Inventory in Vertical Relationships with Private Information and Interdependent Values

Abstract

We study the use of inventory when a distributor is better informed about demand than a manufacturer. We find that when distributor and manufacturer values are interdependent it is optimal to endow the distributor with some inventory before it obtains its private information. We characterize the final allocation of the good and show that the distributor may have too few (many) units relative to the efficient allocation when demand is high (low).

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1 Introduction

The predominant vehicle for manufacturing and selling commodities is the supply or distribution agreement. Examples of these vertical relationships include supply channel relationships, franchise agreements, and leasing and option contracts. What nearly all of these arrangements have in common is that decisions regarding the sale and distribution of the commodity or service are delegated to the better equipped and informed party, the retailer in the case of supply channel relationships or the franchisee in the case of franchise agreements.

Economists have devoted considerable attention to understanding the rationale for various supply contracts and organizations.[1] Agency theory suggests that the standard supply

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[1] See Blair and Lafontaine (2005) on franchising and supply chain management.

contract clauses we observe, such as franchise fees, inventory adjustment provisions, and vertical restraints are put in place to align the parties incentives and discourage opportunistic behavior. We focus on an adverse selection problem with interdependent values: the market value of product for the manufacturer and the distributor (retailer) is positively correlated. In this setting, the opportunistic behavior that needs to be discouraged is the misreporting of market demand by the distributor to obtain inventories on better terms.[2] The literature has also examined moral hazard problems, including provision of incentives for the retailer to expend sufficient effort to gather information on local markets and to establish a distribution network and provide customer services (see, for example, Mathewson and Winter (1984)).

A wide variety of inventory supply arrangements are employed in retail distribution channels. At one extreme are "just in time" agreements where the retailer holds no initial stocks but acquires inventory from the manufacturer as it is needed. This places most of the risk on the manufacturer to absorb unforeseen variations in retail demand. At the other extreme are "outright purchase" agreements whereby the distributor makes a one time purchase of inventory which cannot be returned to the manufacturer, unless it is damaged. In between these two extremes are a multitude of "quantity flexible" agreements that allow actual supply deliveries to vary by some amount around the distributor's forecasted demand for inventory. Such arrangements, which also include "take or pay" contracts (a buyer payment is required independent of delivery), spread the risk of demand uncertainty more evenly between the manufacturer and distributor.

We analyze an agency model of retail distribution to explain the occurrence of inventory supply provisions in resolving adverse selection problems. In particular, we examine settings where a manufacturer and distributor's valuations for goods are positively correlated, and the distributor eventually has private information about both firms' valuations. We have two

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[2] See Krishnan, H. and R. Winter (2007) for an analysis of contractual incentives when a manufacturer sells to two retailers with uncertain demand.

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main findings. First, if the value of goods for the manufacturer vary sufficiently little with the distributor's private information, then any assignment of inventory prior to when the distributor obtains private information is optimal. On the other hand, if the manufacturer's valuations for the good do vary with the private information, then a distributor should be allocated a unique positive amount of inventory before he learns about demand. We characterize the optimal allocation of the good once demand is realized. For an interior set of demand realizations, there is no adjustment in inventory, and for a unique demand realization goods are optimally allocated. Relative to the first-best allocation, for low realizations of demand the distributor has too many goods, while for high realizations of demand the distributor has too few goods. In effect, the distributor is allocated a positive inventory to create countervailing incentives (e.g., Lewis and Sappington (1989)). When endowed with a positive stock, the incentive of the distributor to claim a low value when the value is high is muted since the manufacturer can agree to repurchase the initial stock at a low price thus denying the distributor of profits from resale.

In section 2, we describe the model and establish basic properties of incentive mechanisms. The optimal contract under private information is characterized in section 3. We provide conclusions in section 4.

2   Model and Basic Properties

A risk neutral manufacturer, denoted by M, has a stock $Q > 0$ of a commodity to allocate to different users. M can allocate part of his inventory $q \in [0, Q]$ to a risk neutral retail distributor D to sell. The remaining inventory, $Q - q$ is employed by the manufacturer for self production or direct sales. The net revenues earned by the distributor and by the manufacturer, respectively, from this inventory allocation are $\rho^D(\theta) R^D(q)$ and $\rho^M(\theta) R^M(Q - q)$. We assume the gross revenue functions $R^i(\cdot)$, for $i = D, M$, are increasing and concave

functions of sales. The gross revenues are each weighted by a demand effect $\rho^i(\theta)$ which indicates the value of sales for the distributor and manufacturer, as a function of the parameter θ which reflects demand conditions in the market. The stochastic component of demand θ is distributed as $F(\theta)$ with strictly positive density $f(\theta)$ for $\theta \in [\underline{\theta}, \bar{\theta}]$ with $\underline{\theta} > 0$. We assume, for analytical convenience, that the inverse hazard rates going forwards and backwards are monotone, or

$$\frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) \geq 0, \qquad \frac{d}{d\theta}\left(\frac{1 - F(\theta)}{f(\theta)}\right) \leq 0.$$

The demand effect parameters for the distributor and the manufacturer are given by

$$\rho^i(\theta) = \bar{\rho} + \rho^i\theta$$

$$\text{with } \bar{\rho} > 0, \ \rho^D > \rho^M \geq 0.$$

As demand, θ, increases both direct (via M) and indirect (via D) sales revenues increase. Direct sales rise at a slower rate to reflect that they are less sensitive to variations in market conditions. A special case arises when $\rho^M = 0$ so that the value of inventory for self production is fixed and distributor demand then varies independently. Otherwise, when $\rho^M > 0$, direct and indirect revenues are interdependent so that high demand for distribution sales signals high demand for manufacturer sales as well.

2.1 A distribution mechanism for revealing demand information

M and D initially share the same ex ante information about market conditions. Since D specializes in the sale and promotion of manufactured goods, he eventually observes demand as characterized by θ. M wishes to design a distribution agreement that delegates to D the choice of allocating inventory between direct and indirect sales. D may then utilize his superior knowledge about demand to maximize total surplus, if the appropriate incentives

can be provided. The mechanism allows the distributor to receive an initial allocation for its own sales; this may be adjusted once he observes θ.

Formally, we follow Myerson and Satterthwaite (1983) in modeling the distribution agreement as a mechanism that is designed to maximize expected sales revenue, subject to participation and incentive constraints. The agreement is governed by the menu, $\{q_o, \tau_0, \; q(\theta), \tau(\theta)\}$ which specifies the following:

$q_0 \in [0, Q]$ is the initial inventory allocated to D for distribution sales,

τ_0 is the lump-sum payment to M from D for the initial inventory,

$q(\theta) \in [0, Q]$ is the final inventory D requests based on his report of demand,

$\tau(\theta)$ is the final payment to M from D.

The timing for the agreement is that first an exchange of inventory $q_0 \in [0, Q]$ from M to D for a sum of τ_0 occurs before θ is observed by D. Once D observes demand θ he may adjust his initial allocation by the amount $q(\theta) - q_0 \in [-q_0, Q - q_0]$ at a charge of $\tau(\theta)$. This allows inventory to respond to changes in demand conditions. For example, if demand turns out to be high, D may apply for more inventory for an additional fee, whereas he may return some of his stock to the manufacturer for a credit when demand is low.

To be implementable, agreements must satisfy (i) D truthfully reports his information on demand and (ii) voluntary participation of M and D at all stages. We begin with (i). The adjustments in inventory based on a report of θ' when true observed demand is θ will cause a change in (unweighted) revenues from indirect and direct sales as given by

$$
\begin{aligned}
\Delta R^D\left(q\left(\theta'\right), q_0\right) &\equiv R^D\left(q\left(\theta'\right)\right) - R^D\left(q_0\right) \\
\Delta R^M\left(q\left(\theta'\right), q_0\right) &\equiv R^M\left(Q - q\left(\theta'\right)\right) - R^M\left(Q - q_0\right).
\end{aligned}
$$

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The resulting profit for D from reporting θ' when θ is the true demand signal is denoted by

$$\pi^D\left(\theta' \mid \theta\right) \equiv \rho^D\left(\theta\right) \Delta R^D(q(\theta'), q_0) - \tau(\theta').$$

To implement the distribution agreement requires that it be *incentive compatible,* with

$$\pi^D\left(\theta\right) \equiv \pi^D\left(\theta \mid \theta\right) \geq \pi^D\left(\theta' \mid \theta\right) \text{ for all } \theta, \theta', \tag{1}$$

so that D is induced to truthfully report observed demand, θ. Note that the terms involving q_0 cancel in (IC).

The incentive compatibility constraint (IC) limits the allocations one can implement as described in the following lemma:

Lemma 1: *Necessary and sufficient conditions for (IC) are that for almost all θ*

(a) $q\left(\theta\right)$ *is weakly increasing*

(b) $\frac{d}{d\theta}\pi^D\left(\theta\right) = \rho^D \Delta R^D\left(q\left(\theta\right), q_0\right)$

Proof: The proofs of all formal results appear in the Appendix.

Condition (a) turns out to have particular importance in determining the sales allocations that can be implemented when D is privately informed. In equilibrium, D will be induced to acquire greater inventory the greater is distributor demand. This means that if D discovers demand is high, pretending that demand is low in order to reduce the wholesale payment to M will be costly for D since the additional inventory he receives will decrease when he understates demand. This restricts $D's$ ability to profit from private information.

In addition we require that the distribution agreement be *interim individually rational* (IIR^i) for both parties $i = D, M$. This reflects the reality that either party may dissolve the

relationship and leave with its share of the inventory to sell on its own if it is unprofitable to continue. For the distributor, $\left(IIR^D\right)$ requires,

$$\pi^D\left(\theta\right) \geq 0 \text{ for all } \theta. \tag{IIR^D}$$

Because π^D is defined by the revenue difference, $\left(IIR^D\right)$ implies that D will never find it optimal to leave the mechanism and collect $\rho^D\left(\theta\right) R^D(q_0)$. For the manufacturer, who does not observe θ at the interim stage, $\left(IIR^M\right)$ implies,

$$\Pi^M(q_0) \equiv E_\theta \left\{\Delta R^M\left(q\left(\theta\right), q_0\right) + \tau(\theta)\right\} \geq 0. \tag{IIR^M}$$

We also require that the distribution agreement must be ex ante individually rational $\left(EIR^D\right)$ for D to participate before he learns about demand. This implies,

$$\Pi^D\left(q_0\right) - \tau_0 \geq 0, \tag{EIR^D}$$

where $\Pi^D\left(q_0\right) = E_\theta \pi^D\left(\theta\right)$ is the expected rent, given q_0, that D earns before he observes θ.

The following lemma provides necessary and sufficient conditions for satisfying $(IC), \left(IIR^M\right),$ and $\left(IIR^D\right).$

Lemma 2: *A given distribution agreement* $\{q_0, \tau_0, q\left(\theta\right), \tau\left(\theta\right)\}$ *satisfies* $(IC), \left(IIR^M\right),$ *and* $\left(IIR^D\right),$ *if and only if*

$$(a) \qquad \Pi^M(q_0) = \int_{\underline{\theta}}^{\bar{\theta}} \left[\rho^D\left(\theta\right) \Delta R^D\left(q\left(\theta\right), q_0\right) + \rho^M\left(\theta\right) \Delta R^M\left(q\left(\theta\right), q_0\right)\right] dF - \Pi^D\left(q_0\right) \geq 0,$$

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$$(b) \qquad \Pi^D(q_0) = b - \int_{\underline{\theta}}^{\theta^-} F(\theta) \rho^D \Delta R^D(q(\theta), q_0) \, d\theta + \int_{\theta^+}^{\bar{\theta}} (1 - F(\theta)) \rho^D \Delta R^D(q(\theta), q_0) \, d\theta,$$

where $\pi^D(\theta^-) = \pi^D(\theta^+) \equiv b \geq 0$ *is a positive constant and* $\theta^- \equiv \sup\{\theta \mid q(\theta) < q_0\}$ *and*

$\theta^+ \equiv \inf\{\theta \mid q(\theta) > q_0\}$.[3]

(c) $q(\theta)$ *is weakly increasing.*

The rationale of Lemma 2 is that the manufacturer will agree to the exchange only if he expects to break even. The compensation available to M consists of the total surplus minus $D's$ expected information rents which is given by the expression in part (b).

The manufacturer's problem is to design an agreement $\{q_0, \tau_0, q(\theta), \tau(\theta)\}$ that maximizes his expected sales revenue and transfers subject to satisfying (IC), (IIR^M), (IIR^D) and (EIR^D). By Lemmas 1 and 2, the manufacturer's problem may formally be written as below, where $\lambda \geq 0$ is the Lagrange multiplier attached to (IIR^D) constraint.

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[3] Set $\theta^- = \underline{\theta}$ if $q(\theta)$ never falls below q_0 and set $\theta^+ = \bar{\theta}$ if $q(\theta)$ never rises above q_0.

$$(M) \quad \max_{\{q_0, \tau_0, q(\theta), \theta^-, \theta^+\}} \int_{\underline{\theta}}^{\theta^-} \{\rho^M(\theta) R^M(Q - q(\theta)) + \rho^D(\theta) R^D(q(\theta))$$

$$+ \frac{\lambda}{1+\lambda} \frac{F(\theta)}{f(\theta)} \rho^D \Delta R^D(q(\theta), q_0)\} dF(\theta)$$

$$+ \int_{\theta^-}^{\theta^+} \{\rho^M(\theta) R^M(Q - q_0) + \rho^D(\theta) R^D(q_0)\} dF(\theta)$$

$$+ \int_{\theta^+}^{\bar{\theta}} \{\rho^M(\theta) R^M(Q - q(\theta)) + \rho^D(\theta) R^D(q(\theta))$$

$$- \frac{\lambda}{1+\lambda} \frac{1 - F(\theta)}{f(\theta)} \rho^D \Delta R^D(q(\theta), q_0)\} dF(\theta)$$

$$s.t. \ q(\theta) \text{ is weakly increasing.}$$

3   Characterization of Optimal Agreements

In proceeding to solve the manufacturer's problem we first identify as a benchmark the social surplus maximizing arrangement (first best). Given our assumptions, the ex-post surplus maximizing inventory allotment denoted by $q^*(\theta)$ is unique and non-decreasing in θ with,

$$q^*(\theta) \begin{cases} = 0 \\ \in (0, Q) \\ = Q \end{cases} \text{if } -\rho^M(\theta) R_q^M(Q - q^*(\theta)) + \rho^D(\theta) R_q^D(q^*(\theta)) \begin{cases} \leq 0 \\ = 0 \\ \geq 0 \end{cases}.$$

We say that q is *essential* if $q^*(\theta)$ is interior for all θ. This occurs for instance when the marginal revenues for direct and indirect sales are very large as sales go to zero.

The surplus maximizing agreement is characterized by the following:

Surplus Maximizing Agreement: The following agreement maximizes surplus for the manufacturer

(a) $q_0 = 0$

(b) $\tau_0 = \Pi^D(0)$

(c) $q(\theta) = q^*(\theta)$

(d) $\tau(\theta) = \int_{\underline{\theta}}^{\theta} \rho^D\left(\tilde{\theta}\right) R_q^D\left(q^*\left(\tilde{\theta}\right)\right) \left(\frac{dq^*(\tilde{\theta})}{d\tilde{\theta}}\right) d\tilde{\theta}$.

Let a^* be the agreement defined by (a)-(d).

This agreement is implemented by a two-part pricing arrangement. The arrangement involves a fixed fee equal to the expected profit D expects to earn from reallocating inventory once he learns demand. The initial inventory allocation may be set at zero (although any other initial allocation will suffice). The subsequent adjustment in inventory corresponds with the optimal surplus maximizing inventory based on demand conditions. The payment made by D equals the increase in value of the additional inventory to distribution sales. This arrangement is similar to the two-part pricing arrangements that monopoly sellers implement to capture the full surplus from consumers. Consumers are charged the marginal cost of service provision and a fixed fee equal to the consumer surplus they earn under efficient marginal cost pricing.

3.1   When Can the First Best Be Implemented?

While the surplus maximizing agreement is clearly desirable, the settings in which it is possible to implement such agreements are limited. One such setting is characterized in the following lemma.

Lemma 3: *Suppose ρ^M is sufficiently small. Then the surplus maximizing allocation can be implemented with agreement a^*.*

In settings where ρ^M is small, an increase in demand increases the distributor's stock value without affecting the manufacturer's value. The gains from adjusting inventories are large and sufficient in magnitude to finance the rents which D earns from its private information about demand under these conditions. As a result, the ex-post participation of the manufacturer is unconstrained, thus allowing for a surplus maximizing allocation to be implemented by the two-part pricing agreement a^*.

In contrast, when ρ^M is large, an increase in demand increases both the manufacturer's and distributor's revenue valuation. The gains from trade are inherently reduced in this case. A demand increase which causes D to demand more inventory also increases M's value for inventory. This results in a 'lemons' exchange problem whereby the distributor only wishes to reduce his inventory holdings in the same states of the world where the manufacturer also wishes to reduce his holdings, thus decreasing the potential gains from trade. It may not be possible to implement the surplus maximizing allocation in this case, because the gains from exchange are insufficient to cover $D's$ information rents. When this occurs M is unwilling to participate ex-post after D has observed demand, so that the adjustments that can be implemented are constrained and not surplus maximizing. In this instance, whatever surplus is generated by shifting sales to the more profitable market segment is taxed away by the distributor as an information rent. Consequently, the manufacturer earns negative returns from participating in the agreement and would therefore be better off selling his existing inventory. In this event, it is not possible to implement the surplus maximizing allocation with a distribution agreement. We now turn to this case to examine what agreements are possible in these settings.

3.2 The Lemon's Problem and Inflexible Agreements

When one cannot implement a^* due to a lemon's problem in exchange, inventories must be distorted from their optimal levels to constrain the rents of the distributor and allow the manufacturer to earn a break even return in the second phase of the agreement. The following proposition describes the type of agreements that are possible in this case.

Proposition 1 *Suppose q is essential. Let $\lambda > 0$ be the multiplier corresponding to the monotonicity constraint in $[M]$. The optimal allocation exhibits these properties:*

(a) *The distributor is allocated an intermediate initial inventory $q_0 \in \left(q^* \left(\underline{\theta} \right), q^*(\bar{\theta}) \right)$ that is surplus maximizing, $q_0 = q^* \left(\theta^D \right)$ for some intermediate demand $\theta^D \in \left(\underline{\theta}, \bar{\theta} \right)$.*

(b) *No inventory adjustments are made when demand is close to θ^D. An interval of demands $\left(\theta^-, \theta^+ \right)$ exists with $\underline{\theta} < \theta^- < \theta^D < \theta^+ < \bar{\theta}$ such that*

$$q(\theta) = q_0 \text{ for } \theta \in \left(\theta^-, \theta^+ \right).$$

(c) *Partial adjustments in inventory are made when demand is sufficiently small or large, with*

$$q(\theta) \in \begin{array}{l} \left(q^*(\theta), q_0 \right) \text{ for } \theta \in \left(\underline{\theta}, \theta^- \right) \\ \left(q_0, q^*(\theta) \right) \text{ for } \theta \in \left(\theta^+, \bar{\theta} \right) \end{array}.$$

The solution is illustrated in Figure 1. Property (a) addresses the crucial question of how to design the initial allocation to optimally distribute inventories. The initial allocation is always interior so the distributor is given some initial inventory. Subsequently, once D discovers the state of demand, he is allowed to make adjustments in his initial stock.
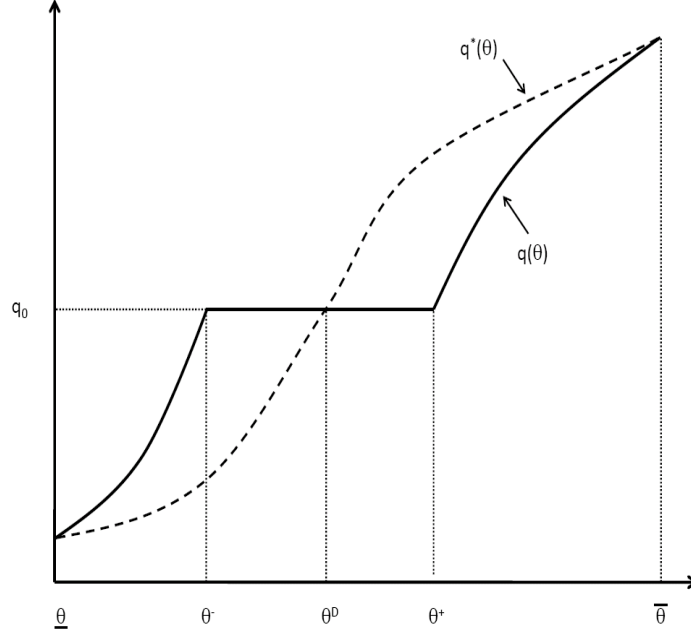
Figure 1: Optimal Allocation

When the inventory is more valuable on average for D, a larger initial share is allocated. Interestingly, it is the relative expected demand between direct and indirect sales, not which party is better informed, that determines the initial inventory allocation between the parties.

Properties (b) and (c) describe how the final asset allocation differs from q^*. When $M's$ compensation constraint is not binding, the first best allocation is implemented with $\Delta q\left(\theta\right) = q^*\left(\theta\right) - q_0$. The allocation begins with the optimal expected division, with D recommending adjustments from there to reach the efficient division $q^*\left(\theta\right)$ once he learns θ.

When the first best is not implementable, it is because the inventory allocation is constrained to generate enough surplus to ensure participation of M. As illustrated in Figure 1, properties (b) and (c) indicate that inventories are not adjusted for states θ close to θ^D. It is only for θ sufficiently different from θ^D that inventory is adjusted in the direction of the efficient levels. To understand this feature, consider the incentives for D to recommend

inventory adjustments when θ is less than θ^D. D is tempted to exaggerate demand to obtain a higher payment from M for returning some inventory. Therefore, reducing the amount of inventory that D may return to M when he claims a high value close to θ^D will discourage D from overstating demand. When θ is greater than θ^D, D has the opposite incentive and now understates demand to reduce the amount he must pay M to acquire more inventory. In effect, $D's$ ability to profit from private information about demand is limited by reducing the adjustments in inventory allocation he is allowed.

4 Conclusions

We presented a simple model of demand interdependencies in a bilateral relationship between a manufacturer and a distributor. We demonstrated that the allocation of inventory before demand is observed by the distributor is a useful device for creating countervailing incentives. By endowing the distributor with a positive stock, he is discouraged from claiming a low value, when the value is high, for otherwise he will be required to sell the inventory back to the manufacturer at a price below the true value of the inventories. Regarding future work, the model potentially has broader applications including the dissolution of partnerships, the sale of real property, and the protection of intellectual property rights.

5 Appendix

Proof of Lemma 1:

(a) By (IC) we require that

$$
\begin{aligned}
\rho^D\left(\theta\right)\Delta R^D\left(q\left(\theta\right),q_0\right)\ -\tau\left(\theta\right) &\geq\ \rho^D\left(\theta\right)\Delta R^D\left(q\left(\theta'\right),q_0\right)-\tau\left(\theta'\right)\\
\rho^D\left(\theta'\right)\Delta R^D\left(q\left(\theta'\right),q_0\right)-\tau\left(\theta'\right) &\geq\ \rho^D\left(\theta'\right)\Delta R^D\left(q\left(\theta\right),q_0\right)-\tau\left(\theta\right)
\end{aligned}
$$

Subtracting one condition from the other and rearranging implies,

$$\rho^D \left(\theta - \theta'\right) \left(\Delta R^D \left(q\left(\theta\right), q_0\right) - \Delta R^D \left(q\left(\theta'\right), q_0\right)\right) \geq 0$$

Since R^D is increasing this implies $q\left(\theta\right)$ must be weakly increasing.

(b) Since $q\left(\theta\right)$ is weakly increasing it is differentiable almost everywhere. This implies that $\tau\left(\theta\right)$ is differentiable wherever $q\left(\theta\right)$ is. Hence the first order condition for truth telling may be written as

$$\pi_1^D \left(\theta \mid \theta\right) = 0 \text{ a.e.}$$

Totally differentiating $\pi^D \left(\theta \mid \theta\right)$ we find that the rate of increase in retailer rents is given by

$$\begin{aligned} \frac{d\pi^D \left(\theta \mid \theta\right)}{d\theta} &= \pi_1^D \left(\theta \mid \theta\right) + \pi_2^D \left(\theta \mid \theta\right) \\ &= \rho^D \Delta R^D \left(q\left(\theta\right), q_0\right) \text{ a.e.} \end{aligned}$$

Proof of Lemma 2

(a) $\left(IIR^M\right)$ requires that

$$\begin{aligned} \Pi^M \left(q_0\right) &= \int_{\underline{\theta}}^{\bar{\theta}} \left\{\rho^M \left(\theta\right) \Delta R^M \left(q\left(\theta\right), q_0\right) + \tau(\theta)\right\} dF(\theta) \\ &= \int_{\underline{\theta}}^{\bar{\theta}} \left\{\rho^M \left(\theta\right) \Delta R^M \left(q\left(\theta\right), q_0\right) + \rho^D \left(\theta\right) \Delta R^D \left(q\left(\theta\right), q_0\right)\right\} dF(\theta) - \Pi^D \left(q_0\right) \geq 0. \end{aligned}$$

(b) By definition $\Pi^D \left(q_0\right) = \int_{\underline{\theta}}^{\bar{\theta}} \pi^D \left(\theta\right) dF(\theta)$. (IC) requires that $q\left(\theta\right)$ be weakly increasing and $\pi^{D\prime} \left(\theta\right) = \rho^D \Delta R^D \left(q\left(\theta\right), q_0\right)$ This implies that there are three possible subintervals of the support $\left[\underline{\theta}, \bar{\theta}\right]$, including $A = \left[\underline{\theta}, \theta^-\right)$, $B = \left[\theta^-\ \theta^+\right]$, and $C = \left(\theta^+, \bar{\theta}\right]$ with the properties

$$\pi^D(\theta) = b - \int_\theta^{\theta^-} \rho^D \Delta R^D(q(x), q_0)\, dx \quad \text{for } \theta \in A$$

$$\pi^D(\theta) = \pi^D(\theta^-) = \pi^D(\theta^+) \equiv b \geq 0 \quad \text{for } \theta \in B \ \cdot$$

$$\pi^D(\theta) = b + \int_{\theta^+}^\theta \rho^D \Delta R^D(q(x), q_0)\, dx \quad \text{for } \theta \in C$$

It follows upon integrating by parts that

$$\int_{\underline{\theta}}^{\theta^-} \pi^D(\theta)\, dF(\theta) = \int_{\underline{\theta}}^{\theta^-} \left\{ b - \int_\theta^{\theta^-} \rho^D \Delta R^D(q(x), q_0)\, dx \right\} dF(\theta)$$

$$= bF(\theta^-) - \int_{\underline{\theta}}^{\theta^-} F(\theta) \rho^D \Delta R^D(q(\theta), q_0)\, d\theta.$$

$$\int_{\theta^+}^{\bar{\theta}} \pi^D(\theta)\, dF(\theta) = \int_{\theta^+}^{\bar{\theta}} \left\{ b + \int_{\theta^+}^\theta \rho^D \Delta R^D(q(x), q_0)\, dx \right\} dF(\theta)$$

$$= b(1 - F(\theta^+)) + \int_{\theta^+}^{\bar{\theta}} (1 - F(\theta)) \rho^D \Delta R^D(q(x), q_0)\, d\theta.$$

So combining the above and noting that $\int_{\theta^-}^{\theta^+} \pi^D(\theta)\, dF(\theta) = b(F(\theta^+) - F(\theta^-))$, we obtain

$$\Pi(q_0) = b - \int_{\underline{\theta}}^{\theta^-} F(\theta) \rho^D \Delta R^D(q(\theta), q_0)\, d\theta + \int_{\theta^+}^{\bar{\theta}} (1 - F(\theta)) \rho^D \Delta R^D(q(\theta), q_0)\, d\theta.$$

Proof of Lemma 3

Consider the first best, surplus maximizing program a^* as defined in the text. We show that (a), (b), and (c) from Lemma 2 hold as ρ^M goes to 0. Since $q^*(\theta) \geq q_0 = 0$, we can simplify the expression for Π^M in Lemma 2 to obtain

$$\Pi^M = \int_{\underline{\theta}}^{\bar{\theta}} \left\{ \rho^M(\theta) R^M(Q - q^*(\theta)) + \rho^D(\theta) R^D(q^*(\theta)) \right\} dF(\theta) - \int_{\underline{\theta}}^{\bar{\theta}} \rho^D(\theta)(1 - F(\theta)) R^D\left(q^*(\theta)\right) d\theta.$$

Trivially, if $q^*(\theta) = 0$ for all θ, then $\Pi^M > 0$ so that a^* can be implemented. More interesting is the case where $q^*(\theta) > 0$ for some θ. Then integrating the above expression for Π^M by

parts, we have

$$\Pi^M = \rho^M\left(\bar{\theta}\right)R^M(Q-q^*\left(\bar{\theta}\right))+\rho^D(\bar{\theta})R^D(q^*\left(\bar{\theta}\right))-\int_{\underline{\theta}}^{\bar{\theta}}\left\{\rho^M R^M(Q-q^*\left(\theta\right))+\rho^D R^D(q^*\left(\theta\right))\right\}F\left(\theta\right)d\theta$$

$$-\int_{\underline{\theta}}^{\bar{\theta}}\rho^D(\theta)\left(1-F\left(\theta\right)\right)R^D\left(q^*\left(\theta\right)\right)d\theta$$

$$=\left(\bar{\rho}+\rho^M\bar{\theta}\right)R^M(Q-q^*\left(\bar{\theta}\right))+(\bar{\rho}+\rho^D\bar{\theta})R^D(q^*\left(\bar{\theta}\right))-\int_{\underline{\theta}}^{\bar{\theta}}\rho^M R^M(Q-q^*\left(\theta\right))F\left(\theta\right)d\theta$$

$$-\int_{\underline{\theta}}^{\bar{\theta}}\rho^D R^D\left(q^*\left(\theta\right)\right)d\theta$$

$$\geq\left(\bar{\rho}+\rho^M\bar{\theta}\right)R^M(Q-q^*\left(\bar{\theta}\right))+\bar{\rho}R^D(q^*\left(\bar{\theta}\right))-\int_{\underline{\theta}}^{\bar{\theta}}\rho^M R^M(Q-q^*\left(\theta\right))F\left(\theta\right)d\theta$$

$$\to\bar{\rho}R^M(Q-q^*\left(\bar{\theta}\right))+\bar{\rho}R^D(q^*\left(\bar{\theta}\right)),$$

as $\rho^M\to 0$. Since $q^*(\bar{\theta})>0$ holds, this last expression is strictly positive. Hence this demonstrates that $\Pi^M>0$ will hold once ρ^M falls below a threshold.

Proof of Proposition 1

To begin we shall assume the following condition (m) is satisfied

$$q_0\in[q^*\left(\underline{\theta}\right),q^*\left(\bar{\theta}\right)]$$

Later we will verify that (m) is in fact satisfied in the solution to $[M]$

Assuming an interior solution, pointwise maximization of $[M]$ with respect to $q\left(\cdot\right)$ yields the following: If $\theta\in[\underline{\theta},\theta^-)$

$$\left(\rho^D\left(\theta\right)+\frac{\lambda\rho^D}{1+\lambda}\frac{F\left(\theta\right)}{f\left(\theta\right)}\right)R_q^D\left(q\left(\theta\right)\right)-\rho^M\left(\theta\right)R_q^M\left(Q-q\left(\theta\right)\right)=0$$

and for $\theta \in (\theta^+, \bar{\theta}]$

$$\left(\rho^D(\theta) - \frac{\lambda \rho^D}{1+\lambda} \frac{1-F(\theta)}{f(\theta)}\right) R_q^D(q(\theta)) - \rho^M(\theta) R_q^M(Q - q(\theta)) = 0.$$

We note that for $\theta \in [\underline{\theta}, \theta^-)$

$$\rho^D(\theta) R_q^D(q(\theta)) - \rho^M(\theta) R_q^M(Q - q(\theta)) < 0,$$

which implies $q_0 > q(\theta) > q^*(\theta)$. A similar argument applied to $\theta \in (\theta^+, \bar{\theta}]$ establishes that $q_0 < q(\theta) < q^*(\theta)$. This completes the proof of part (c) of Proposition 1.

Let $H(\theta^-, \theta^+, q_0)$ be the maximal value of the objective function defined in $[M]$ for given values of θ^-, θ^+ and q_0. Applying Leibniz's rule we obtain

$$
\begin{aligned}
\frac{\partial H}{\partial \theta^-} =\ & \left\{\rho^M(\theta^-) R^M(Q - q(\theta^-)) + \rho^D(\theta^-) R^D(q(\theta^-))\right\} f(\theta^-) \\
& + \frac{\lambda \rho^D}{1+\lambda} \frac{F(\theta^-)}{f(\theta^-)} \Delta R^D(q(\theta^-), q_0) f(\theta^-) \\
& - \left\{\rho^M(\theta^-) R^M(Q - q_0) + \rho^D(\theta^-) R^D(q_0) f(\theta^-)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial \theta^+} =\ & \left\{\rho^M(\theta^+) R^M(Q - q_0) + \rho^D(\theta^+) R^D(q_0)\right\} f(\theta^+) \\
& - \left\{\rho^M(\theta^+) R^M(Q - q(\theta^+)) + \rho^D(\theta^+) R^D(q(\theta^+))\right\} f(\theta^+) \\
& + \frac{\lambda \rho^D}{1+\lambda} \frac{1-F(\theta^+)}{f(\theta^+)} \Delta R^D(q(\theta^+), q_0) f(\theta^+).
\end{aligned}
$$

Let $\theta_0^- = \sup(\theta \mid q(\theta) < q_0)$ where $q(\theta)$ is determined by the above pointwise condition.

Then it follows that $\frac{\partial H}{\partial \theta^-}\big|_{\theta_0^-} = 0$.

$$\frac{\partial H}{\partial \theta^-} \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} = 0 \text{ for } \theta^- \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} \theta_0^-$$

so that H is maximized at θ_0^-. A similar argument applied to θ^+ establishes that $\theta_0^+ = \inf$ $(\theta \mid q(\theta) > q_0)$ maximizes H when $q(\theta)$ is determined by the above pointwise condition.

To establish property (b) of Proposition 1 notice that property (m) implies

$$q^*\left(\underline{\theta}\right) < q_0 < q^*\left(\bar{\theta}\right).$$

Define θ^D by $q_0 = q^*(\theta)$. Therefore, $\underline{\theta} < \theta^D < \bar{\theta}$ since q^* is an increasing function of θ. Recalling the definition of θ_0^-, note that

$$\frac{d}{dq}\left(\rho^M\left(\theta_0^-\right)R^M(Q-q) + \rho^D\left(\theta_0^-\right)R^D(q)\right) < 0,$$

so we have $\theta_0^- < \theta^D$. A similar argument for θ_0^+ establishes $\theta^D < \theta_0^+$. It is also evident that $\underline{\theta} < \theta_0^-$; otherwise, we would have $q^*\left(\underline{\theta}\right) = q_0$ which violates (m). Similarly, $\theta_0^+ < \bar{\theta}$.

To complete the proof, we show property (m) holds. Differentiating $H\left(\theta^-, \theta^+, q_0\right)$ with respect to q_0 and employing the Envelope Theorem, one can show that the necessary condition for maximization with respect to q_0 is given by

$$\int_{\theta^-}^{\theta^+} \left\{\rho^D(\theta)R_q^D(q_0) - \rho^M(\theta)R_q^M(Q-q_0)\right\}dF(\theta) - \int_{\underline{\theta}}^{\theta^-} \frac{\lambda}{1+\lambda}\rho^D(\theta)F(\theta)R_q^D(q_0)d\theta$$

$$+\int_{\theta^+}^{\bar{\theta}} \frac{\lambda}{1+\lambda} \rho^D(\theta) \left(1 - F\left(\theta\right)\right) R_q^D(q_0) d\theta = 0.$$

To see that this implies (m), assume that $q_0 \leq q^*\left(\underline{\theta}\right)$. Then, $\left(IIR^D\right)$ would bind over an interval $[\underline{\theta}, \theta^+)$ with $-\rho^M\left(\theta\right) R_q^M(Q-q_0) + \rho^D\left(\theta\right) R_q(q_0) > 0$ for all $\theta \in [\underline{\theta}, \theta^+)$ which leads to a contradiction of the first order condition. A similar argument applies if we have $q_0 \geq q^*\left(\bar{\theta}\right)$. Hence (m) must hold.

References

Blair, R. and F. Lafontaine, *The Economics of Franchising*, Cambridge: Cambridge University Press 2005.

Cachon, G. and M. Lariviere, (2005), "Supply chain coordination with revenue sharing: strengths and limitations," *Management Science*, 51(1), 30-44.

Krishnan, H. and R. Winter, (2007), "Vertical Control of Price and Inventory, *American Economic Review*, 97(5), 1840–1857.

Lewis, T. and D. Sappington, (1989), "Countervailing Incentives in Agency Models," *Journal of Economic Theory*, 49, 294-313.

Mathewson, G. F. and R. Winter, (1984), "An Economic Theory of Vertical Restraints," *The Rand Journal of Economics*, Spring, 27-38.