Split–Award Procurement Auctions with Uncertain Scale Economies: Theory and Data*

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Abstract

In a number of observed procurements, the buyer has employed an auction format that allows for a split-award outcome. We focus on settings where the range of uncertainty regarding scale economies is large and, depending on cost realizations, the efficient allocations include split-award outcomes as well as sole-source outcomes (one active supplier). We examine the price performance and efficiency properties of split-award auctions under asymmetric information. In equilibrium, both award outcomes can occur: the split-award outcome arises only when it minimizes total costs; sole-source outcomes, however, occur too often from an efficiency viewpoint. Equilibrium bids involve pooling at a common price for the split award, and separation for sole-source awards. We provide conditions under which the buyer and suppliers all benefit from a split-award format relative to a winner-take-all unit auction format. Model predictions are assessed with data on submitted 'step-ladder' bid prices for a US defense split-award procurement. (JEL Classification C72, D44, D82, L13)

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1 Introduction

We often observe procurements in which a buyer employs an auction format that allows for a split-award outcome. Split-award auctions are used frequently in defense procurement, where examples include fighter engines, missiles and submarines. Procurement contracts that divide production awards across suppliers are also very common in the private sector. A prominent recent example involves Singapore Airlines who simultaneously solicited bids from Boeing and Airbus and chose to purchase from both.[1] A number of commercial buyers have also begun to use Web-based bidding processes that result in divided production awards.[2] Together with other public and private sector procurements, these examples highlight an important and recurrent problem in procurement, namely, the appropriate format for competition among suppliers. At a basic level, this involves a choice between competition on a 'winner-take-all' basis, where only one firm produces, or competition that allows for split awards, where multiple firms produce. In addition to procurement, this is an important issue for a number of regulated industries in which the potential for multiple-provider service exists (e.g. cable TV and managed competition in health care). With regard to basic efficiency considerations, we expect scale economies to favor winner-take-all while scale diseconomies favor split awards. In practice, however, there are serious information problems associated with an *ex ante* determination of the extent of scale economies.[3]

In this paper we focus on environments with asymmetric cost information where the range of uncertainty is large enough that efficient allocations include both sole-source awards (all items to one supplier) and split awards. We examine the price performance and efficiency properties of split-award auctions in this setting. Our analysis provides an equilibrium view of the bidding incentives and allows us to address several questions and concerns that have been raised about observed prices and outcomes. Public officials who have been involved in split-award auctions have not always reacted with similar opinions. In fact, a number of officials and observers have been quite enthusiastic about the auction results, but several others have been critical. A common concern regarding bidding incentives in a split-award format is the potential use of a 'bid-to-lose' strategy in which a supplier chooses not to bid aggressively but rather to submit 'high' bid prices for the sole-source award and aim at obtaining only a share of the total award with more competitive bids for split-award outcomes.[4] In our analysis, we examine how the incentives to bid for a split and a sole-source award are related to both each other and the extent of scale economies, and we assess the impact on the

[1] In the private sector, other procurements involving split awards include General Motors (auto parts), and IBM (computer chips); for additional examples and discussion, see Burnett and Kovacic [7] and Anton and Yao [2]. In the public sector, defense remains the most important single application (see Rogerson [26]).

[2] For additional examples along these lines, see recent work on supply chains in the operations management literature, including Elmaghraby [11], Tunca and Wu [27], Cachon and Zhang [8] and Bernstein and de Vericourt [6].

[3] A number of competing forces are often present. Burnett and Kovacic [7] discuss several, including learning effects, spillovers, duplication of fixed assets, technology transfer costs, and incentives for cost minimization.

[4] Pyatt [24], for example, describes favorable outcomes and dollar benefits for a number of specific projects; Beltramo [4] and Meeker [22] are more critical. Burnett and Kovacic [7] as well as Pyatt [24] discuss the 'bid-to-lose' issue further.

buyer and suppliers' welfare.

The model, which is based on Anton and Yao [2], involves a procurement auction where two suppliers submit sealed bids to a buyer who seeks to procure a fixed total number of items.[5] A bid specifies two prices, one for supplying all of the items (sole-source award) and one for supplying the split quantity (split award). Each supplier has private cost information. Once the bids are submitted, the buyer chooses an award that minimizes the total payment to the bidders. This auction format allows for discriminatory pricing and, as the buyer chooses optimally *ex post*, the award scheme is consistent with a limited ability to commit for the buyer.

The fundamental departure from Anton and Yao [2] is the presence of scale economies. In that analysis, the split is always first-best efficient (global diseconomies of scale) and only the split is awarded in equilibrium. In the present paper, scale economies are present and, hence, the efficient award choice now depends crucially on the aggregation of the privately observed cost information held by the individual suppliers. As we show, scale economies have a strong effect on the bidding equilibrium and may lead to inefficient outcomes. In addition, the present framework allows us to address the 'bid-to-lose' effect noted above, since sole-source outcomes occur in equilibrium with positive probability.

Our main results are as follows. We identify a set of equilibria parametrized by a threshold type that demarcates the range of split outcomes from sole-source ones. This set also includes the equilibrium of the standard one-unit auction as a special case. In general, the sole-sourcing outcome occurs too often from an efficiency viewpoint. Thus, price competition in a split-award auction allows the buyer and suppliers to capture some but not all of the available efficiency gains.

In equilibrium, bidding involves pooling at a common split price for an interval of high cost types, and separation with a sole-source award for low cost types. Intuitively, the incentive to bid aggressively and undercut an opponent is stronger for a supplier (cost type) when a sole-source award is likely to be efficient. In contrast, this incentive is muted when the split outcome is likely to be efficient, because this allows both suppliers to earn a positive profit (for all cost types). As in Anton and Yao [2], pooling at a common split price can be viewed as an implicit form of equilibrium bidding coordination.

Turning to the welfare properties, we provide conditions under which a split-award auction format generates a Pareto improvement over the winner-take-all auction format. The unit auction is useful as a benchmark because it is commonly employed in practice and, with splits excluded by definition, it helps to isolate the differential effect of split outcomes. We find that the buyer pays a lower price in the range of split outcomes, but a higher price in the sole-source range (relative to bid prices in the unit auction). Thus, when the likelihood of a split outcome is sufficiently large, the buyer will pay a lower expected price.

At the same time, supplier profits (interim) are also higher. Because a split can occur only when

[5]The model is structured to capture several essential features of split-award auctions as conducted by the U.S. Government; for more on this point, see Anton and Yao [2]. See also the discussion in Burnett and Kovacic [7], regarding legislative reforms and mandates for a competitive procurement format.

it is cost efficient relative to a sole-source outcome, supplier profits in the split range can exceed the corresponding profits in a unit auction (where a supplier who wins must produce the full production award). As noted above, higher bid prices over the sole-source range necessarily improve supplier profits. Thus, suppliers in each range earn higher profits. Intuitively, the joint surplus for a Pareto improvement is provided by the efficiency gains in the split region. Although pooling in the split region allows prices in the sole-source region to be higher, the efficiency gains allow for relatively lower prices in the split region and all parties can benefit in equilibrium.

These results provide a useful framework for evaluating the performance of split-award auctions. First, two features of equilibrium bids in the split range are consistent with a bid-to-lose interpretation. Due to pooling, the split price does not vary with realized cost types and, therefore, lower cost bidders are not more aggressive. Further, in the split range the suppliers do submit a 'high' bid price for a sole-source award and this is designed to steer the buyer towards the split. However, even though pooling at a common price represents a form of equilibrium bidding coordination and suppliers earn relatively higher profits, this is not necessarily an undesirable outcome for the buyer. This is because the split is the equilibrium award only when it is efficient and the price structure allows the buyer to obtain part of the efficiency gains.

The second point concerns prices in the sole-source range. Suppose, for example, a buyer found that prices for sole-source awards in a split-award format exceeded those when the procurement was conducted on a winner-take-all basis. Here, it would be wrong to take this as evidence of poor overall price performance. Rather, we would expect such a price shift (as part of a Pareto improvement) and a full evaluation would weight this as a partial offset to savings over the split range.

In the literature, our paper relates most directly with two streams of work. First, beginning with the work of Wilson [29] on share auctions, several studies focus on explicit bidding strategies for auctions with divisible or multiple objects. For the case of full information (among bidders), Bernheim and Whinston [5] examine menu auctions, and Anton and Yao [1] examine split-award auctions and, more recently, Inderst [14] examines split awards with multiple buyers. As noted above, Anton and Yao [2] consider incomplete information and study equilibrium bidding when a split award is full-information efficient. Klotz and Chatterjee [13] study a model of repeated procurement with costly entry. Perry and Sákovics [23] consider a related problem but with a different auction format and bidders who are subject to constant returns to scale. They analyze a sequential auction in which a procurement contract is split into two possibly asymmetric parts and a bidder can only win one of the two parts. Their emphasis is on the optimal size of the two sub-contracts and on the effects on entry of the sequential auction format with respect to a pure single-source auction. In these papers, as in our own, the emphasis is on a positive analysis of equilibrium in a specific auction format.[6]

The second stream of related work deals with regulation and monopoly versus duopoly market structure. This includes Auriol and Laffont [3], Dana and Spier [9], and McGuire and Riordan

[6]For related work involving market experiments see Davis and Wilson [10] and Li and Plott [16]. On the empirical side, see the analysis of cost savings from dual sourcing in defense procurement by Lyons [17].

[20].[7] These papers examine optimal regulatory policy and auction design and assess the impact of asymmetric cost information on the extent of monopoly or duopoly allowed in the market.[8] Relative to the first stream of work, there is more emphasis on normative dimensions and a stronger set of assumptions regarding the commitment abilities of the buyer. Note that the monopoly versus duopoly distinction corresponds (roughly) to a sole-source versus a split award.

On the closely related topic of market design, McMillan [21] provides an excellent recent discussion of theory and policy. Taking inspiration from his discussion of defense procurement, we examine bidding data from defense contractors in relation to our equilibrium analysis. The novel feature of the data is that we have the full set of submitted 'step-ladder' bid prices, as well as the realized award choice, for several award rounds. Noting the limitations for inference based on one procurement project, we find that the bids exhibit several properties that are consistent with an equilibrium interpretation of the 'bid-to-lose' strategy noted above.

We present the model in Section 2. The bidding equilibrium, our primary result, is presented and discussed in Section 3. The welfare analysis is carried out in Section 4. Next, we examine the bidding data. We conclude in Section 6. All proofs are in the Appendix.

2 The Model

We examine a sealed-bid, low-price auction format in which a buyer seeks to procure a given total quantity, normalized to one unit, from two suppliers, $i = A, B$. All parties are risk neutral and the suppliers are *ex ante* symmetric. The three possible auction awards are denoted by SS_A, SS_B, and Σ. At the 'sole-source' award SS_i, firm i supplies one unit while firm $j \neq i$ supplies zero. At the 'split' award Σ, each firm supplies the buyer one-half of the quantity. With three potential awards, the split-award auction is as simple as possible.

Let θ_i and $C(\theta_i)$ be the total cost of supplier i of producing a quantity of one and one-half, respectively, where θ_i is private information of supplier i. The cost parameter for each supplier is an independent draw from a distribution F with a positive continuous density f and interval support $\Theta := \left[\underline{\theta}, \overline{\theta}\right]$. We assume that for each type costs increase with quantity: $\theta > C(\theta) > 0$ for all $\theta \in \Theta$. The values θ and $C(\theta)$ should be interpreted as the increase in cost with respect to a status quo of supplying zero to the buyer; this is equivalent either to assuming that the cost of no production is zero or to assuming that, over the period analyzed, there is a fixed cost that cannot be avoided. Next, we assume that $0 < C'(\theta) < 1$ holds for all $\theta \in \Theta$. Intuitively, this means that "marginal cost" shifts up with θ, as higher cost types have a greater cost of supplying the increased quantity (from 0 to $\frac{1}{2}$, and from $\frac{1}{2}$ to 1).

[7] See, also, Riordan [25] in which a buyer chooses the number of qualified suppliers, and Wolinsky [28] where the market structure can vary continuously between monopoly and duopoly.

[8] Laffont and Tirole [15] discuss mechanism design and dual sourcing in this context. Maskin and Riley [19] treat the problem of designing a multiple object auction when buyer valuations are concave in quantity, so that efficiency favors dividing the award (equally, for equal valuations).

Our third cost assumption is pivotal for scale economies and formalizes the notion that the range of cost uncertainty is significant. Let $H(\theta_A, \theta_B)$ denote the cost difference between SS_A or SS_B, whichever has lower costs, and Σ across the range of cost types, as defined by

$$H(\theta_A, \theta_B) \equiv \min\{\theta_A, \theta_B\} - [C(\theta_A) + C(\theta_B)]. \tag{1}$$

Intuitively, H is a direct measure of the efficiency gains that are generated by the possibility of awarding split production. We make the following assumption.

Assumption 1 $\exists \theta_m \in \left(\underline{\theta}, \overline{\theta}\right)$ *such that* $H\left(\theta_m, \overline{\theta}\right) = 0$.

The assumption implies that the cost range is sufficiently large that each award is efficient for some pair of cost types: since $H\left(\theta, \overline{\theta}\right) > 0$ holds for $\theta > \theta_m$, diseconomies of scale are present and Σ is efficient for sufficiently high types. These cost assumptions are sufficient for the bidding analysis that follows. Narrowly interpreted, they pertain to production costs. More generally, however, a variety of reduced form interpretations are possible. We discuss these and develop the efficiency properties in more detail further below.

Throughout the paper we will work with Assumption 1. Since the assumption may sometimes be difficult to interpret, the reader may want to consider the following stronger and more intuitive assumption.

Assumption 2 $0 < C'(\theta) < \frac{1}{2}$ *and* $\underline{\theta} - 2C(\underline{\theta}) < 0 < \overline{\theta} - 2C\left(\overline{\theta}\right)$.

The assumption implies that there is $\theta^* \in \left(\underline{\theta}, \overline{\theta}\right)$ such that economies of scale prevail when $\theta < \theta^*$ while diseconomies prevail when $\theta > \theta^*$. Assumption 2 implies $H\left(\underline{\theta}, \overline{\theta}\right) < 0 < H\left(\overline{\theta}, \overline{\theta}\right)$; since $H\left(\theta, \overline{\theta}\right)$ is continuous in θ it follows that Assumption 1 is satisfied whenever Assumption 2 is. Thus, all the results of the paper hold under the stronger assumption.

A bid in the auction is an ordered-pair (p, p_Σ), where p is the sole-source price at which a supplier offers to deliver one unit and p_Σ is the split price at which one half is offered. In response to bids of (p, p_Σ) and $(\hat{p}, \hat{p}_\Sigma)$ submitted by i and j, respectively, the buyer chooses the auction award SS_A, SS_B or Σ that achieves $\min\{p, p_\Sigma + \hat{p}_\Sigma, \hat{p}\}$, so that for any submitted bids the buyer chooses the award with the lowest total price. In the event of a tie the buyer is indifferent between two or more awards. We assume that ties are broken in favor of splitting, that is whenever $\min\{p, \hat{p}\} = p_\Sigma + \hat{p}_\Sigma$ each firm supplies $\frac{1}{2}$. All our results hold for any tie-breaking rule, and this particular rule is chosen only to simplify the presentation. We leave unspecified the tie-breaking rule when the two sole-source bids are identical and strictly less than $p_\Sigma + \hat{p}_\Sigma$.

To specify payoffs, suppose supplier i submits a bid (p, p_Σ) while j submits $(\hat{p}, \hat{p}_\Sigma)$. Then, for a realized cost type of θ_i, the payoff function for bidder i induced by the auction rules satisfies

$$u((p, p_\Sigma), (\hat{p}, \hat{p}_\Sigma), \theta_i) = \begin{cases} 0 & \text{if} \quad \hat{p} < \min\{p, p_\Sigma + \hat{p}_\Sigma\}; \\ p_\Sigma - C(\theta_i) & \text{if} \quad p_\Sigma + \hat{p}_\Sigma \le \min\{p, \hat{p}\}; \\ p - \theta_i & \text{if} \quad p < \min\{\hat{p}, p_\Sigma + \hat{p}_\Sigma\}. \end{cases} \tag{2}$$

We examine symmetric Bayesian-Nash equilibria (hereafter, bidding equilibria) for this auction game. A bidding strategy for a supplier is a pair of F-measurable functions $(P, P_\Sigma) : \left[\underline{\theta}, \overline{\theta}\right] \to \Re^2_+$. Thus, we seek a bidding strategy (P, P_Σ) such that

$$\left(P\left(\theta_i\right), P_\Sigma\left(\theta_i\right)\right) \in \arg \max_{(p, p_\Sigma) \in \Re^2_+} \int_{\underline{\theta}}^{\overline{\theta}} u\left(\left(p, p_\Sigma\right), \left(P\left(\theta_j\right), P_\Sigma\left(\theta_j\right)\right), \theta_i\right) dF\left(\theta_j\right) \qquad \forall \theta_i \in \left[\underline{\theta}, \overline{\theta}\right]. \tag{3}$$

Before turning to the derivation of the bidding equilibrium set, we describe, and illustrate in Figure 1 below, the full information efficient award choice. This is useful for our bidding and welfare analysis. Essentially, the split is efficient (minimizes the sum of supplier costs) in a 'band' around the 45° line in the $\left[\underline{\theta}, \overline{\theta}\right] \times \left[\underline{\theta}, \overline{\theta}\right]$ type square, while a sole-source award is efficient otherwise. Figure 1 provides a typical graph of the efficient allocation.

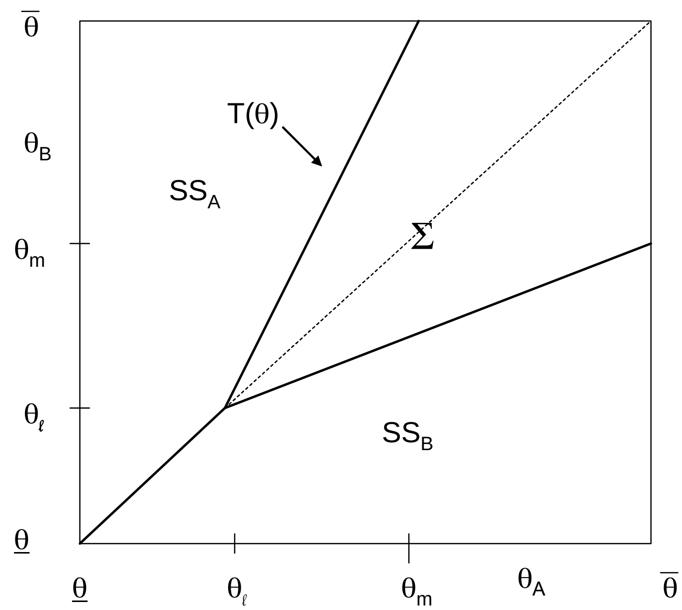

Figure 1: Efficient Allocation

When both suppliers are 'high cost' types, each above θ_m, we have $H > 0$ and the split minimizes costs. This is due to the presence of diseconomies of scale for each supplier in this range of cost types.

Scale economies come into play when at least one supplier is a 'low cost' type, below θ_m, and a sole-source award is efficient outside a band around the 45° line. Here, a large difference in θ_A and θ_B translates into a large cost advantage. Scale economies can arise in a procurement setting when experience is a dominant influence on costs. Another effect, analyzed in Auriol and Laffont [3] and Klotz and Chatterjee [13], is the duplication of fixed costs which may arise with two producers.

Formally, the above cost assumptions imply that there is a lower threshold type θ_ℓ, where $\underline{\theta} \leq \theta_\ell < \theta_m$, such that only sole-source awards are efficient when both θ_A and θ_B are below θ_ℓ. In the middle range, $\theta_\ell < \theta < \theta_m$, there is a critical opponent type, $T(\theta)$, such that

$$H\left(\theta, T\left(\theta\right)\right) = \theta - C\left(\theta\right) - C\left(T\left(\theta\right)\right) = 0.$$

The function T is defined over the range (θ_ℓ, θ_m); it is continuous, strictly increasing and it satisfies $T(\theta) > \theta$ and $T(\theta_m) = \overline{\theta}$. Essentially, $T(\theta)$ traces out the efficiency boundary between split and sole-source awards. If $\theta_B > T(\theta_A)$ then a sole-source award for A is efficient; if $T(\theta_A) > \theta_B > \theta_A$, then the split is efficient. Figure 1 displays the case of $\theta_\ell > \underline{\theta}$ in which $T(\theta_\ell) = \theta_\ell$, but the case $\theta_\ell = \underline{\theta}$ is also possible, depending on the shape of the cost function C.[9]

To summarize, the existence of the critical type θ_m captures formally the notion that the range of cost uncertainty is significant. The split award Σ is necessarily efficient for high cost types. For low cost types, efficiency necessarily involves sole-source awards when cost types are sufficiently far apart, and it is possible that efficiency requires a sole-source award for types that are close (or identical) in value. Thus, each award is efficient for some realized cost types.

3 Equilibrium Bidding

In this section we characterize the equilibrium bidding strategies. The structure of prices is discussed first, and this is followed by an examination of the threshold type that demarcates the separating and pooling regions of each equilibrium. Equilibrium awards are then considered relative to efficient awards and the role of prices in supporting the equilibrium is discussed. Finally, we discuss how the equilibrium relates to the bid-to-lose effect.

The structure of the bidding equilibrium is motivated by the relationship between efficient awards and costs. Intuitively, we might expect types with high θ to submit bids that are more likely to induce the buyer to choose the split, as costs are lower when each supplier produces the split quantity. For types with low θ, efficiency considerations point to sole-source awards, hence we expect such suppliers to bid aggressively for the full award.

Consider, then, a candidate bidding equilibrium with the following structure. Let $\tau \in \left[\theta_m, \overline{\theta}\right]$ denote a fixed threshold level, and suppose that the equilibrium award is Σ when both suppliers are 'high cost' relative to τ, as defined by $\min\{\theta_A, \theta_B\} > \tau$; and a sole-source award when at least one supplier is 'low cost' relative to τ, as defined by $\min\{\theta_A, \theta_B\} \leq \tau$. Suppose further that the price P^τ_Σ offered by the bidders is constant, and that the bidding function $P^\tau(\theta)$ for the sole-source award is continuous.

In equilibrium the bid prices must make τ, the threshold type, indifferent between a split-award and a sole-source award. This is necessary with a continuum of types. With a sole-source award the expected payoff is $[1 - F(\tau)][P^\tau(\tau) - \tau]$, since the sole-source award is won only when the opponent's type is $\theta > \tau$. Similarly, with a split award the expected payoff is $[1 - F(\tau)][P^\tau_\Sigma - C(\tau)]$. This leads to the condition

$$P^\tau(\tau) - \tau = P^\tau_\Sigma - C(\tau). \tag{4}$$

In words, the sole-source price at τ must exceed the split price by exactly the incremental production

[9] When Assumption 2 holds, we necessarily have $\theta_\ell > \underline{\theta}$, so the case displayed in Figure 1 is the only one that applies. We thank a referee for pointing this out.

cost for τ between split and full production. Furthermore, the continuity of the function $P^{\tau}(\theta)$ implies that in equilibrium the buyer must be indifferent between choosing the split or a sole-source award at the threshold, that is

$$P^{\tau}(\tau) = 2P^{\tau}_{\Sigma}. \tag{5}$$

Together, these two properties pin down the split price, as a function of the threshold type τ, at

$$P^{\tau}_{\Sigma} \equiv \tau - C(\tau). \tag{6}$$

Thus, in all bidding equilibria of the type we are discussing, the offer for the split-award price must be given by (6) for all $\theta \geq \tau$, the types that receive the split award with positive probability. Types $\theta < \tau$ never win a split award, and any price $P_{\Sigma} \geq \tau - C(\tau)$ would support the outcome. For simplicity, we assume that all types offer $\tau - C(\tau)$ as split-award price.

Finally, we describe the equilibrium bids for the sole-source award. For a given value of τ, define the continuous function

$$P^{\tau}(\theta) = \begin{cases} \theta + H(\tau,\tau)\frac{1-F(\tau)}{1-F(\theta)} + \int_{\theta}^{\tau}\frac{1-F(x)}{1-F(\theta)}dx & \text{if } \theta < \tau; \\ \\ 2[\tau - C(\tau)] & \text{if } \theta \geq \tau. \end{cases} \tag{7}$$

The following proposition characterizes the set of equilibria parameterized by τ.

Proposition 1 *For any $\tau \in [\theta_m, \overline{\theta}]$ the pair $(P^{\tau}, P^{\tau}_{\Sigma})$, where P^{τ}_{Σ} is given by (6) and P^{τ} is given by (7), is a symmetric Bayesian equilibrium.*

The buyer chooses the award with the lowest total price and, for these bid prices, finds it optimal to follow the award pattern described above. Thus, the equilibrium award is Σ when both suppliers are high cost types, and the (interim) expected payoff for $\theta > \tau$ is $\Pi(\theta) = [P^{\tau}_{\Sigma} - C(\theta)][1 - F(\tau)]$. A low cost type receives a sole-source award whenever the other supplier is a higher cost type, thus for $\theta \leq \tau$ we have $\Pi(\theta) = [P^{\tau}(\theta) - \theta][1 - F(\theta)]$. Split prices from low cost types and sole-source prices from high cost types are 'off the equilibrium path' and play a role in supporting the equilibrium.

In equilibrium low cost types separate while high cost types pool at a constant split price P^{τ}_{Σ}. Low cost suppliers face a trade-off between a higher sole-source price and the probability of winning a sole-source award. In equilibrium the sole-source price, $P^{\tau}(\theta)$ rises with θ, while the probability $1 - F(\theta)$ declines. In contrast, this trade-off is absent for high cost types, as both the probability of a split award $1 - F(\tau)$ and the split-award price P^{τ}_{Σ} are constant.

The threshold parameter $\tau \in [\theta_m, \overline{\theta}]$ indexes a family of equilibria. Letting $\tau \to \overline{\theta}$, we see from (7) that P^{τ} converges to the familiar bidding equilibrium of a standard single object auction. To understand the structure of P^{τ} when $\tau < \overline{\theta}$, consider the separate terms in (7). The sum $\theta + \int_{\theta}^{\tau}\frac{1-F(x)}{1-F(\theta)}dx$ is the bid price in a unit auction (for the full quantity) when the range of types is $[\underline{\theta}, \tau]$. Thus, the additional term $H(\tau,\tau)\frac{1-F(\tau)}{1-F(\theta)}$ adjusts sole-source prices to account for the incentives created by the presence of the split award Σ.

The relationship between the split price and the threshold depends on bidding incentives and efficiency. A comparison with the bidding equilibria in Anton and Yao [2] helps to identify this interaction. In that analysis, the cost structure is such that the split is always efficient and always awarded in equilibrium. Thus, if we consider only types in the interval $\left[\tau, \overline{\theta}\right]$ and eliminate the ones below τ, the equilibrium bids in Anton and Yao [2] also involve pooling at a constant split price. In this case, any split price between $C\left(\overline{\theta}\right)$ and $\tau - C(\tau)$ can support this equilibrium.

This degree of freedom is eliminated in the present context by the occurrence of sole-source outcomes in equilibrium. Reintroducing types below τ, we see from Proposition 1 that the split price must now be $\tau - C(\tau)$. Significantly, it is the 'high' split price that emerges in equilibrium. As argued above, this is because the type τ must be indifferent between the split and a sole-source award; otherwise, low cost types could profitably raise sole-source prices. Thus, relative to Anton and Yao [2], sole-source outcomes occur in equilibrium and the bidding incentives of low cost types pin down the equilibrium split price.

Now consider the range for τ. The lower bound θ_m reflects the efficiency distinction between high cost and low cost types. For type $\overline{\theta}$ to earn non-negative profits, we must have $P_\Sigma^\tau - C\left(\overline{\theta}\right) \geq 0$. Substituting for P_Σ^τ, this reduces to $H\left(\tau, \overline{\theta}\right) \geq 0$ and, consequently, $\theta_m \leq \tau$. Otherwise, if τ were below θ_m, the split would be inefficient for τ, and the split price would induce low cost types (around τ) to exploit the underlying scale economy and deviate to capture a sole-source award instead of the split. Thus, the presence of scale economies limits the extent to which the split can occur in equilibrium.

Since Σ is the efficient award when $\min\{\theta_A, \theta_B\} \geq \theta_m$, we see that whenever Σ is the equilibrium award it is also the (full information) efficient award. Sole source awards, however, occur too often in equilibrium (relative to the first best). When $\tau > \theta_m$, types between θ_m and τ receive a sole-source award when the other supplier is also a high cost type, but the split is the efficient award in this case. Further, for types below θ_m we know that the split is efficient when the type pair lies inside the efficiency boundary, as given by T. This leads to the following.

Corollary 1 *Equilibrium award allocations have a strict bias in favor of sole-source awards relative to the efficient allocation.*

It is interesting to observe that some bids are made 'in order to lose', such as the bid P_Σ^τ for types $\theta < \tau$ and the bid $P^\tau(\theta) = 2P_\Sigma^\tau$ for types $\theta > \tau$. In split-award auctions it is often observed that bidders tend to make some non-serious bids, i.e. for some ways of splitting the award the bids are inferior (from the buyer's point of view) to other bids made by the same seller. This is exactly what happens in the class of equilibria that we have discussed. Upon learning θ, a bidder with $\theta < \tau$ strictly prefers to win a sole-source award, so the bid for the split-award is designed so that the buyer will prefer to sole-source; similarly, a bidder with $\theta > \tau$ strictly prefers the split award and designs the sole-source bid to make it less attractive than the split bid. In other words, the two bids are designed to rule out the 'undesirable' award. Notice also that this implicit coordination on the split supports positive profits for all types.

When some bids are made only 'to lose' an additional source of multiplicity appears. In our equilibrium we have specified the sole-source bid for types $\theta \geq \tau$ to be constant at $2P_{\Sigma}^{\tau}$, which is barely enough for the buyer to prefer the split award, but in fact it is not difficult to construct bidding equilibria in which $P^{\tau}(\theta) > 2P_{\Sigma}^{\tau}$ for $\theta \geq \tau$. this confirms that our tie-breaking rule giving preference to splits is not crucial for our results. On the other hand, low cost types receive only sole-source awards in equilibrium, and for them split-award prices are irrelevant (as long as they are high enough). Thus, types $\theta < \tau$ could announce split prices $p_{\Sigma}^{\tau}(\theta) > P_{\Sigma}^{\tau}$.

Notice further that sole-source prices are not necessarily "aggressive." The presence of the term $H(\tau,\tau)\frac{1-F(\tau)}{1-F(\theta)}$ in (7), which is strictly positive whenever $\tau < \overline{\theta}$, implies that sole-source prices exceed bid prices in a corresponding unit auction. What happens here is that the rents created by the split at constant prices for high cost types necessarily increase the incentive rents for the low cost types.

4 Welfare Properties

The main goal of this section is to identify conditions under which the split-award auction is Pareto superior to the "winner-take-all" (WTA), unit auction format. For each equilibrium with threshold $\tau \in \left[\theta_m, \overline{\theta}\right]$, let $V(\tau)$ denote the ex ante expected payment for the buyer and $\Pi(\theta,\tau)$ denote the interim expected profit for a supplier of type θ. From (6) and (7), we calculate (here $F(\tau) \equiv F_{\tau}$)

$$\frac{1}{2}V(\tau) = P_{\Sigma}^{\tau}(1-F_{\tau})^2 + H(\tau,\tau)F_{\tau}(1-F_{\tau}) + \int_{\underline{\theta}}^{\tau}[1-F(x)]\left[x + \frac{F(x)}{f(x)}\right]dF(x)\,, \tag{8}$$

$$\Pi(\theta,\tau) = \begin{cases} [\tau - C(\tau) - C(\theta)](1-F_{\tau})\,, & \text{for } \theta \geq \tau; \\ \\ H(\tau,\tau)(1-F_{\tau}) + \int_{\theta}^{\tau}[1-F(x)]\,dx, & \text{for } \theta < \tau. \end{cases} \tag{9}$$

As previously observed, when $\tau \to \overline{\theta}$ both the payment and profit functions converge to those of the standard single object auction.

4.1 An Example

To illustrate the behavior of interim seller profit and ex-ante buyer payment, let $F(\theta) = \theta - 1$ for $\theta \in [1,2]$ and $C(\theta) = .5\theta - .2$. Calculating, we find $\theta_m = 1.2$ and $\tau \in [1.2,\ 2]$ is the range for the threshold parameter. For the buyer, from (8), we find $V(\tau) = 0.33 + 0.8\tau + 0.6\tau^2 - 0.33\tau^3$. For the unit auction benchmark ($\tau = \overline{\theta} = 2$), the buyer pays $V(2) = 1.67$ and, over all τ, the lowest buyer payment is at $\theta_m = 1.2$ with $V(1.2) = 1.58$. Refer to Figure 2 where we graph the difference $V(\tau) - V\left(\overline{\theta}\right)$. As long as $\tau \leq 1.32$, we have $V(\tau) < V\left(\overline{\theta}\right)$. It is easy to check that, for this example, $V(\tau)$ is concave in τ. Thus, at relatively low values for τ the buyer has a lower expected payment than in a unit auction.

For low cost types, we use (9) to calculate the profit difference between the equilibrium with τ and the unit-auction benchmark (winner-take-all format), and find $\Pi(\theta,\tau) - \Pi\left(\theta,\overline{\theta}\right) = -1.2 + 1.6\tau - 0.5\tau^2$.

This profit difference is also graphed in Figure 2. From (9), we find that the profit differential for high cost types (which does vary with θ) is positive whenever the low cost differential is positive. Thus, if an equilibrium with threshold τ has $\Pi(\theta, \tau) - \Pi\left(\theta, \overline{\theta}\right) > 0$ for low cost types, $\theta < \tau$, then we know all types earn a higher interim profit than in a unit auction. Referring to Figure 2, we see that the profit difference graph is concave, and has value zero at the τ threshold extremes, $\theta_m = 1.2$ and $\overline{\theta} = 2$.[10] Thus, in this example, all seller types prefer any split equilibrium to the unit auction outcome.

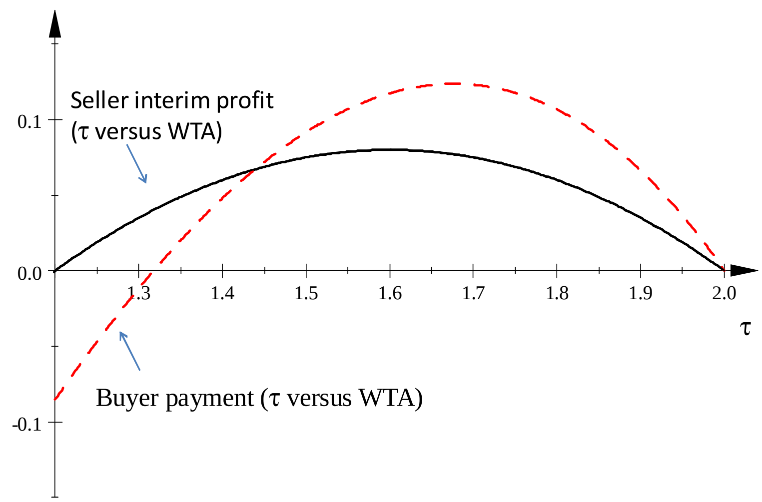

Figure 2: Buyer (dash) and Seller (solid) Payoffs for Uniform Distribution and Linear Cost

Combining these properties, we see that each split-award bidding equilibrium with a threshold τ in the range of 1.2 up to 1.32 yields a Pareto improvement for the buyer and the suppliers (for all θ) relative to a unit auction. We thus ask what factors determine the existence of a Pareto improvement over a unit auction.

4.2 General Properties of Prices and Profits

Intuitively, any Pareto improvement must be the result of gains from trade that arise when the split award is the efficient choice. In turn, equilibrium bids must involve a price structure that transfers some of these gains to the buyer. Thus, we focus on bid prices and how they relate to V and Π over the range of τ.

Consider first how bid prices compare when τ is close to $\overline{\theta}$. We know that the split price $P_{\Sigma}^{\tau} = \tau - C(\tau)$ rises smoothly from $\theta_m - C(\theta_m)$ to $\overline{\theta} - C\left(\overline{\theta}\right)$ as we vary τ, and the payment by the

[10]This holds at $\overline{\theta}$ by definition of the profit difference. We selected the support and cost parameters so that the difference is also zero at θ_m. Varying the support or the cost paramters will move the profit difference into the positive or negative range at θ_m.

buyer is $2P^{\tau}_{\Sigma}$ whenever Σ is awarded. Assumption 1 implies that

$$2\left[\overline{\theta} - C\left(\overline{\theta}\right)\right] > \overline{\theta}.$$

In a single-unit auction the bid rises smoothly to a bid price of $\overline{\theta}$ from the type $\overline{\theta}$. Therefore $2P^{\tau}_{\Sigma}$ must exceed $\overline{\theta}$ as τ rises and there is a unique $\theta_r \in \left(\theta_m, \overline{\theta}\right)$ such that

$$\overline{\theta} = 2\left[\theta_r - C\left(\theta_r\right)\right]. \tag{10}$$

This implies that, when $\tau > \theta_r$, split-award bidding equilibria have high prices for the buyer relative to a unit auction. To see this, fix τ above θ_r, and consider the bids from θ types above τ. Then $2P^{\tau}_{\Sigma}$ exceeds $\overline{\theta}$ and, hence, it exceeds the unit auction bid price for each θ type where $\theta > \tau$. The same holds for $\theta \leq \tau$: from $2P^{\tau}_{\Sigma} > \overline{\theta}$ and (7), we see that sole-source prices exceed the unit auction bid price for $\theta \leq \tau$. Figure 3 provides a graph of the situation: for $\tau_H > \theta_r$ the equilibrium bid prices with the 'high' threshold τ_H are always above the unit auction bid prices.

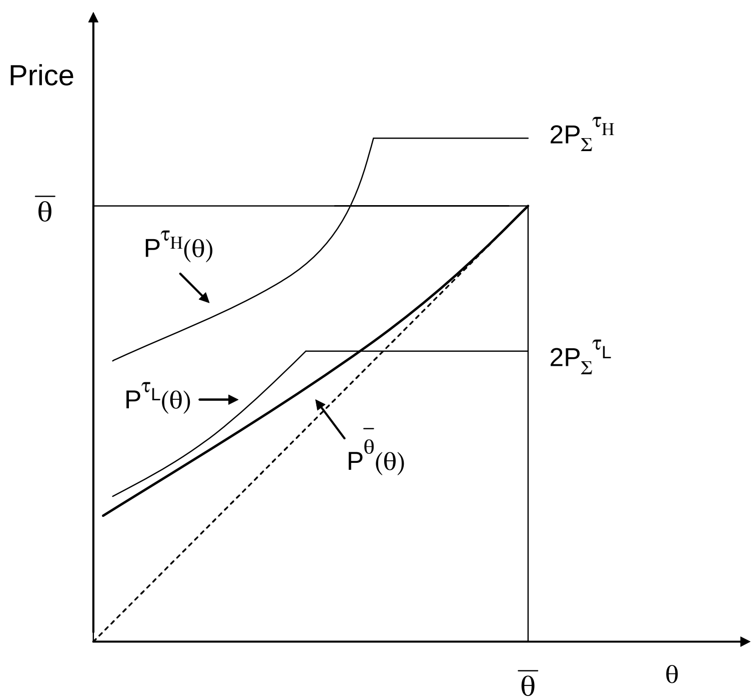

Figure 3: Equilibrium Bid Functions

The range of $\tau < \theta_r$ is associated with relatively lower prices for the buyer. For τ below θ_r, the split price for the buyer of $2P^{\tau}_{\Sigma}$ is below $\overline{\theta}$. The result, as illustrated with the 'low' τ_L threshold in Figure 3, is that unit auction bid prices from types near $\overline{\theta}$ are now greater than $2P^{\tau_L}_{\Sigma}$, and this works to the buyer's benefit.

Figure 3 is also helpful for understanding how the price schedule $P^{\tau}(\theta)$ varies with τ. Consider first the flat part on the interval $\left[\tau, \overline{\theta}\right]$. On that interval we have $P^{\tau}(\theta) = 2\left[\tau - C(\tau)\right]$. Since, by assumption, $C'(\tau) < 1$, the flat part goes up as τ increases. On the other hand, the interval $\left[\tau, \overline{\theta}\right]$

shrinks as τ increases. Consider now the strictly increasing part on the interval $[\underline{\theta}, \tau)$. For each θ belonging to this interval we have

$$\frac{\partial P^{\tau}(\theta)}{\partial \tau} = 2\left[1 - C'(\tau)\right]\frac{1 - F(\tau)}{1 - F(\theta)} - H(\tau, \tau)\frac{f(\tau)}{1 - F(\theta)}.$$

Since $H(\tau, \tau) > 0$ over the range of values for τ that we are considering, the sign of the derivative is indeterminate. Thus, the function may go up or down on this interval. Notice however that as $\tau \to \overline{\theta}$ the first term goes to zero, so the sign must be negative. Thus, consider a value of τ close to $\overline{\theta}$. When we increase the value of τ to $\tau + \Delta$, then we must have $P^{\tau+\Delta}(\theta) < P^{\tau}(\theta)$ on the interval $[\underline{\theta}, \tau)$ (i.e. the function goes down) and $P^{\tau+\Delta}(\theta) > P^{\tau}(\theta)$ on the interval $\left[\tau + \Delta, \overline{\theta}\right]$. Since continuity must be preserved, the function $P^{\tau+\Delta}(\theta)$ must become steeper in the interval $[\tau, \tau + \Delta]$. Furthermore, as $\tau \to \overline{\theta}$ the function converges to the bidding function for the single unit auction. This implies that the function is below $\overline{\theta}$ over most of the interval $[\underline{\theta}, \tau)$ and then it becomes very steep. In the limit the interval $\left[\tau, \overline{\theta}\right]$ shrinks to zero and the function is always below $\overline{\theta}$ except at the point $\overline{\theta}$.

In the next proposition we state both a necessary and a sufficient condition for the expected price to be lower in a split-award auction than in a single unit auction.

Proposition 2 *Consider the expected prices to the buyer $V\left(\overline{\theta}\right)$ for the threshold $\overline{\theta}$ (i.e., winner-take-all format), and $V(\tau)$ for the threshold τ. Let θ_r be the unique solution to (10). Then:*

1. *If $V(\tau) < V\left(\overline{\theta}\right)$ then $\tau < \theta_r$.*
2. *If $\left[1 - F(\tau)^2\right] H(\tau, \tau) \;<\; \int_{\tau}^{\overline{\theta}} \left[1 - F(x)^2\right] dx$ then $V(\tau) < V\left(\overline{\theta}\right)$*

As discussed above, if $\tau \geq \theta_r$ then the equilibrium prices in the split-award auction are higher than in the winner-take-all format for each value of θ. It follows that a necessary condition for the expected price to be lower is $\tau < \theta_r$.

To gain intuition for the sufficient condition presented in point 2 it is worth exploring a few variations. First, consider the distribution of cost types and the effect of shifting mass to the right within the high cost range. Let F^1 and F^2 be two distribution functions that satisfy $F^1(\tau) = F^2(\tau)$ and $F^1(\theta) \leq F^2(\theta)$ for $\theta > \tau$. Thus, F^1 and F^2 have the same total mass in each of the low cost and high cost ranges, but with F^1 "higher" high cost types are more likely (first order stochastic dominance over the high cost range). Because the integral term is larger under F^1, the sufficient condition becomes easier to satisfy.

To see the economic force at work here, note that the buyer pays $2P_{\Sigma}^{\tau}$ for the split award and this price is independent of the distribution above the threshold of τ. In contrast, supplier rents in a unit auction depend on the distribution and all supplier types earn greater rents under F^1 due to the rightward shift of mass above τ. Thus, the buyer benefits relative to a unit auction when the distribution places more weight at the top of the cost type range.

Next, consider the effect of varying the relative cost difference between sole-source awards and the split award. To do this, we focus on $H(\tau, \tau)$ in the sufficient condition and suppose that $C(\tau)$

declines. Then the left-hand side increases and the sufficient condition is harder to satisfy. In other words, as the split becomes more cost efficient, the buyer does not benefit relative to a unit auction. This may seem counterintuitive as one might expect the buyer to share directly in any such efficiency gain. To see why not, recall that $P_{\Sigma}^{\tau} = \tau - C(\tau)$ is set to remove the bidding incentive of type τ to deviate and capture a sole-source award, and this depends on the incremental cost between split and sole-source production rather than the split cost by itself. Thus, P_{Σ}^{τ} rises by the same amount as the decline in $C(\tau)$ as the split becomes relatively more efficient. Consequently, holding τ fixed, the direct effect of efficiency gains works to the disadvantage of the buyer. Furthermore, since the expected profit increases for types $\theta \geq \tau$, the incentive rents received by lower types must also increase. In fact, since $H(\tau, \tau)$ increases when $C(\tau)$ declines, we can see that the price offered by types $\theta < \tau$ increases. This implies that the indirect strategic effect also works to the disadvantage of the buyer.

Finally, note that variations in the threshold τ are associated with both of the above effects since rents and cost efficiency depend on τ. When there exists a threshold value, necessarily below θ_r, at which the sufficiency condition holds then we know that the buyer benefits by paying a lower expected price as compared to a unit auction.

Supplier (interim) profits vary with the type θ and we will focus on conditions under which all supplier types earn greater profits, that is $\Pi(\theta, \tau) \geq \Pi\left(\theta, \overline{\theta}\right)$, $\forall \theta \in \left[\underline{\theta}, \overline{\theta}\right]$. Drawing on the above discussion for the case of $\tau > \theta_r$, it is obvious from the comparison of equilibrium prices that a unit auction provides lower interim profits when $\tau > \theta_r$. In addition, the profit inequality remains strict at $\tau = \theta_r$ and, by continuity, this must extend to a range of threshold values below θ_r. We therefore have the following result.

Proposition 3 *Consider the interim profits $\Pi(\theta, \overline{\theta})$ for the threshold $\overline{\theta}$ (i.e., winner-take-all format), and $\Pi(\theta, \tau)$ for the threshold τ. Let θ_r be the unique solution to (10). Then:*

1. *There exists a value $\tau^* < \theta_r$ such that $\Pi(\theta, \tau) > \Pi\left(\theta, \overline{\theta}\right)$, $\forall \theta \in \left[\underline{\theta}, \overline{\theta}\right]$ whenever $\tau \geq \tau^*$.*
2. *If $\frac{1-F(\theta)}{C'(\theta)}$ is strictly decreasing in θ and $H(\tau, \tau) > \int_{\tau}^{\overline{\theta}} \frac{1-F(x)}{1-F(\tau)} dx$ then $\Pi(\theta, \tau) > \Pi\left(\theta, \overline{\theta}\right)$, $\forall \theta \in \left[\underline{\theta}, \overline{\theta}\right]$.*
3. *If $\frac{2[1-C'(x)]}{x-C(x)-C\left(\overline{\theta}\right)} < \frac{f(x)}{1-F(x)}$ $\forall x > \tau$, then $\Pi(\theta, \tau) > \Pi(\theta, x)$ $\forall x > \tau$ and $\forall \theta \in \left[\underline{\theta}, \overline{\theta}\right]$.*

Since point 1 was discussed above we discuss the proposition starting with point 2. In words, the sufficient condition is that the threshold type of τ earns a higher profit than in a unit auction. The greater the efficiency gain from the split, as given by the size of the cost differential $H(\tau, \tau)$, the more likely it is that this is the case. Given a positive profit differential for τ, we know that all lower cost types earn the same positive differential.

For high cost types, the profit differential varies with θ. The hypothesis in point 2 involves a relatively mild regularity condition on the distribution and the cost function. This ensures that the profit differential, which is positive for types τ and $\overline{\theta}$, remains positive over the high cost range.

For reference, note that the regularity condition is automatically satisfied in the familiar case of multiplicative cost uncertainty.

A stronger version of point 2 is provided by point 3, as supplier profits now decline (uniformly in type) as the threshold rises. Under this sufficient condition, declines in award probabilities will offset the higher prices associated with a rising threshold. For example, with an increasing hazard for F, the right hand side is increasing; with $C(\theta) = c\theta$ for $c < 1/2$, the left hand side is decreasing.

Together, Propositions 2 and 3 provide the conditions for a Pareto improvement relative to a unit auction benchmark. They can be combined to obtain the following corollary.

Corollary 2 *Let τ^* be the value defined in point 1 of Proposition 3. If the condition*

$$\left[1 - F(\tau^*)^2\right] H(\tau^*, \tau^*) < \int_{\tau^*}^{\overline{\theta}} \left[1 - F(x)^2\right] dx$$

holds, then there are equilibria in which, compared to the unit auction (winner-take-all format), all seller types have a higher interim expected profit and the buyer pays a lower expected price.

To summarize, the buyer benefits when a relatively low price in the split region offsets relatively high prices in the sole-source region. Low-cost suppliers benefit from high sole-source prices and, due to efficiency gains at the split, high cost suppliers also benefit despite the relatively low split price.

4.3 The Equilibrium Set

Proposition 1 characterizes a fairly wide range of equilibria, indexed by the threshold τ. Are there many other equilibria? Here we provide a partial answer to the question by establishing that all symmetric equilibria in which the split award occurs for an interval of types have to be of the form described in Proposition 1.

We say that a symmetric Bayesian equilibrium *has pooling intervals* if there is a collection of intervals $\{[\tau_1, \tau_2], \ldots, [\tau_{k-1}, \tau_k]\}$ such that a type $\theta \in [\tau_i, \tau_{i+1}]$ wins the split award if and only if the type of the opponent is also in the interval $[\tau_i, \tau_{i+1}]$, and wins the sole-source award if the type of the opponent is higher than τ_{i+1}. These candidate equilibria exhibit an intuitive monotonicity property with respect to θ: as the cost of a rival bidder j goes up, the amount of the award (0 versus Σ versus SS_i) that bidder i receives also goes up.

The next proposition shows that symmetric Bayesian equilibria with pooling intervals must have the form of the equilibria discussed in Proposition 1.

Proposition 4 *If a symmetric Bayesian equilibrium has pooling intervals, then there is a single interval of the form $\left[\tau_\ell, \overline{\theta}\right]$, where $\tau_\ell \geq \theta_m$.*

The logic behind the result is the following. If the split is awarded whenever $(\theta_A, \theta_B) \in [\tau_\ell, \tau_h]^2$ then the split price must be constant, otherwise some type could strictly increase the price and still win the split award. If $\tau_h < \bar{\theta}$, then the split price cannot be above $C(\tau_h)$ or else incentives with respect to higher cost types (above τ_h) will unravel. With this ceiling on the split price, however, the type τ_h can profitably capture a sole-source award with probability $1 - F(\tau_\ell)$ thus destroying the equilibrium. Once $\tau_h = \bar{\theta}$ is established, the argument for $\tau_\ell \geq \theta_m$ is the same as in Proposition 1.

Notice that we are not providing a full characterization of the equilibrium set; there may be asymmetric equilibria and there may be symmetric equilibria which are non-monotonic. However, monotonic and symmetric equilibria discussed in Proposition 4 appear to be the most natural ones, and it is interesting to know that the equilibria discussed in Proposition 1 coincide with the equilibria in this class.

4.4 Auction Format and Reserve Prices

Is there any sensible way to select among equilibria described in Proposition 1? Suppose first that the sellers are able to coordinate on a particular threshold $\tau \in \left[\theta_m, \overline{\theta}\right]$. If they can do this before learning their cost, then there will typically be a unique value τ that maximizes the ex-ante expected profit

$$\Pi^*(\tau) = \int_{\underline{\theta}}^{\tau} \left[P^\tau(\theta) - \theta\right]\left[1 - F(\theta)\right] dF(\theta) + \int_{\tau}^{\overline{\theta}} \left[P_\Sigma^\tau - C(\theta)\right]\left[1 - F(\tau)\right] dF(\theta)$$

where the functions P_Σ^τ and P^τ are given by (6) and (7), respectively. It is easily verified that $\Pi^*(\tau)$ is strictly decreasing at $\tau = \overline{\theta}$ and, hence, that the unit auction is never preferred by sellers on an ex-ante basis. The function $\Pi^*(\tau)$ is typically not monotonic, however, and the maximizer τ^* may take different values below $\overline{\theta}$ depending on the distribution F and costs C (see the Appendix for a formal treatment of $\Pi^*(\tau)$).

Relative to the winner-take-all format, the buyer prefers a split-award auction with a low threshold, below θ_r (Proposition 2). A natural concern for the buyer is that suppliers may find a high-threshold equilibrium to be focal with respect to profits. Thus, if the buyer expects the sellers to choose the equilibrium characterized by τ^* and τ^* is high, the winner-take-all format will be selected.

However, the buyer can typically do better by choosing appropriate reserve prices. With a winner-take-all format, a reserve price below $\overline{\theta}$ will reduce the expected payment for the buyer. Under this format, however, reserve prices may be problematic if there is a high opportunity cost for the seller of not awarding the contract (e.g., the procurement involves a critical component for the buyer). Reserve prices imply a positive probability of not awarding the contract and under limited commitment the 'threat' not to procure above the reserve price may not be credible. In such cases, reserve prices are ineffective. By contrast, reserve prices in a split-award auction need not imply a positive probability of no award.

In a split-award auction the buyer can set two different reserve prices, one for the split award and the other for a sole-source award. Call these prices $\overline{P}$ and $\overline{P}_\Sigma$. The two prices can be chosen to change the probabilities of a sole-source or split award outcome. For example, a winner-take-all auction is obtained by setting $\overline{P} = +\infty$ and $\overline{P}_\Sigma = 0$.

A buyer interested in efficiency can choose a reserve price for the split award of $\overline{P}_{\Sigma}^{m} = \theta_m - C(\theta_m)$. With such a reserve price, the only symmetric and monotonic split equilibrium that survives is the one with the lowest possible threshold $\tau = \theta_m$. Recalling that higher values of the threshold imply that a sole-source outcome occurs too often with respect to efficiency, we see that efficiency is maximized at the lowest threshold. More generally, by setting $\overline{P} = +\infty$ and picking $\overline{P}_{\Sigma} \in \left(\theta_m - C(\theta_m), \overline{\theta} - C\left(\overline{\theta}\right)\right)$, the buyer can eliminate the equilibria with a threshold τ such that $\tau - C(\tau) > \overline{P}_{\Sigma}$.

If the buyer wants to choose reserve prices to minimize ex-ante payments then things are more complicated since now both reserve prices, $\overline{P}$ and $\overline{P}_{\Sigma}$, can be used to reduce the incentive rents going to the sellers. A full discussion of this problem, and more generally of the optimal auction design problem, goes beyond the scope of this paper. We can, however, provide an assessment of cost efficiency in relation to an optimal reserve price policy. Thus, let us consider when the most efficient equilibrium outcome ($\tau = \theta_m$) is also the outcome where the buyer's ex-ante payment is minimized.

Proposition 5 *Suppose that*

1. $\frac{1-C'(x)}{x-2C(x)}$ *is strictly decreasing in* x*;*
2. $\frac{F(x)f(x)}{1-F(x)^2}$ *is strictly increasing in* x*;*
3. $\int_{\theta_m}^{\overline{\theta}} F(x)^2\, dx < \overline{\theta} - 2C(\overline{\theta}) + \left[C(\overline{\theta}) - C(\theta_m)\right] F(\theta_m)^2$.

Then the buyer payment $V(\tau)$ *is strictly quasi-concave for* $\tau \in [\theta_m, \overline{\theta}]$*, and minimized at* $\tau = \theta_m$*.*

Condition 2 is relatively mild and is implied by the standard monotone hazard rate assumption. Together, Conditions 1 and 2 are sufficient to imply that the buyer payment $V(\tau)$ is quasi-concave. We know directly from (8) that $V'\left(\overline{\theta}\right) < 0$. Thus, by quasi-concavity, the ex-ante buyer payment is minimized either at the most efficient equilibrium, $\tau = \theta_m$, or at the unit auction equilibrium, $\tau = \overline{\theta}$. Condition 3 then implies that the buyer prefers the split equilibrium with threshold θ_m to the unit auction (threshold $\overline{\theta}$) as well as all other equilibria.

Given Proposition 5, let us reconsider a reserve price policy with $\overline{P}_{\Sigma} = \theta_m - C(\theta_m)$. The efficient split equilibrium, $\tau = \theta_m$, survives while all split equilibria with a threshold $\tau > \theta_m$ are eliminated. The unit auction equilibrium, $\tau = \overline{\theta}$, is robust to the introduction of any reserve price on the split award, and will survive under a sufficiently high sole-source reserve price, such as $\overline{P} = +\infty$. Therefore, a policy with $\overline{P} = +\infty$ and $\overline{P}_{\Sigma} = \theta_m - C(\theta_m)$ will leave two equilibria from our original set. With respect to interim seller profits, Proposition 3 provides conditions such that all seller types will prefer the θ_m equilibrium; alternatively, in the Appendix, we note that an analog to Proposition 5 holds for ex-ante seller profits, and we provide a sufficient condition for ex ante profit in the θ_m split equilibrium to exceed ex ante profit in the unit auction equilibrium, $\tau = \overline{\theta}$. To summarize, we have identified when the reserve price policy with $\overline{P} = +\infty$ and $\overline{P}_{\Sigma} = \theta_m - C(\theta_m)$ reduces the candidate set of equilibria to two, the split equilibrium is preferred by the sellers (interim or ex ante), and the split equilibrium is both the most efficient equilibrium outcome and the one with the lowest ex-ante payment by the buyer. As is easily verified, our earlier example, with a uniform distribution and linear cost structure, satisfies the conditions in Proposition 5, and the resulting properties for the ex-ante buyer payment are exhibited in Figure 2; the seller interim and ex ante profit conditions are also valid for the example.

Another option for the buyer is to choose a more aggressive reserve price for the sole-source award. For any $\overline{P} < \overline{\theta}$, the unit auction equilibrium, $\tau = \overline{\theta}$, will not survive; however, as is well known, there exists an equilibrium in which low cost types, $\theta < \overline{P}$, win the sole-source award with probability $1 - F(\theta)$, while high cost types never win. Consider then a reserve price policy with $\overline{P}_{\Sigma} = \theta_m - C(\theta_m)$ and $\overline{P} = 2\overline{P}_{\Sigma}$. By Assumption 1, we have $\underline{\theta} < \theta_m < 2\overline{P}_{\Sigma} < \overline{\theta}$. This policy still leaves two equilibria, including the θ_m split equilibrium, but now the unit auction equilibrium is the one with a binding reserve price.[11] It is straightforward to verify that, for any distribution F, Assumption 1 implies that every seller type has a higher interim profit in the θ_m split equilibrium and, hence, the ex ante profit is always larger as well. Intuitively, no additional conditions on seller payoffs are required because the aggressive sole-source reserve price reduces profits in the sole-source equilibrium while leaving the θ_m split equilibrium intact.

As a final remark, one way of achieving a unique equilibrium is to allow the buyer to amend the auction rules: whenever both the sole-source and the split reserve prices bind (i.e., all bid prices are above their reserve levels) the bidders are required to accept the split award and are paid $\overline{P}_{\Sigma}^{m}$. With this rule, the 'winner-take-all' equilibrium disappears and only the efficient equilibrium survives. Note that, since the bidders always make non-negative profits, it is (ex post) individually rational to participate.

In sum, these welfare comparisons provide an evaluation of split-award auctions in terms of price performance. While price is clearly a crucial dimension, it is important to point out that additional factors such as maintaining a supplier base, innovation and investment incentives, and incumbency advantages, must be considered in a number of procurement applications.[12] With respect to a full policy evaluation, our welfare results are intended to provide guidance in relation to the price and profit dimension.

5 Bidding Data

In this section, we examine bidding data for several rounds of a split-award procurement conducted by the U. S. Department of Defense. The novel feature of the data is that we have the actual

[11]Formally, the bid function for the sole-source equilibrium, with reserve price $\overline{P}$, is given by $\theta + \int_{\theta}^{\overline{P}} \frac{1-F(x)}{1-F(\theta)} dx$.

[12]For example, Riordan [25] examines supplier qualification and selection in a setting where investment in cost reduction, posterior to the award decision, generates a scale economy. Anton and Yao [1] point out that *ex ante* investment incentives can be stimulated by the downstream profit potential in a split award format. Greenstein [12] provides empirical results regarding incumbency advantages in computer procurement by the federal government; see also Marshall, Meurer and Richard [18] on federal computer procurement and litigation settlements.

submitted bid prices from the suppliers for the full range of split quantities, as well as the observed buyer award choice. Typically, publicly available data only includes information on bidder identity, price and quantity for the award outcome. With the full set of submitted bid prices, we are able to examine the buyer's award choice in relation to the full set of procurement options and assess how each supplier chose to structure bid prices across sole-source and split quantities. To be sure, we must be careful with interpretation as we only have these bids for one procurement auction.[13]

Year 1					
Split %	0	13.7	27.5	35.8	40.5
A's bid	-	80.46	148.23	185.49	201.96
B's bid	523.53	463.59	403.92	367.2	354.51
Total Price	523.53	544.05	552.15	552.69	556.47

Year 1 (continued)							
Split %	45.6	54.2	59.5	64.1	72.5	**86.3**	100
A's bid	220.32	248.13	265.41	281.07	309.96	**355.59**	403.92
B's bid	324.0	289.44	262.71	244.62	210.06	**144.45**	-
Total Price	544.32	537.57	528.12	525.69	520.02	**500.04**	403.92

Year 2							
Split %	0	17	34	50	**66**	83	100
A's bid	0	118.53	193.86	267.57	**340.47**	406.35	480.33
B's bid	484.38	418.23	350.73	280.53	**207.09**	131.22	0
Total Price	484.38	536.76	544.59	548.1	**547.56**	537.3	480.33

Year 3						
Split %	0	28.6	**42.9**	57.1	71.4	100
A's bid	0	146.07	**193.86**	242.73	293.76	391.23
B's bid	381.51	283.5	**234.09**	186.03	135.54	0
Total Price	381.51	429.57	**427.95**	428.76	429.3	391.23

[13] The data were obtained as a result of discussions with procurement officials who had expressed concerns that they were being 'gamed' by the suppliers. In addition, there are many well-known complicating factors including dynamic cost effects (learning curve) as well as quality, reliability, and delivery. Thus, we will confine ourselves to an examination of the bidding structure.

The data is for 3 annual rounds of a procurement auction. The buyer is the U. S. Department of Defense, the bidders are two large, well-known defense contractors, and the item being procured is a missile. In each year, the buyer specified a set of quantities and solicited a bid from each supplier. Each bid specified a set of prices, with one price for each of the specified quantities. To maintain confidentiality, the bidders are labeled A and B, the quantities (splits) are expressed as a percentage of the total annual award, and prices have been multiplied by a constant (converted to millions of 2005 dollars).[14] For reference, each total annual award was well over 1000 missiles; Years 2 and 3 had the same quantity while Year 1 was about 75% of that level. The 'bold' entry in each of the tables is the award selected by the buyer, with a typical total payment in the neighborhood of 500 million dollars.

To read the table, each entry for a bidder shows the total price the bidder offered for the corresponding quantity (split) and the buyer can choose any pair of prices under a split to obtain the total quantity. Thus, in Year 1, Bidder A offers to supply the split quantity of 13.7% (of the total) at a price of 80.46 (million dollars). If the buyer were to choose the split of 13.7%, then A produces 13.7% for a payment of 80.46, B produces the residual of 86.3% for a payment of 463.59, and the buyer pays 544.05 in total. Sole-source awards for A and B correspond to the entries at 100% and 0%, respectively.

Several features of the bids stand out. In all years and for all bids, the unit price declines with quantity. Bids are uniformly declining over time (with respect to the absolute quantity).[15] In Year 1, Bidder A is more aggressive than B, but this reverses by Year 3. This is reflected in the award sequence as A's share declines from 86.3% to 66% to 42.9%. In each year, the buyer chooses an interior split rather than a sole-source award. Note that in Year 1, the buyer chose 86.3% and paid 500.04; a sole-source award to A, however, only had a bid price of 403.92. Thus, the possible saving of $500.04 - 403.92 = 96.12$ was ignored by the buyer[16].

This revealed preference suggests strongly that the buyer attaches some extra value to an interior split versus a sole-source outcome. How would the bidders respond if this were common knowledge? Consider, then, our model of a split-award auction with the modification that the buyer has preferences given by payoffs of $(V - p)$ for a sole-source award with payment p and $(V + k - \hat{p})$ for the split award with payment $\hat{p}$. That is, the buyer enjoys an added surplus of k from splitting the award (even though total quantity is the same).

Proposition 1 extends directly to cover this case. Suppose now that the buyer has additional utility k when the award is split. Consider an equilibrium characterized by a threshold τ and assume

[14] Bidder A is the original developer and Bidder B received a transfer of technology from A to enable production. At the time of the first round of bidding, A had prior production experience of approximately one year's worth of (subsequent) annual production while B had only produced about 15% of A's volume. Lyons [17] provides an excellent discussion of defense procurement practices, especially with regard to missile programs.

[15] This is common and often taken as evidence of learning curve effects. See Lyons [17] for a careful empirical study of procurement costs with a sample of missile procurement programs.

[16] According to procurement officials, the bidders were aware that while price would be a very important consideration other factors could also influence the award choice. This is always true of competitive defense procurement.

that the sole source prices for types $\theta_i \geq \tau$ keep the buyer indifferent between a sole source and split award, as in the equilibrium of Proposition 1; call $\left(P_k^{\tau}(\theta_i), P_{\Sigma,k}^{\tau}\right)$ the equilibrium price functions. To keep the seller of type τ indifferent between a sole source and split award, condition (4) must still be satisfied, and therefore

$$P_k^{\tau}(\tau) - \tau = P_{\Sigma,k}^{\tau} - C(\tau). \tag{11}$$

The indifference condition for the buyer however is different, given the additional utility k obtained from splitting. Thus we have

$$P_k^{\tau}(\tau) = 2P_{\Sigma,k}^{\tau} - k. \tag{12}$$

Combining (11) and (12) we obtain

$$P_{\Sigma,k}^{\tau} \equiv k + \tau - C(\tau). \tag{13}$$

This is intuitive: given that the buyer places an additional value k on the split award, types above τ lift their split price by k. Incentive compatibility implies that the higher surplus obtained by types with $\theta_i > \tau$ must increase the surplus of types $\theta_i < \tau$. Thus, the sole-source award prices for $\theta_i < \tau$ must also go up. Consider the following price function:

$$P_k^{\tau}(\theta_i) \equiv \begin{cases} \theta_i + [k + H(\tau,\tau)]\frac{1-F(\tau)}{1-F(\theta_i)} + \int_{\theta_i}^{\tau}\frac{1-F(x)}{1-F(\theta_i)}dx & \text{if } \theta_i < \tau; \\ \\ 2P_{\Sigma,k}^{\tau} & \text{if } \theta_i \geq \tau. \end{cases} \tag{14}$$

According to this function types below τ raise their sole-source price by a variable amount depending on how likely they are to receive a sole-source award.

The proof of Proposition 1 can now be adapted to show that the pair $\left(P_k^{\tau}(\theta_i), P_{\Sigma,k}^{\tau}\right)$ constitutes an equilibrium.

Proposition 6 *For any $\tau \in \left[\theta_m(k), \overline{\theta}\right]$ the pair $\left(P_k^{\tau}, P_{\Sigma,k}^{\tau}\right)$, where $P_{\Sigma,k}^{\tau}$ is given by (13) and P_k^{τ} is given by (14), is a symmetric Bayesian equilibrium.*

Notice that the function $P_k^{\tau}(\theta_i)$ described in (14) is not continuous. In fact, this is an immediate consequence of condition (12): at $\theta_i = \tau$ there must be a gap of k between sole source and split prices in order to keep the buyer indifferent. More generally, recall that the sole source award bid price for types $\theta_i \geq \tau$ only plays a supporting role in equilibrium and need only satisfy the condition that the buyer prefers the split award, so indifference is not necessary. Thus, any price function such that $P_k^{\tau}(\theta_i) \geq 2P_{\Sigma,k}^{\tau} - k$ for $\theta_i \geq \tau$ will support the same equilibrium outcome. In the case $k = 0$ we were able to construct an equilibrium with a continuous sole–source price function, but there were many other equilibria with the same outcome but discontinuous sole-source price functions.

In our basic model ($k \equiv 0$), it is not an equilibrium outcome for the buyer to choose a split award when the submitted bids have a minimum price at a sole-source award. With $k > 0$, however, it is readily verified that the equilibrium bids can result in realizations with sole-source prices below the split prices and the buyer choosing a split award. Intuitively, the buyer preference provides the bidders with an opportunity to leverage their bids upward. In the simpler case of complete information, the above result reduces to an increase of k at each bidder's sole-source and split price.

With this bidding perspective in hand, let us return to the data. Recall that in year 1, Bidder A offered a very low sole-source price while B did not and the buyer chose an interior split at 86.3%, passing up the savings from sole-sourcing with A. Perhaps A was bidding aggressively in the hope of obtaining a sole-source award; on the other hand, it may be that Bidder B was being very cautious due to a lack of experience. Whatever the reason, when we turn to Year 2 we see that both bidders now offer low sole-source prices relative to the combined prices at interior splits. The variation in total price across interior splits is also much smaller than in Year 1. In Year 3 this pattern is even stronger and the price variation across splits is now almost nil.

The interpretation suggested by the equilibrium is that both bidders are pursuing a form of the 'bid to lose' strategy. Specifically, they are padding their sole-source price and split prices relative to costs to account for the buyer's split preference. The award pattern moves strongly toward equal division from Year 1 to 3. Also, consistent with this view, Bidder A clearly becomes less aggressive as we move to Year 3.[17] Noting that we must always be cautious about drawing conclusions from a single procurement project, the equilibrium model provides a foundation for this bidding behavior (implicit collusion in the non-cooperative sense) in which prices at the splits sustain the incentive to avoid undercutting to a sole-source award, (i.e., the 'bid to lose' incentive) and at the same time capture the buyer's added surplus from the split.

6 Conclusion

We derived equilibrium bids for a split-award auction when the range of uncertainty regarding cost scale economies is large and each type of award is efficient over different ranges of cost realizations. In equilibrium, low cost suppliers separate and receive a sole-source award while high cost suppliers pool at a common split price and receive a split award. Whenever the equilibrium involves a split of the total award, it is the efficient choice. Sole-source awards, however, occur too often relative to a first-best setting. We also identified when a split-award format can yield a Pareto improvement relative to a winner-take-all unit auction benchmark and we assessed the price and efficiency properties of split-award auctions. Finally, our examination of submitted bids for a defense procurement revealed patterns consistent with coordination on split-award outcomes.

[17] Note that with knowledge of their own bid and public information on the selected award, a bidder can compare their own sole-source price to the observed outcome; there is no need to observe the opponent's bid prices to infer that the buyer is willing to a pay a premium for the split.

Appendix

Proof of Proposition 1. As a preliminary observation, notice that under the proposed strategies all types make a positive expected profit, as long as $\tau \geq \theta_m$ (the only exception is type $\overline{\theta}$, who makes zero profit in the equilibrium with $\tau = \overline{\theta}$). This follows from the fact that the expected profit for the highest type is $\left[P^\tau_\Sigma - C\left(\overline{\theta}\right)\right][1 - F(\tau)]$. The condition $P^\tau_\Sigma - C\left(\overline{\theta}\right) \geq 0$ is equivalent to $\tau \geq \theta_m$. Thus, it does not pay to deviate announcing bids that ensure that the firm never participates in production.

We now show that (P^τ, P^τ_Σ) defined in (6) and (7) is a best response when the opponent is also using (P^τ, P^τ_Σ). By construction, P^τ is increasing, strictly on the interval $(\underline{\theta}, \tau)$, and continuous; P^τ_Σ is constant, and $P^\tau(\theta) = 2P^\tau_\Sigma$ for each $\theta \geq \tau$. Consider bidder A of type θ, who has to decide a bid (p, p_Σ). The set of all feasible bids can be divided in two categories:

1. if $p_\Sigma + P^\tau_\Sigma > p$, then the buyer never chooses Σ, and A can only win a sole-source award, which occurs when $P^\tau(\theta_B) > p$; call such a bid a sole-source deviation;

2. if $p_\Sigma + P^\tau_\Sigma \leq p$, then A can only win a split award, which occurs when $P^\tau(\theta_B) \geq p_\Sigma + P^\tau_\Sigma$; call this a split deviation.

We can, without loss of generality, restrict the choice of p in $[P^\tau(\underline{\theta}), P^\tau_\Sigma + p_\Sigma]$ and similarly we can restrict p_Σ in a split deviation to the interval $\left[P^\tau(\underline{\theta}) - P^\tau_\Sigma, P^\tau\left(\overline{\theta}\right) - P^\tau_\Sigma\right]$. Given the definition of P^τ, we have $P^\tau\left(\overline{\theta}\right) = P^\tau(\tau) = 2P^\tau_\Sigma$, so that the relevant interval for the split deviation is $[P^\tau(\underline{\theta}) - P^\tau_\Sigma,\ P^\tau_\Sigma]$. Since P^τ is continuous and strictly increasing over $(\underline{\theta}, \tau)$, for each $p \in [P^\tau(\underline{\theta}),\ 2P^\tau_\Sigma)$ in a sole-source deviation there is a unique $\theta_p \in [\underline{\theta}, \tau)$ such that $p = P^\tau(\theta_p)$. Similarly, p_Σ in a split deviation has a unique $\theta_s \in [\underline{\theta}, \tau)$ with $p_\Sigma = P^\tau(\theta_s) - P^\tau_\Sigma$. Define

$$p^\tau_\Sigma(\theta) = P^\tau(\theta) - P^\tau_\Sigma.$$

Taking θ_p and θ_s as choice variables, we first show that each type $\theta \in [\underline{\theta}, \tau)$ maximizes expected utility selecting $(\theta_p, \theta_s) = (\theta, \tau)$. Define the expected utility of a type θ who selects (θ_p, θ_s) as

$$U(\theta_p, \theta_s | \theta) = \begin{cases} [p^\tau_\Sigma(\theta_s) - C(\theta)][1 - F(\theta_s)] & \text{if } P^\tau(\theta_p) \geq P^\tau_\Sigma + p^\tau_\Sigma(\theta_s) \\ \\ (P^\tau(\theta_p) - \theta)(1 - F(\theta_p)) & \text{if } P^\tau(\theta_p) < P^\tau_\Sigma + p^\tau_\Sigma(\theta_s). \end{cases}$$

Consider first the set of announcements (θ_p, θ_s) such that $P^\tau(\theta_p) < P^\tau_\Sigma + p^\tau_\Sigma(\theta_s)$. If the optimal announcement is in this set, then

$$U(\theta_p, \theta_s | \theta) = [P^\tau(\theta_p) - \theta][1 - F(\theta_p)], \tag{15}$$

and it must be the case that

$$\frac{\partial U(\theta_p, \theta_s | \theta)}{\partial \theta_p} = 0.$$

Using the definition of P^{τ} we obtain

$$\frac{\partial U(\theta_p, \theta_s|\,\theta)}{\partial \theta_p} = \frac{dP^{\tau}(\theta_p)}{d\theta_p}[1 - F(\theta_p)] - [P^{\tau}(\theta_p) - \theta] f(\theta_p) = (\theta - \theta_p) f(\theta_p).$$

We conclude that the unique maximum over this set is $\theta_p = \theta$ and observe that, since $\theta < \tau$, the announcement (θ, τ) maximizes utility over this set. Next, suppose that the optimal announcement (θ_p, θ_s) is such that

$$P^{\tau}(\theta_p) - P^{\tau}_{\Sigma} > p^{\tau}_{\Sigma}(\theta_s),$$

and notice that this is possible only if $\theta_s < \tau$. We will show that this gives a lower utility than announcing (θ, τ). The expected utility is

$$U(\theta_p, \theta_s|\,\theta) = [P^{\tau}(\theta_s) - P^{\tau}_{\Sigma} - C(\theta)][1 - F(\theta_s)].$$

Using the definition of P^{τ} we obtain

$$U(\theta_p, \theta_s|\,\theta) = [\theta_s - P^{\tau}_{\Sigma} - C(\theta)][1 - F(\theta_s)] + H(\tau, \tau)[1 - F(\tau)] + \int_{\theta_s}^{\tau} (1 - F(x))\, dx. \tag{16}$$

Observe that the derivative

$$\frac{\partial U(\theta_p, \theta_s|\,\theta)}{\partial \theta_s} = (P^{\tau}_{\Sigma} - [\theta_s - C(\theta)])\;\; f(\theta_s),$$

is positive for $\theta_s < P^{\tau}_{\Sigma} + C(\theta)$ and negative afterwards. Thus, using the definition of P^{τ}_{Σ}, the unique maximizer is

$$\theta^*_s = \tau + C(\theta) - C(\tau). \tag{17}$$

Notice that $\theta < \theta^*_s < \tau$. The first inequality follows from the fact that $\theta < \tau$ and the function C is increasing, while the second follows from the fact that $\theta - C(\theta)$ is increasing. Plugging the value θ^*_s from (17) into the value of the expected utility (16) we obtain

$$U(\theta_p, \theta^*_s|\,\theta) = H(\tau, \tau)[1 - F(\tau)] + \int_{\theta^*_s}^{\tau} (1 - F(x))\, dx. \tag{18}$$

We want to show that

$$U(\theta_p, \theta^*_s|\,\theta) < [P^{\tau}(\theta) - \theta][1 - F(\theta)]. \tag{19}$$

Using the definition of $P^{\tau}(\theta)$ and $U(\theta_p, \theta^*_s|\,\theta)$ from (18) the inequality turns out to be equivalent to

$$\int_{\theta^*_s}^{\tau} (1 - F(x))\, dx < \int_{\theta}^{\tau} (1 - F(x))\, dx$$

which is satisfied because $\theta^*_s > \theta$ and the argument of the integral is positive. This completes the proof that the prescribed bidding strategy is optimal for types $\theta \in [\underline{\theta}, \tau)$.

Consider now a type $\theta \geq \tau$. First notice that the prescribed strategy is optimal among all announcements (p, p_{Σ}) such that $p_{\Sigma} + P^{\tau}_{\Sigma} \leq p$. Any price for the split award higher than P^{τ}_{Σ} yields a profit of zero (remember that the highest price for sole sourcing offered by the other bidder is $2P^{\tau}_{\Sigma}$, thus implying that by bidding $p_\Sigma > P^\tau_\Sigma$ splitting never occurs). On the other hand, consider a lower price and let $p_\Sigma(\theta_s) < P^\tau_\Sigma$. In this case the expected utility is given by (16), and since now $\theta \geq \tau$ we have that the expected utility is strictly increasing in θ_s over the interval $[\underline{\theta}, \tau]$. Thus, selecting $\theta_s = \tau$, i.e. $p_\Sigma(\theta_s) = P_\Sigma$, is optimal. The only thing left to show is that for any announcement (p, p_Σ) such that $p_\Sigma + P_\Sigma > p$ the expected utility is inferior to $[P_\Sigma - C(\theta)][1 - F(\tau)]$.

For any announcement in this class the expected utility can be written as in (15), with $\theta_p \leq \tau$. Using the same logic as above we have

$$\frac{\partial U(\theta_p, \theta_s | \theta)}{\partial \theta_p} = (\theta - \theta_p) f(\theta_p).$$

Since now $\theta \geq \tau$, this means that the derivative is strictly positive for each $\theta_p < \tau$. Therefore, the optimal choice in this class of announcements is $\theta_p = \tau$. Thus, we have to show

$$[P_\Sigma - C(\theta)][1 - F(\tau)] \geq [P^\tau(\tau) - \theta][1 - F(\tau)].$$

Using $P^\tau(\tau) = 2P_\Sigma = 2(\tau - C(\tau))$ the inequality becomes equivalent to

$$\theta - C(\theta) \geq \tau - C(\tau)$$

which is satisfied because $\theta - C(\theta)$ is increasing and $\theta \geq \tau$. ■

Proof of Proposition 2. Let $F_\tau \equiv F(\tau)$ and $C_\tau \equiv C(\tau)$. Using (8) we can compute

$$V(\tau) - V(\overline{\theta}) = (2 - F_\tau^2)\tau - 2(1 - F_\tau^2) C_\tau - \overline{\theta} + \int_\tau^{\overline{\theta}} F(x)^2 dx.$$

Since F is increasing, $F_\tau^2(\overline{\theta} - \tau)$ is a lower bound on the integral. This implies

$$V(\tau) - V(\overline{\theta}) > (1 - F_\tau^2)\left[2(\tau - C_\tau) - \overline{\theta}\right].$$

For $\tau > \theta_r$ the bracketed term on the right in positive. Hence, $\tau < \theta_r$ is necessary. For the sufficient condition, note that $\overline{\theta} = \int_\tau^{\overline{\theta}} 1 dx + \tau$ and collect terms. ■

Proof of Proposition 3. We prove point 1 first. Let $\Delta(\theta) \equiv \Pi(\theta, \tau) - \Pi(\theta, \overline{\theta})$, and let $\tau \in (\theta_r, \overline{\theta})$. The claim is immediate for low cost types since $\Delta(\theta) = \Delta(\tau)$ for $\theta < \tau$ and

$$\Delta(\tau) = H(\tau, \tau)(1 - F_\tau) - \int_\tau^{\overline{\theta}} [1 - F(x)] dx > (1 - F_\tau)\left[H(\tau, \tau) - \overline{\theta} + \tau\right] > 0,$$

where the last step follows from $\tau > \theta_r$. For high cost types, we have

$$\Delta(\theta) = H(\tau, \theta)(1 - F_\tau) - \int_\theta^{\overline{\theta}} [1 - F(x)] dx > (1 - F_\tau) H(\tau, \theta) - [1 - F(\theta)](\overline{\theta} - \theta).$$

Since $1 - F(\theta)$ is decreasing, $\theta - C(\theta)$ is increasing, and $C(\theta)$ is increasing, we have $\Delta(\theta) > [1 - F_\tau]\left[2(\tau - C_\tau) - \overline{\theta}\right]$, which is positive for $\tau > \theta_r$. We thus have $\Delta(\theta) > 0 \quad \forall \theta \in \left[\underline{\theta}, \overline{\theta}\right]$ when

$\tau \in (\theta_r, \overline{\theta})$. The profit inequality is also strict when $\tau = \theta_r$, and it is identically zero at $\tau = \overline{\theta}$. Thus, by continuity of the profit functions, the strict profit inequality holds for τ in some neighborhood of the form $(\theta', \overline{\theta})$ where θ' is strictly less than θ_r.

Now consider point 2. From (9), the sufficient condition in point 2 implies $\Delta(\tau) > 0$. From above, we know $\Delta(\theta) = \Delta(\tau)$ for $\theta < \tau$. This leaves the case of high cost types, $\theta > \tau$. By calculation, we have $\Delta'(\theta) = -C'(\theta)[1 - F_\tau] + 1 - F(\theta)$. Hence, we see that $\Delta'(\tau) > 0$ and $\Delta'\left(\overline{\theta}\right) < 0$. Under the regularity condition, it can be checked that $\Delta'(\theta)$ crosses 0 exactly once between τ and $\overline{\theta}$. Thus, as θ varies from τ to $\overline{\theta}$, $\Delta(\theta)$ increases from $\Delta(\tau)$ and then eventually falls to $\Delta\left(\overline{\theta}\right)$. Therefore, $\Delta(\theta)$ cannot fall below the smaller of $\Delta(\tau)$ and $\Delta\left(\overline{\theta}\right)$. From above we know $\Delta(\tau) > 0$. We also have $\Delta\left(\overline{\theta}\right) = H(\tau, \overline{\theta})[1 - F_\tau] > 0$. The claim then follows directly.

For point 3, calculate $\partial\Pi(\theta, x)/\partial x$ for high cost and low cost types, noting that $\Pi(\theta, x)$ has a kink when $\theta = x$. Under the sufficient condition, the partial is negative at x for all types, and the claim in 3 follows directly. ■

Proof of Proposition 4. Given bid functions (P, P_Σ), let $\mathbf{1}_\Sigma(\theta_A, \theta_B)$ and $\mathbf{1}_{SS}(\theta_A, \theta_B)$ be the indicator functions for the award choice. Then $q_\Sigma(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} \mathbf{1}_\Sigma(\theta_A, x) dF(x)$ is the interim probability of Σ for θ, and $q_{SS}(\theta)$ is defined similarly for SS.

Consider the highest interval $[\tau_\ell, \tau_m]$. In the candidate equilibrium the outcome is split award whenever $(\theta_A, \theta_B) \in [\tau_\ell, \tau_m] \times [\tau_\ell, \tau_m]$. It follows that $P_\Sigma(\theta) \leq \min_{\theta' \in [\tau_\ell, \tau_m]} (P(\theta') - P_\Sigma(\theta'))$ for each $\theta \in [\tau_\ell, \tau_m]$. This implies that $P_\Sigma(\theta)$ is constant over the interval, since otherwise some type could increase profits by increasing the price. Let $P_\Sigma(\theta) = P_\Sigma^*$ be the constant price.

Standard incentive-compatibility arguments can be used to establish that in a Bayesian equilibrium we must have

$$\Pi(\theta) = \Pi\left(\bar{\theta}\right) + \int_\theta^{\bar{\theta}} \left[q_{SS}(x) + C'(x) q_\Sigma(x)\right] dx. \tag{20}$$

Let $F_m \equiv F(\tau_m)$ and $F_\ell \equiv F(\tau_\ell)$. For $\theta \in [\tau_\ell, \tau_m]$ we have

$$\Pi(\theta) = [P(\theta) - \theta](1 - F_m) + [P_\Sigma^* - C(\theta)](F_m - F_\ell). \tag{21}$$

Furthermore, for $\theta \in [\tau_\ell, \tau_m]$ we have $q_{SS}(\theta) = 1 - F_m$ and $q_\Sigma(\theta) = F_m - F_\ell$. Thus, (20) can be written as

$$\Pi(\theta) = \Pi\left(\bar{\theta}\right) + \int_\theta^{\tau_m} \left[1 - F_m + C'(x)(F_m - F_\ell)\right] dx + \int_{\tau_m}^{\bar{\theta}} \left[q_{SS}(x) + C'(x) q_\Sigma(x)\right] dx. \tag{22}$$

Equating (21) and (22) we conclude that the expected revenue

$$R(\theta) = P(\theta)(1 - F_m) + P_\Sigma^*(F_m - F_\ell)$$

is constant over the interval $[\tau_\ell, \tau_m]$. Since expected revenue is constant, $P(\theta)$ must be constant whenever $F_m < 1$. Letting P^* and P_Σ^* denote these constant prices, we then find $P^* = 2P_\Sigma^*$ as otherwise a small increase in the split price would raise profits.

The next step is to show $P_\Sigma^* = C(\tau_m)$. Since for $\theta > \tau_m$ we have $q_\Sigma(\theta) = 0$, condition (20) implies

$$\Pi(\theta) = [P(\theta) - \theta][1 - F(\theta)] = \int_\theta^{\bar{\theta}} [1 - F(x)]dx.$$

It has to be the case that $\lim_{\theta \downarrow s_m} P(\theta) = P^*$; otherwise, type τ_m can find ϵ and δ such that the bid $(P^* + \delta, P_\Sigma^* - \epsilon)$ increases profits. Continuity of sole-source prices at τ_m then implies $P_\Sigma^* = C(\theta_m)$ since Π is continuous at τ_m . Hence, $\Pi(\tau_m) = (P^* - \tau_m)(1 - F_m)$. But then, if $F_m < 1$ type τ_m can increase profit by inducing a sole-source award more often, as with $(P^* - \epsilon, P_\Sigma^* + \epsilon)$. Hence, $\tau_m = \bar{\theta}$ and the highest pooling interval has to be of the form $\left[\tau_\ell, \bar{\theta}\right]$.

Suppose now that there are multiple intervals. Given the previous analysis, there must be two intervals of the type $[\tau_a, \tau_b]$ and $\left[\tau_\ell, \bar{\theta}\right]$ with $\tau_b < \tau_\ell < \bar{\theta}$. A type $\theta \in [\tau_a, \tau_b]$ splits the reward if and only if the type of the opponent belongs to $[\tau_a, \tau_b]$, while the type wins the sole-source award if the type of the opponent is strictly higher than θ. Then we can apply the same arguments to establish that the split price P_Σ^{**} and the sole source price P^{**} have to be constant on the interval $[\tau_a, \tau_b]$. The rest of the argument also goes through, so that we conclude $\tau_b = \bar{\theta}$. This contradicts the fact that there are two intervals. Thus, all pooling equilibria have a single pooling interval of the form $\left[\tau_\ell, \bar{\theta}\right]$. Having established this, the inequality $\tau_\ell \geq \theta_m$ follows from the argument in Proposition 1. ■

Proof of Proposition 5. Simplifying (8) from the text and then differentiating, we have, respectively,

$$\begin{aligned} V(\tau) &= 2[\tau - C(\tau)]\left(1 - F_\tau^2\right) + 2\int_{\underline{\theta}}^{\tau} xF(x)\,dF(x), \\ V'(\tau) &= 2\left[1 - C'(\tau)\right]\left(1 - F_\tau^2\right) - 2[\tau - 2C(\tau)]\,F_\tau f_\tau. \end{aligned}$$

Noting that $V'\left(\overline{\theta}\right) = -2\left[\overline{\theta} - 2C(\overline{\theta})\right] f(\overline{\theta}) < 0$, by Assumption 1, we see that V is decreasing at $\overline{\theta}$. Next, observe that $V'(\tau) \gtrless 0 \Leftrightarrow$

$$\frac{1 - C'(\tau)}{\tau - 2C(\tau)} \gtrless \frac{F(\tau)\,f(\tau)}{1 - F(\tau)^2}.$$

Hence, the two monotonicity conditions in the proposition imply that V' is either always negative or that it is initially positive and then crosses crosses zero once before τ reaches $\overline{\theta}$. Hence, V is (strictly) quasi-concave and has a unique minimum in the half-open interval $[\theta_m, \overline{\theta})$. Finally, the last condition of the proposition holds $\Leftrightarrow V(\theta_m) < V(\overline{\theta})$. By quasi-concavity, the unique minimum is at θ_m. ■

Ex-Ante Seller Profits. Simplifying $\Pi^*(\tau)$ with the equilibrium prices $P^\tau(\theta)$ and P_Σ^τ, we have

$$\Pi^*(\tau) = (1 - F_\tau)\left\{[\tau - 2C(\tau)]\,F_\tau + [\tau - C(\tau)]\,(1 - F_\tau) - \int_\tau^{\overline{\theta}} C(\theta)\,dF(\theta)\right\} + \int_{\underline{\theta}}^{\tau} [1 - F(\theta)]F(\theta)\,d\theta.$$

Differentiating, we find

$$\frac{d}{d\tau}\Pi^*(\tau) = \left[1 - C'(\tau)\right]\left(1 - F_\tau^2\right) - f_\tau\left[\tau - C(\tau)(1 + F_\tau) + \int_\tau^{\overline{\theta}} C(\theta)\, dF(\theta)\right].$$

Evaluating the derivative at $\tau = \overline{\theta}$, the expression reduces to $-f(\overline{\theta})[\overline{\theta} - 2C(\overline{\theta})]$, which is negative by Assumption 1. Hence, $\Pi^*(\tau)$ is decreasing at $\tau = \overline{\theta}$ and sellers will always have an ex-ante preference for a range of split outcomes over the WTA outcome. Under the two monotonicity conditions,

$$\frac{f(x)}{1 - F(x)^2} \text{ is strictly increasing in } x,$$
$$\frac{1 - C'(x)}{x - C(x)[1 + F(x)] - \int_x^{\overline{\theta}} C(\theta)\, dF(\theta)} \text{ is strictly decreasing in } x,$$

it is straightforward to check that $\Pi^*(\tau)$ is strictly quasi-concave over $\tau \in [\theta_m, \overline{\theta}]$. Finally, note that $\Pi^*(\theta_m) > \Pi^*\left(\overline{\theta}\right) \Leftrightarrow$

$$\overline{\theta} - C(\theta_m)F(\theta_m) - \int_{\theta_m}^{\overline{\theta}} C(\theta)\, dF(\theta) > \frac{\int_{\theta_m}^{\overline{\theta}} [1 - F(\theta)]F(\theta)\, d\theta}{1 - F(\theta_m)}.$$

Thus, in a rough analogy to Proposition 5, the three conditions above imply that there is an interval, starting at θ_m, such that any split equilibrium with a threshold τ in this interval will be preferred to the unit auction equilibrium with respect to seller ex-ante profits. We note that all three conditions above hold for the example in the text. ■

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