on the circle at an angle 2θ in the CCW direction from points X and Y, respectively.

Is this easier than $S' = T^T \cdot S \cdot T$?

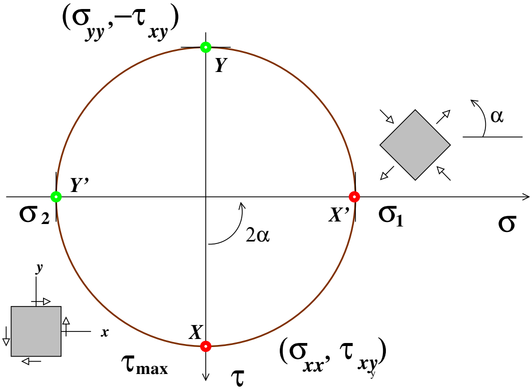
Mohr's Circle Procedure - transform the stress



notes on Mohr's circle

- Why is a counter-clockwise rotation of the coordinate axes translated into a counter-clock-wise rotation on Mohr's circle?
- Why are the rotations of the coordinate axes doubled when translated to Mohr's circle?

The answer to these questions can be answered by considering the state of stress caused by pure torsion:



notes on Mohr's circle

- This state of stress is described by a Mohr's circle centered at the origin of the $\sigma - \tau$ axes with a radius equal to $\bar{\tau}$.
- Points X and Y lie on the τ-axis
- Point X is located at $(0, +\bar{\tau})$.
  Point Y is located at $(0, -\bar{\tau})$.
- If a new set of axes $(x'y')$ are drawn at an angle of 45 degrees (CCW) to the xy axes, no shear stresses will act on surfaces normal to the $x'y'$ axes.
- The normal stress acting along the x'-axis will be in tension. The normal stress acting along the y'-axis will be in compression.
- These stresses correspond to points X' and Y', rotated in a CCW direction of 90 degrees from points X and Y, respectively.
- Point X' shows a tensile stress $+\bar{\tau}$.
  Point Y' shows a compressive stress $-\bar{\tau}$.

So what's easier, Mohr's Circle or $S' = T^T \cdot S \cdot T$?

```
S = [ 2 -5 ;                     % 2D stress tensor
     -5  4 ];

[ evecS , evalS ] = eig(S)  % principle directions & stresses

tau_max = (max(diag(evalS)) - min(diag(evalS)))/2

t = 47;                          % rotation angle theta (degrees)
T  = [ cosd(t) , -sind(t) ; % coordinate transformation matx
       sind(t) ,  cosd(t) ];

St = T'*S*T                      % transformed stress tensor
```

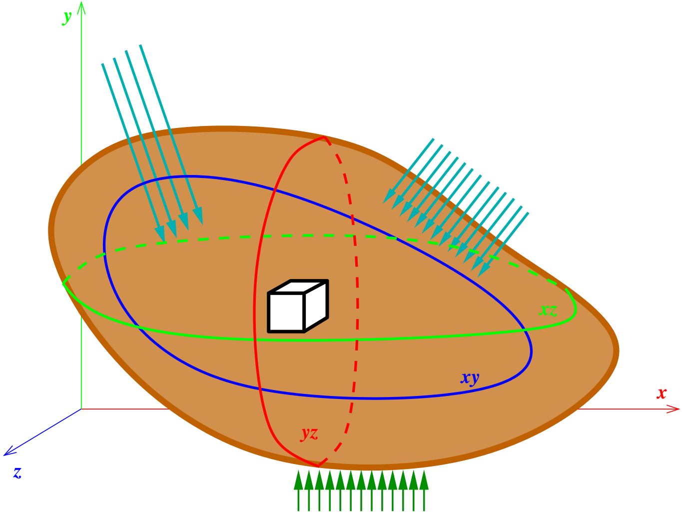
... you may decide for yourself.

It's good to know about Linear Algebra.

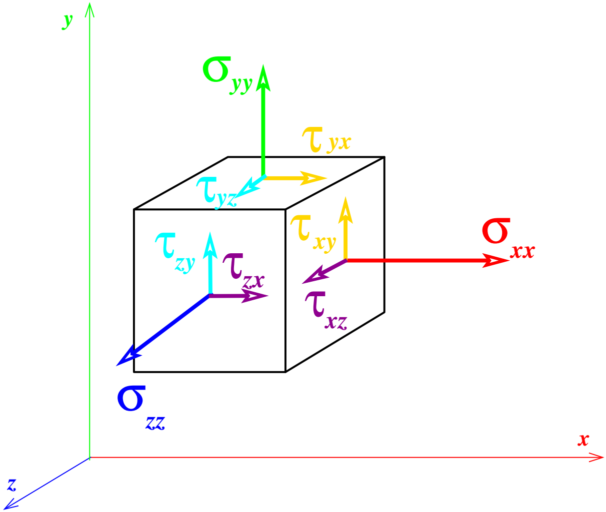
stress transformation in 3D

the continuum potato

cuts on three planes (*yz*, *zx*, *zy*)
and an internal elemental cube with surface areas ΔA.



The elemental stress cube showing "positive-facing" faces



$$\sigma_{ii} = \lim_{\Delta A \to 0} \frac{\Delta N_{ii}}{\Delta A_i}$$

$$\tau_{ij} = \lim_{\Delta A \to 0} \frac{\Delta V_{ij}}{\Delta A_i}$$

$$S = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

the stress tensor

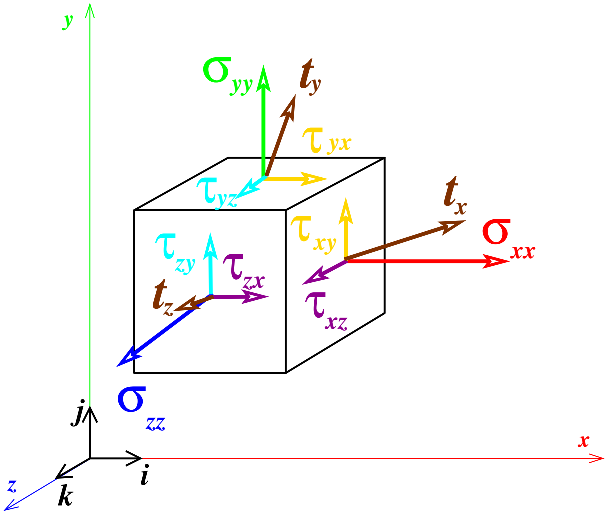
The stresses on the surfaces in the *xyz* coordinate system are the rows of the stress tensor.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\begin{bmatrix} \tau_{yx} & \sigma_{yy} & \tau_{yz} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\begin{bmatrix} \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

stress vectors, or “tractions” on the coordinate planes



$$\vec{t}_x = \vec{\imath} \cdot S$$

$$\vec{t}_y = \vec{\jmath} \cdot S$$

$$\vec{t}_z = \vec{\kappa} \cdot S$$

stress vectors, or "tractions" on the coordinate planes

The traction, $\vec{t}$, on a surface has *xyz* components.
The traction on the surface with unit normal $\vec{\imath}$ is

$$\vec{t}_x = \sigma_{xx}\vec{\imath} + \tau_{xy}\vec{\jmath} + \tau_{xz}\vec{k} = \left[\begin{array}{ccc} \sigma_{xx} & \tau_{xy} & \tau_{xz} \end{array} \right] = \vec{\imath} \cdot S$$

The traction on the surface with unit normal $\vec{\jmath}$ is

$$\vec{t}_y = \tau_{yx}\vec{\imath} + \sigma_{yy}\vec{\jmath} + \tau_{yz}\vec{\kappa} = \left[\begin{array}{ccc} \tau_{yx} & \sigma_{yy} & \tau_{yz} \end{array} \right] = \vec{\jmath} \cdot S$$

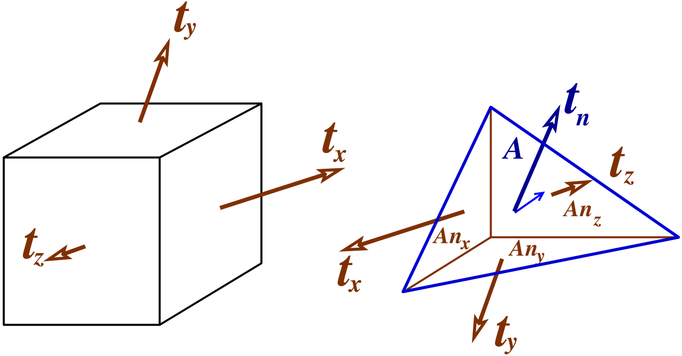
The traction on the surface with unit normal $\vec{\kappa}$ is

$$\vec{t}_z = \tau_{zx}\vec{\imath} + \tau_{zy}\vec{\jmath} + \sigma_{zz}\vec{\kappa} = \left[\begin{array}{ccc} \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{array} \right] = \vec{\kappa} \cdot S$$

Likewise ... (because the stress tensor is a linear operator)
The traction on **any** surface with unit normal $\vec{n}$ is

$$\vec{t}_n = \vec{n} \cdot S$$

traction on an arbitrary surface $\vec{t}_n = \vec{n} \cdot S$



$\vec{t}_x = \vec{\imath} \cdot S$

$\vec{t}_y = \vec{\jmath} \cdot S$

$\vec{t}_z = \vec{\kappa} \cdot S$

From equilibrium on the tetrahedron,

$$A\vec{t}_n = An_x\,\vec{t}_x + An_y\,\vec{t}_y + An_z\,\vec{t}_z$$

recall from 2D: $A\cos\theta = An_x$ and $A\sin\theta = An_y$

traction on an arbitrary surface $\vec{t}_n = \vec{n} \cdot S$

- Consider the three coordinate planes passing through a point with unit normal vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{\kappa}$
- The unit normal vector $\vec{n}$ to any plane can be expressed in terms of $\vec{\imath}$, $\vec{\jmath}$, and $\vec{\kappa}$.

$$\vec{n} = n_x \, \vec{\imath} + n_y \, \vec{\jmath} + n_z \, \vec{\kappa}$$

- From equilibrium on the tetrahedron:

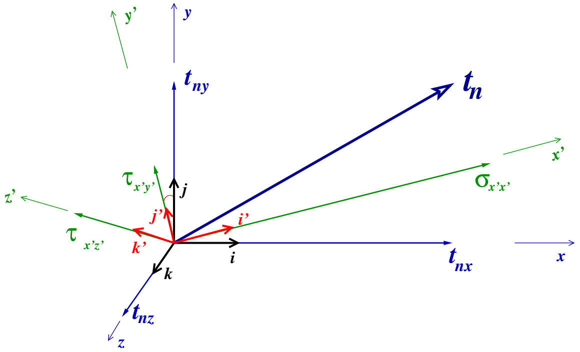
$$\vec{t}_n = n_x \, \vec{t}_x + n_y \, \vec{t}_y + n_z \, \vec{t}_z$$

- So the traction $\vec{t}_n$ on a plane with unit normal $\vec{n}$

$$\vec{t}_n = n_x \vec{\imath} \cdot S + n_y \vec{\jmath} \cdot S + n_z \vec{\kappa} \cdot S = (n_x \vec{\imath} + n_y \vec{\jmath} + n_z \vec{\kappa}) \cdot S = \vec{n} \cdot S$$

traction stress vectors in other coordinates: $\vec{t}'_n = \vec{t}_n \cdot T$

- Let's express this traction $\vec{t}_n$ in a new coordinate system $x'y'z'$ where $\vec{\imath}, \vec{\jmath}, \vec{\kappa}$ are unit vectors in the x, y, z axes, and $\vec{i}', \vec{j}', \vec{k}'$ are unit vectors in the x', y', z' axes.
- These rotated components of $\vec{t}_n$ **are** the normal stress and shear stress in the rotated coordinate system : $t'_n = [\sigma_{x'x'} \ , \ \tau_{x'y'} \ , \tau_{x'z'}]$



$$\vec{t}'_n = \vec{t}_n \cdot T$$

$$T = \begin{bmatrix} \vec{\imath} \cdot \vec{i}' & \vec{\imath} \cdot \vec{j}' & \vec{\imath} \cdot \vec{k}' \\ \vec{\jmath} \cdot \vec{i}' & \vec{\jmath} \cdot \vec{j}' & \vec{\jmath} \cdot \vec{k}' \\ \vec{\kappa} \cdot \vec{i}' & \vec{\kappa} \cdot \vec{j}' & \vec{\kappa} \cdot \vec{k}' \end{bmatrix}$$

$$T^{-1} = T^T$$

recall coordinate transformation matrix in 2D

transformation of tractions and stresses

apply coordinate transformation to a traction

- the traction on surface "*n*" is
  ... $\vec{t}_n = \vec{n} \cdot S$
- the coordinate transformation from unit normal $\vec{n}'$ to unit normal $\vec{n}$ is
  ... $\vec{n} = \vec{n}' \cdot T^T$
- the coordinate transformation from traction vectors $\vec{t}_n$ to $\vec{t}'_n$ is
  ... $\vec{t}'_n = \vec{t}_n \cdot T$

$$\begin{aligned}
\vec{t}'_n &= \vec{t}_n \cdot T \\
&= \vec{n} \cdot S \cdot T \\
&= \vec{n}' \cdot T^T \cdot S \cdot T \\
&= \vec{n}' \cdot (T^T \cdot S \cdot T) \\
&= \vec{n}' \cdot S'
\end{aligned}$$

So, the stress tensor *S*' in a coordinate system with axis $\vec{n}'$ can be expressed in terms of the stress tensor *S* in a coordinate system with axis $\vec{n}$ where $\vec{n}' = \vec{n} \cdot T$,

$$S' = T^T \cdot S \cdot T$$

Mohr’s circle in 3D

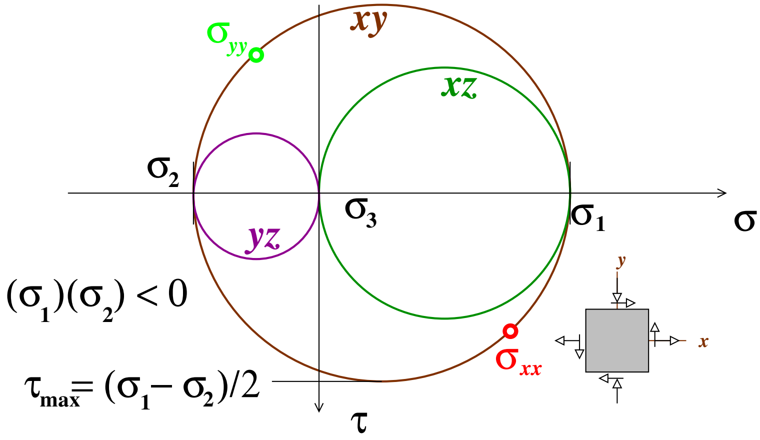
Mohr's circle in 3D

- Make three Mohr's circles,
  - one for the *xy* stress square,
  - one for the *yz* stress square, and
  - one for the *xz* stress square.

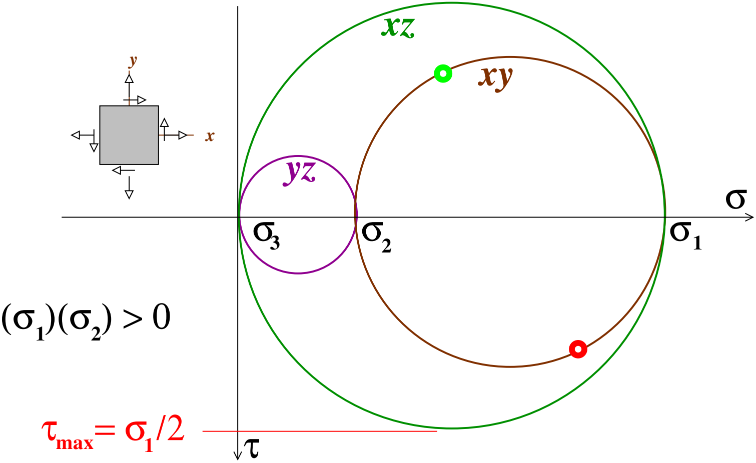
- The three circles will share three principle stresses.

- If there is no traction on one face of the stress cube, then then one of the principle stresses is zero.
  In this case:
  - If $\sigma_1 > 0$ and $\sigma_2 < 0$, then $\tau_{max} = (\sigma_1 - \sigma_2)/2$
  - If $\sigma_1 > 0$ and $\sigma_2 > 0$, then $\tau_{max} = +\sigma_1/2$
  - If $\sigma_1 < 0$ and $\sigma_2 < 0$, then $\tau_{max} = -\sigma_2/2$

Mohr's circle in 3D for $\sigma_1 > 0$, $\sigma_2 < 0$, and $\sigma_3 = 0$



Mohr's circle in 3D for $\sigma_1 > 0$, $\sigma_2 > 0$, and $\sigma_3 = 0$



What are the 3D coordinate rotations
to get from the x, y, z coordinates to the τ_{max} coordinates?

... (α about z, 45 deg. about y').

Biaxial tension: ductile shear failure. Triaxial tension: brittle tensile failure.