On “Resource Flexibility with Responsive Pricing”

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Abstract

We noticed several derivation errors in the article “Resource Flexibility with Responsive Pricing” by Chod and Rudi (2005) (*Operations Research* **53**, 532-548). The errors occurred when the authors use the bivariate normal (BVN) demand-intercept distribution to establish some comparative statics while assuming the effect of the negative values of BVN is negligible. We point out these errors and provide corrections. Unfortunately, with the correct expressions, the proofs cannot go through. Some of the qualitative properties still hold under different assumptions on the demand-intercept distribution, as shown by Bish, Liu and Bish (2009) using a stochastic comparison approach, but not all. Our analysis shows the limitation of truncated BVN distribution in this context.

1 Introduction

Chod and Rudi (2005) (C&R) investigate the value of two types of flexibility a firm can employ to better match demand and supply: resource flexibility and responsive pricing. Assuming bivariate normal (BVN) demand intercepts in a linear demand model, they develop several comparative statics. We read this article with great interest. However, we noticed derivation errors in the proofs of the comparative statistics. The problems occurred when the authors assume the effect of the negative values of BVN is negligible. In this note, we point out these errors and provide corrections. Unfortunately, with these corrections, we can no longer prove these qualitative properties. On the other hand, for tractability, some of C&R’s proofs are based on certain approximation of the original problem, i.e., relaxing the nonnegative product quantity constraint. Failing to show these results using the approximate model does not necessarily mean the properties do not hold in the original problem. In fact, working on the original problem setting (with nonnegative product quantities and demand intercepts), Bish, Liu and Bish (2009) (BLB) are able to prove some related properties using the notion of convex order, although these results are restrictive in some other way as explained later. Also, BLB's approach can recover some (but not all) of C&R's results under BVN demand intercepts. However, different from C&R, BLB allow these intercepts to take negative values. Our analysis here shows that the truncated BVN distributions does not work in this context.

In order to clearly state C&R's propositions and point out the errors, in Section 2 we briefly review their model and introduce notation and some preliminaries. Then, in Section 3, we state the propositions in error and provide corrections and counterexamples.

2 Model and Preliminaries

C&R consider a firm selling two distinct products, indexed by $i = 1, 2$. The demand curves are linear with the intercepts $\boldsymbol{\xi} = (\xi_1, \xi_2)^T$, identical slopes $-b$, and a linearly additive cross-price effect d. To sell the outputs $\mathbf{Q} = (Q_1, Q_2)^T$, the firm charges the prices $\mathbf{p} = (p_1, p_2)^T$ according to the following relationship

$$p_1(Q_1|p_2) = \xi_1 - bQ_1 + dp_2,$$
$$p_2(Q_2|p_1) = \xi_2 - bQ_2 + dp_1.$$

A positive d indicates that the products are substitutes, while a negative d indicates that they are complements. Because a product's own price effect on its demand should be more significant than the cross-price effect, $|d| < 1$.

Assume $\boldsymbol{\xi} = (\xi_1, \xi_2)$ has a continuous probability distribution with joint density function $f(\xi_1, \xi_2)$ on $\Re^2_+$, mean vector (μ_1, μ_2), variance-covariance matrix $\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$, and correlation coefficient $\rho = \sigma_{12}/(\sigma_1\sigma_2)$. To study comparative statics, assume further that $\boldsymbol{\xi} = (\xi_1, \xi_2)$ follows a BVN distribution, but the coefficients of variation are not large, so that the effect of the negative values of $\boldsymbol{\xi}$ is negligible.

The firm faces a two-stage decision problem: in the first stage, based on the distribution of $\boldsymbol{\xi}$, it decides the resource capacity q which is acquired at a unit cost w; in the second stage, based on the realization of $\boldsymbol{\xi}$, it decides how to allocate q between the two products. This results in the stochastic program

$$\max_{q \geq 0} \pi(q, S) = \max_{q \geq 0} [-qw + \Pi(q, S)]$$

and

$$\Pi(q, S) = \mathsf{E} \max_{\mathbf{Q} \in S} \mathbf{Q}^T \mathbf{p}(\mathbf{Q}, \boldsymbol{\xi}),$$

where E denotes expectation and S is the set of feasible output vectors $\mathbf{Q}$ that depends on the firm's decision rule for the output quantities. If the firm employs the holdback rule, the feasibility set is $S^h = \{\mathbf{Q} : \mathbf{Q} \geq \mathbf{0}, \mathbf{1}^T \mathbf{Q} \leq q\}$. If the firm employs the clearance rule, the feasibility set is $S^c = \{\mathbf{Q} : \mathbf{Q} \geq \mathbf{0}, \mathbf{1}^T \mathbf{Q} = q\}$. To derive analytical results related to demand variability and correlation, the authors relax the nonnegativity constraints imposed on the output vector $\mathbf{Q}$. This means ro replace S^h and S^c by $\widetilde{S}^h = \{\mathbf{Q} : \mathbf{1}^T \mathbf{Q} \leq q\}$ and $\widetilde{S}^c = \{\mathbf{Q} : \mathbf{1}^T \mathbf{Q} = q\}$, respectively.

In Section 6 of C&R, the authors consider a decentralized model, in which the firm purchase the flexible resource from a supplier. The supplier is the first mover in a Stackelberg game and sets w to maximize its profit. Propositions 9 and 10 concern this model.

In the next section, we point out errors in the proofs of Proposition 2 and the second part of Proposition 9 and provide corrections. We also show that Propositions 3-5, as stated in C&R, which is based on the approximate feasible region $\widetilde{S}^h$, do not hold. Finally, we show that the general case of Proposition 10, as stated in C&R, does not hold. For each of these propositions, we provide the correct analytical expressions of interest (also based on $\widetilde{S}^h$) and give counterexamples.

A common error made in C&R can be summarized as follows. When establishing the comparative statics, these authors took the following steps: 1) Assume that the demand intercepts $\boldsymbol{\xi}$ follow a BVN distribution. 2) Express $\boldsymbol{\xi}$ in terms of two independent standard normal random variables. 3) Reexpress the expected profit function π according to the variable transformation in the previous step. 4) Take the derivative of π with respect to the parameter of interest, observe the sign of the derivative, and obtain comparative statics. Because the BVN random variables can take negative values, the authors make a further assumption that the region corresponding to the negative values of these variables can be ignored. Unfortunately, some calculation errors occurred in step 3 by allowing some (but not all) negative values of $\boldsymbol{\xi}$, a violation of the authors' assumption. This miscalculation, in effect, yields an approximation of π, denoted by $\widehat{\pi}$. Consequently, in step 4, the conclusion on the comparative statics is based on the derivatives of $\widehat{\pi}$. But, the derivatives of $\widehat{\pi}$ can be very different from those of π. This can be seen from the following example.

Example 1. For any $\varepsilon > 0$, let $\varepsilon_n = \frac{n\varepsilon}{3}$, $n = 0, 1, \cdots$. Define $H_1(x) = x$, $x \in [0,\ \infty)$,

$$H_2(x) = \begin{cases} x + 2(x - \varepsilon_{2n}), & x \in [\varepsilon_{2n},\ \varepsilon_{2n+1}], \\ x + 2(\varepsilon_{2n+2} - x), & x \in [\varepsilon_{2n+1},\ \varepsilon_{2n+2}], \quad n = 0, 1, \cdots. \end{cases}$$

Then, $|H_1(x) - H_2(x)| < \varepsilon$, but $|\frac{\mathsf{d}H_1(x)}{\mathsf{d}x} - \frac{\mathsf{d}H_2(x)}{\mathsf{d}x}| > 2$.

It is worth mentioning that, using the notion of convex order, BLB are able ro establish properties related C&R's Propositions 2-4 and Proposition 10 (ii) on the original feasible region S^h, assuming $\boldsymbol{\xi}$ has positive support. Here, by "related" we mean that the results are not direct generalizations of those in C&R. For example, Proposition 2 states that q^h is increasing in σ, where $\sigma = \sigma_1 = \sigma_2$. For BLB's result to recover this property, we need an additional condition that $\mu_1 = \mu_2$, i.e., the demand intercepts have the same mean, which is rather restrictive. On the other hand, when $\boldsymbol{\xi}$ is BVN, as assumed in C&R, the equal mean assumption can be relaxed. BLB can indeed show Propositions 2, 3, 5, and a special case of Proposition 10(i) under BVN on S^h. However, different from C&R, BLB allow $\boldsymbol{\xi}$ to take negative values. When $\boldsymbol{\xi}$ is restricted to nonnegative values, we obtain a truncated BVN distribution. Our analysis below shows, unfortunately, that Propositions 3-5 and 10 do not hold for this distribution.

In the remainder of this note, we assume $\boldsymbol{\xi}$ follows a BVN distribution. We inherit all the notations from C&R, and we refer the reader to C&R for their definitions. To simplify exposition, we introduce the following additional notation:

$$
\begin{aligned}
&\hat{\rho} = \sqrt{\frac{1+\rho}{2}}, \ \ \tilde{\rho} = \sqrt{\frac{1-\rho}{2}}, \ \ \hat{\sigma} = 2\sigma\hat{\rho}, \ \ \tilde{\sigma} = 2\sigma\tilde{\rho}, \ \ \theta = \frac{\hat{\rho}}{\tilde{\rho}},\\
&\hat{b} = \frac{b}{\hat{\sigma}}, \ \ \tilde{b} = \frac{b}{\tilde{\sigma}}, \ \ \hat{\mu}_i = \frac{\mu_i}{\hat{\sigma}}, \ \tilde{\mu}_i = \frac{\mu_i}{\tilde{\sigma}}, \ i = 1, 2,\\
&\hat{\alpha} = 2q\hat{b} - \hat{\mu}_1 - \hat{\mu}_2, \ \tilde{\alpha} = 2q\tilde{b} - \tilde{\mu}_1 + \tilde{\mu}_2, \ \tilde{\alpha}_1 = -2q\tilde{b} - \tilde{\mu}_1 + \tilde{\mu}_2,\\
&g_1(z,q) = \frac{2bq - \mu_1 - \mu_2 - \hat{\sigma}z}{2(1-d)}, \ g_2(z,q) = \frac{-2bq + \mu_1 - \mu_2 + \tilde{\sigma}z}{2(1+d)},\\
&g_3(z,q) = \frac{-2bq - \mu_1 + \mu_2 - \tilde{\sigma}z}{2(1+d)},\\
&g_4(z_1,z_2,q) = \tilde{\rho}(\sigma_1 - \sigma_2)z_1 + \hat{\rho}(\sigma_1 + \sigma_2)z_2 + \mu_1 + \mu_2 - 2bq,\\
&g_5(z_1,z_2) = \frac{(\sigma_1+\sigma_2)\tilde{\rho}z_2 - (\sigma_1 - \sigma_2)\hat{\rho}z_1}{4\tilde{\rho}\hat{\rho}},\\
&g_6(z,q) = 2\hat{\rho}\sigma_1 z + (\sigma_1 - \sigma_2)\gamma_2 + \mu_1 + \mu_2 - 2bq,\\
&g_7(z,q) = 2\hat{\rho}\sigma_2 z - (\sigma_1 - \sigma_2)\gamma_1 + \mu_1 + \mu_2 - 2bq,\\
&h_i(z) = \frac{z + \gamma_i\hat{\rho}}{\theta(1-\rho)^2}, \ \ \gamma_i = \frac{\mu_i}{\sigma_i}, \ \ b_i = \frac{b}{\sigma_i}, \ \ i = 1, 2,\\
&\hat{\beta}_1 = \frac{2qb_1 - \gamma_1 - \gamma_2}{2\hat{\rho}}, \ \tilde{\beta}_1 = \frac{2qb_1 - \gamma_1 + \gamma_2}{2\tilde{\rho}},\\
&\hat{\beta}_2 = \frac{2qb_2 - \gamma_1 - \gamma_2}{2\hat{\rho}}, \ \tilde{\beta}_2 = \frac{-2qb_2 - \gamma_1 + \gamma_2}{2\tilde{\rho}},\\
&\gamma = \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2}, \quad \ell(q) = \frac{2bq - \mu_1 - \mu_2}{\sigma_1 + \sigma_2},\\
&\Delta(\sigma_1,\sigma_2,\rho) = \mathsf{Pr}(\xi_1 \geq 0, \xi_2 \geq 0) = \int_{-\hat{\mu}_1 - \hat{\mu}_2}^{\infty} \phi(z)\Big[\Phi(z\theta + 2\tilde{\mu}_2) - \Phi(-z\theta - 2\tilde{\mu}_1)\Big]\, \mathsf{d}z.
\end{aligned}
$$

3 Corrections and Counterexamples

Proposition 2 (Chod and Rudi) *If $\boldsymbol{\xi}$ has a bivariate normal distribution with $\sigma_1 = \sigma_2 = \sigma$, then $\mathsf{d}q^h/\mathsf{d}\sigma > 0$.*

Claim 1 *The proof of the above Proposition 2 in C&R is incorrect.*

(Note that BLB are able to show this result when $\boldsymbol{\xi}$ is allowed to take negative values.)

Proof. We first derive the expression for $\mathsf{d}q^h/\mathsf{d}\sigma$ under the truncated BVN distribution and then point out errors. By implicit differentiation, we have

$$\frac{\mathsf{d}q^h}{\mathsf{d}\sigma} = -\frac{\partial^2\pi/\partial q\partial\sigma}{\partial^2\pi/\partial^2 q}\bigg|_{q=q^h}.$$

Because $\partial^2\pi(q,S^h)/\partial^2 q < 0$, it suffices to prove $\partial^2\pi(q,S^h)/\partial q\partial\sigma > 0$.

Note that Chod and Rudi explicitly assume that the regions of the parameter space yielding negative demand curve intercepts can be ignored. This is to assume that

$$1 = \iint_{\Omega_{1234}} f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2, \tag{1}$$

where $\Omega_{1234} = \cup_{i=1}^4 \Omega_i$ (see C&R for the definitions of the areas Ω_i, and see Figure 1 for an illustration). A precise account for this assumption is to assume $\boldsymbol{\xi}$ follows a truncated BVN distribution. That is,

$$\begin{aligned} f(\xi_1,\xi_2) \;&=\; \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ &\quad\times\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{\xi_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{\xi_1-\mu_1}{\sigma_1}\frac{\xi_2-\mu_2}{\sigma_2} + \left(\frac{\xi_2-\mu_2}{\sigma_2}\right)^2\right]\right\}. \end{aligned}$$

Let Ω_{234} be the union of Ω_2, Ω_3 and Ω_4. From C&R equation (16), we have

$$\begin{aligned} \frac{\partial\pi(q,S^h)}{\partial q} \;&=\; -w + \iint_{\Omega_2}\frac{\xi_1+\xi_2-2bq}{2(1-d)}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 + \iint_{\Omega_3}\frac{\xi_1+\xi_2 d-2bq}{1-d^2}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 \\ &\quad + \iint_{\Omega_4}\frac{\xi_1 d+\xi_2-2bq}{1-d^2}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 \\ &=\; -w + \iint_{\Omega_{234}}\frac{\xi_1+\xi_2-2bq}{2(1-d)}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 + \iint_{\Omega_3}\frac{\xi_1-\xi_2-2bq}{2(1+d)}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 \\ &\quad + \iint_{\Omega_4}\frac{\xi_2-\xi_1-2bq}{2(1+d)}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2. \end{aligned} \tag{2}$$

Now, make the following variable change so to transform the integration from the $\boldsymbol{\xi}$-space to the $\mathbf{Z}$-space, where $\mathbf{Z} = (Z_1, Z_2)$ are two independent standard normal random variables:

$$\begin{cases} z_1\hat{\sigma} + z_2\tilde{\sigma} = -2\mu_1 + 2\xi_1, \\ z_1\hat{\sigma} - z_2\tilde{\sigma} = -2\mu_2 + 2\xi_2. \end{cases} \tag{3}$$

Figure 2 illustrates this space transformation, indicating the changing positions of the five division lines of the $\boldsymbol{\xi}$-space in the $\mathbf{Z}$-space. Note that $\hat{\Omega}_i$ in the $\mathbf{Z}$-space, corresponds Ω_i in the $\boldsymbol{\xi}$-space $(i = 1, 2, 3, 4)$, while $\hat{\Omega}_i$ $(i = 5, 6, 7, 8, 9)$ in the $\mathbf{Z}$-space correspond to the parameter regions in the $\boldsymbol{\xi}$-space that lead to negative demand curve intercepts.

We next reexpress $\partial\pi(q, S^h)/\partial q$ by the integrals on the $\mathbf{Z}$-space. It is in this step that C&R made errors (see P545, left column, lines 9-12). Let ϕ and Φ denote the standard normal p.d.f and c.d.f., respectively. The correct expression is:

$$\begin{aligned}
\frac{\partial\pi(q, S^h)}{\partial q} &= -\frac{1}{\Delta(\sigma,\sigma,\rho)}\bigg\{\int_{\hat{\alpha}}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2} g_1(z_1,q)\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
&\quad -\int_{\tilde{\alpha}}^{\infty}\int_{z_2\theta^{-1}-2\hat{\mu}_2}^{\infty} g_2(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
&\quad -\int_{-\infty}^{\tilde{\alpha}_1}\int_{-z_2\theta^{-1}-2\hat{\mu}_1}^{\infty} g_3(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\bigg\} - w \\
&= -\frac{1}{\Delta(\sigma,\sigma,\rho)}\bigg\{\int_{\hat{\alpha}}^{\infty} g_1(z_1,q)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\phi(z_1)\mathsf{d}z_1 \\
&\quad -\int_{\tilde{\alpha}}^{\infty} g_2(z_2,q)\phi(z_2)[1-\Phi(z_2\theta^{-1}-2\hat{\mu}_2)]\mathsf{d}z_2 \\
&\quad -\int_{-\infty}^{\tilde{\alpha}_1} g_3(z_2,q)\phi(z_2)[1-\Phi(-z_2\theta^{-1}-2\hat{\mu}_1)]\mathsf{d}z_2\bigg\} - w.
\end{aligned} \tag{4}$$

From (4), we have

$$\begin{aligned}
\frac{\partial^2\pi(q, S^h)}{\partial q\partial\sigma} &= \frac{1}{\Delta(\sigma,\sigma,\rho)}\bigg[\frac{2\tilde{\mu}_2}{\sigma}\int_{\hat{\alpha}}^{\infty} g_1(z_1,q)\phi(z_1)\phi(z_1\theta+2\tilde{\mu}_2)\mathsf{d}z_1 \\
&\quad +\frac{2\tilde{\mu}_1}{\sigma}\int_{\hat{\alpha}}^{\infty} g_1(z_1,q)\phi(z_1)\phi(-z_1\theta-2\tilde{\mu}_1)\mathsf{d}z_1 \\
&\quad +\frac{\hat{\rho}}{1-d}\int_{\hat{\alpha}}^{\infty} z_1\phi(z_1)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\mathsf{d}z_1 \\
&\quad +\frac{\tilde{\rho}}{1+d}\int_{\tilde{\alpha}}^{\infty} z_2\phi(z_2)[1-\Phi(z_2\theta^{-1}-2\hat{\mu}_2)]\mathsf{d}z_2 \\
&\quad -\frac{2\hat{\mu}_2}{\sigma}\int_{\tilde{\alpha}}^{\infty} g_2(z_2,q)\phi(z_2)\phi(z_2\theta^{-1}-2\hat{\mu}_2)\mathsf{d}z_2 \\
&\quad -\frac{\tilde{\rho}}{1+d}\int_{-\infty}^{\tilde{\alpha}_1} z_2\phi(z_2)[1-\Phi(-z_2\theta^{-1}-2\hat{\mu}_1)]\mathsf{d}z_2 \\
&\quad -\frac{2\hat{\mu}_1}{\sigma}\int_{-\infty}^{\tilde{\alpha}_1} g_3(z_2,q)\phi(z_2)\phi(-z_2\theta^{-1}-2\hat{\mu}_1)\mathsf{d}z_2\bigg] \\
&\quad +\frac{\Delta_\sigma(\sigma,\sigma,\rho)}{\Delta^2(\sigma,\sigma,\rho)}\bigg[\int_{\hat{\alpha}}^{\infty} g_1(z_1,q)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\phi(z_1)\mathsf{d}z_1 \\
&\quad +\int_{\tilde{\alpha}}^{\infty} g_2(z_2,q)\phi(z_2)[1-\Phi(z_2\theta^{-1}-2\hat{\mu}_2)]\mathsf{d}z_2 \\
&\quad +\int_{-\infty}^{\tilde{\alpha}_1} g_3(z_2,q)\phi(z_2)[1-\Phi(-z_2\theta^{-1}-2\hat{\mu}_1)]\mathsf{d}z_2\bigg],
\end{aligned}$$

where

$$\begin{aligned}\Delta_\sigma(\sigma,\sigma,\rho) &= \frac{\mathsf{d}\Delta(\sigma,\sigma,\rho)}{\mathsf{d}\sigma}\\ &= -\frac{2\tilde{\mu}_2}{\sigma}\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\phi(z_1)\phi(z_1\theta+2\tilde{\mu}_2)\mathsf{d}z_1-\frac{2\tilde{\mu}_1}{\sigma}\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\phi(z_1)\phi(-z_1\theta-2\tilde{\mu}_1)\mathsf{d}z_1.\end{aligned}$$

This expression contains several more terms than the one obtained by C&R. From this, we cannot show $\partial^2\pi(q,S^h)/\partial q\partial\sigma$ is positive.

We now provide an explanation of the errors in C&R. Note that in the derivation of (4), we excluded the regions in which ξ_i are negative, i.e., $\hat{\Omega}_i$ $(i=5,\cdots,9)$. In contrast, C&R's expression included these regions, a contradiction to their nonnegative demand assumption. To see this, observe that if the coefficients of variation are not extremely large so that $\Delta(\sigma,\sigma,\rho)\approx 1$, then

$$\iint_{\hat{\Omega}_6\cup\hat{\Omega}_8\cup\hat{\Omega}_9}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\approx 0, \tag{5}$$

$$\iint_{\hat{\Omega}_7\cup\hat{\Omega}_8}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\approx 0, \tag{6}$$

$$\iint_{\hat{\Omega}_5\cup\hat{\Omega}_6}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\approx 0. \tag{7}$$

This leads to the following approximation of (2):

$$\begin{aligned}\frac{\partial\pi(q,S^h)}{\partial q} &\approx -w+\frac{\mu_1+\mu_2-2bq}{2(1-d)}+\iint_{\hat{\Omega}_1}g_1(z_1,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &\quad+\iint_{\hat{\Omega}_6\cup\hat{\Omega}_8\cup\hat{\Omega}_9}g_1(z_1,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &\quad+\iint_{\hat{\Omega}_3}g_2(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2+\iint_{\hat{\Omega}_5\cup\hat{\Omega}_6}g_2(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &\quad+\iint_{\hat{\Omega}_4}g_3(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2+\iint_{\hat{\Omega}_7\cup\hat{\Omega}_8}g_3(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &= -w+\frac{\mu_1+\mu_2-2bq}{2(1-d)}\\ &\quad+\int_{-\infty}^{\hat{\alpha}}g_1(z,q)\phi(z)\mathsf{d}z+\int_{\tilde{\alpha}}^{\infty}g_2(z,q)\phi(z)\mathsf{d}z\\ &\quad+\int_{-\infty}^{\tilde{\alpha}_1}g_3(z,q)\phi(z)\mathsf{d}z=H_2.\end{aligned} \tag{8}$$

(See C&R, p545, left column, lines 14-16). Taking derivative on both side of (8), C&R state that

$$\begin{aligned}\frac{\partial H_2}{\partial\sigma} &= \frac{\mathsf{d}}{\mathsf{d}\sigma}\Big[-w+\frac{\mu_1+\mu_2-2bq}{2(1-d)}+\int_{-\infty}^{\hat{\alpha}}g_1(z,q)\phi(z)\mathsf{d}z\\ &\quad+\int_{\tilde{\alpha}}^{\infty}g_2(z,q)\phi(z)\mathsf{d}z+\int_{-\infty}^{\tilde{\alpha}_1}g_3(z,q)\phi(z)\mathsf{d}z\Big]\end{aligned}$$

$$
\begin{aligned}
&= -\int_{-\infty}^{\hat{\alpha}} \frac{z\sqrt{(1+\rho)/2}}{1-d}\phi(z)\mathsf{d}z + \int_{\tilde{\alpha}}^{\infty} \frac{z\sqrt{(1-\rho)/2}}{1+d}\phi(z)\mathsf{d}z \\
&\quad - \int_{-\infty}^{\tilde{\alpha}_1} \frac{z\sqrt{(1-\rho)/2}}{1+d}\phi(z)\mathsf{d}z > 0.
\end{aligned}
\tag{9}
$$

(see C&R, p545, left column). However, as argued in Example 1 in Section 2, $H_2 \approx \partial\pi(q, S^h)/\partial q$ does not imply $\partial H_2/\partial\sigma = \partial^2\pi(q, S^h)/\partial q\partial\sigma$.

Proposition 3 (Chod and Rudi) *If $\boldsymbol{\xi}$ has a bivariate normal distribution with $\sigma_1 = \sigma_2 = \sigma$, then $\mathsf{d}\pi(\tilde{q}^h, \tilde{S}^h)/\mathsf{d}\sigma > 0$.*

Claim 2 *The above Proposition 3 is false; there exists a counterexample. The expression of $\mathsf{d}\pi(\tilde{q}^h, \tilde{S}^h)/\mathsf{d}\sigma$ given by C&R is erroneous.*

(Note that BLB are able to show $\mathsf{d}\pi(q^h, S^h)/\mathsf{d}\sigma > 0$ when $\boldsymbol{\xi}$ is BVN which is allowed to take negative values.)

Proof. We first drive the expression for $\mathsf{d}\pi(\tilde{q}^h, \tilde{S}^h)/\mathsf{d}\sigma$. Note that

$$
\frac{\mathsf{d}\pi(\tilde{q}^h, \widetilde{S}^h)}{\mathsf{d}\sigma} = \left.\frac{\partial\pi(q, \widetilde{S}^h)}{\partial\sigma}\right|_{q=\tilde{q}^h} + \left.\frac{\partial\pi(q, \widetilde{S}^h)}{\partial q}\right|_{q=\tilde{q}^h} \cdot \frac{\partial\tilde{q}^h}{\partial\sigma}
$$

Because $\partial\pi(q, \widetilde{S}^h)/\partial q|_{q=\tilde{q}^h} = 0$, so

$$
\frac{\mathsf{d}\pi(\tilde{q}^h, \widetilde{S}^h)}{\mathsf{d}\sigma} = \left.\frac{\partial\pi(q, \widetilde{S}^h)}{\partial\sigma}\right|_{q=\tilde{q}^h}. \tag{10}
$$

We notice $\pi(q, \widetilde{S}^h)$ given by C&R has typos (see P545, right column, lines 16-18); the correct expression can be written as

$$
\begin{aligned}
\pi(q, \widetilde{S}^h) &= -wq + \mathsf{E}\left\{\frac{\xi_1^2 + \xi_2^2 + 2d\xi_1\xi_2}{4b(1-d^2)}\mathsf{I}_{\Omega_1}\right\} + \mathsf{E}\left\{\left[\frac{(\xi_1-\xi_2)^2}{8b(1+d)} + \frac{q(\xi_1+\xi_2-bq)}{2(1-d)}\right]\mathsf{I}_{\Omega_{234}}\right\} \\
&= -wq + \mathsf{E}\left\{\frac{\xi_1^2 + \xi_2^2 + 2d\xi_1\xi_2}{4b(1-d^2)}\mathsf{I}_{\Omega_{1234}}\right\} \\
&\quad + \mathsf{E}\left\{\left[\frac{(\xi_1-\xi_2)^2}{8b(1+d)} + \frac{q(\xi_1+\xi_2-bq)}{2(1-d)} - \frac{\xi_1^2 + \xi_2^2 + 2d\xi_1\xi_2}{4b(1-d^2)}\right]\mathsf{I}_{\Omega_{234}}\right\} \\
&= -wq + \mathsf{E}\left\{\frac{\frac{1+d}{2}(\xi_1+\xi_2)^2 + \frac{1-d}{2}(\xi_1-\xi_2)^2}{4b(1-d^2)}\mathsf{I}_{\Omega_{1234}}\right\} - \mathsf{E}\left[\frac{(\xi_1+\xi_2-2bq)^2}{8b(1-d)}\mathsf{I}_{\Omega_{234}}\right].
\end{aligned}
\tag{11}
$$

By the integral transform given by (3), we have

$$
\begin{aligned}
\pi(q,\widetilde{S}^h) &= -wq+\frac{1}{\Delta(\sigma,\sigma,\rho)}\left\{\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2}\frac{(z_1\hat{\sigma}+\mu_1+\mu_2)^2}{8b(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\right.\\
&\quad+\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2}\frac{(z_2\tilde{\sigma}+\mu_1-\mu_2)^2}{8b(1+d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\
&\quad\left.-\int_{\hat{\alpha}}^{\infty}\frac{1-d}{2b}g_1^2(z_1,q)\phi(z_1)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\mathsf{d}z_1\right\}.
\end{aligned}
\tag{12}
$$

This leads to

$$
\begin{aligned}
\frac{\partial\pi(q,\widetilde{S}^h)}{\partial\sigma} &= \frac{1}{\Delta(\sigma,\sigma,\rho)}\left\{\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\frac{-\tilde{\mu}_2}{2b(1-d^2)\sigma}(z_1\hat{\sigma}+\mu_1+\mu_2)^2\phi(z_1)\phi(z_1\theta+2\tilde{\mu}_2)\mathsf{d}z_1\right.\\
&\quad+\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\frac{-\tilde{\mu}_1}{2b(1-d^2)\sigma}(z_1\hat{\sigma}+\mu_1+\mu_2)^2\phi(z_1)\phi(-z_1\theta-2\tilde{\mu}_1)\mathsf{d}z_1\\
&\quad+\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2}\frac{z_1\hat{\rho}(z_1\hat{\sigma}+\mu_1+\mu_2)}{2b(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\
&\quad+\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2}\frac{z_2\tilde{\rho}(z_2\tilde{\sigma}+\mu_1-\mu_2)}{2b(1+d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\
&\quad+\int_{\hat{\alpha}}^{\infty}\frac{\hat{\rho}}{b}g_1(z_1,q)z_1\phi(z_1)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\mathsf{d}z_1\\
&\quad+\int_{\hat{\alpha}}^{\infty}\frac{1-d}{b}\frac{\tilde{\mu}_2}{\sigma}g_1^2(z_1,q)\phi(z_1)\phi(z_1\theta+2\tilde{\mu}_2)\mathsf{d}z_1\\
&\quad\left.+\int_{\hat{\alpha}}^{\infty}\frac{1-d}{b}\frac{\tilde{\mu}_1}{\sigma}g_1^2(z_1,q)\phi(z_1)\phi(-z_1\theta-2\tilde{\mu}_1)\mathsf{d}z_1\right\}\\
&\quad-\frac{\Delta_\sigma(\sigma,\sigma,\rho)}{\Delta^2(\sigma,\sigma,\rho)}\left\{\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2}\frac{(z_1\hat{\sigma}+\mu_1+\mu_2)^2}{8b(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\right.\\
&\quad+\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2}\frac{(z_2\tilde{\sigma}+\mu_1-\mu_2)^2}{8b(1+d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\
&\quad\left.-\int_{\hat{\alpha}}^{\infty}\frac{1-d}{2b}g_1^2(z_1,q)\phi(z_1)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\mathsf{d}z_1\right\}.
\end{aligned}
$$

Counterexample: Letting $\mu_1=230,\mu_2=20,d=0.5,b=0.5,\rho=0.8,\sigma_1=\sigma_2=\sigma=4$ and $w=213.333$, we have $\tilde{q}^h=80$ and $\partial\pi(q,\widetilde{S}^h)/\partial\sigma|_{q=\tilde{q}^h=80}=-7.19496<0$. Also, $\mathsf{d}\pi(\tilde{q}^h,\widetilde{S}^h)/\mathsf{d}\sigma|_{\tilde{q}^h=80}=-7.19496<0$. Here

$$
\iint_{\Omega_{1234}}f(x_1,x_2)\mathsf{d}x_1\mathsf{d}x_2=1,\ \frac{\sigma_1}{\mu_1}=0.0173913,\ \frac{\sigma_2}{\mu_2}=0.2.
$$

□

Proposition 4 (Chod and Rudi). *If* $\boldsymbol{\xi}$ *has a bivariate normal distribution, then* $\mathsf{d}\tilde{q}^h/\mathsf{d}\rho>0$.

Claim 3 *The above Proposition 4 is false; there exists a counterexample. The expression of* $\mathsf{d}\tilde{q}^h/\mathsf{d}\rho$ *given by C&R is erroneous.*

(Note that BLB do not discuss this result under BVN.)

Proof. By implicit differentiation, we have

$$\frac{\mathsf{d}\tilde{q}^h}{\mathsf{d}\rho} = -\frac{\partial^2\pi/\partial q\partial\rho}{\partial^2\pi/\partial^2 q}\bigg|_{\tilde{q}^h}.$$

Because $\partial^2\pi(q,\widetilde{S}^h)/\partial^2 q < 0$, it remains to consider $\partial^2\pi(q,\widetilde{S}^h)/\partial q\partial\rho$. Using the following integral transform (see P544, right column, lines 13-14)

$$\begin{cases} \xi_1 = \tilde{\rho}\sigma_1 z_1 + \hat{\rho}\sigma_1 z_2 + \mu_1, \\ \xi_2 = -\tilde{\rho}\sigma_2 z_1 + \hat{\rho}\sigma_2 z_2 + \mu_2, \end{cases} \tag{13}$$

Figure 4 illustrates this space transformation, we can rewrite $\pi(q,\widetilde{S}^h)$ in (11) as

$$\begin{aligned}
\pi(q,\widetilde{S}^h) \quad = \quad & -wq + \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)} \times \\
& \times \Bigg\{ \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{(\tilde{\rho}(\sigma_1-\sigma_2)z_1 + \hat{\rho}(\sigma_1+\sigma_2)z_2 + \mu_1 + \mu_2)^2}{8b(1-d)} \phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
& + \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{(\tilde{\rho}(\sigma_1+\sigma_2)z_1 + \hat{\rho}(\sigma_1-\sigma_2)z_2 + \mu_1 - \mu_2)^2}{8b(1+d)} \phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
& - \int_{\hat{\beta}_2}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2} \frac{g_4^2(z_1,z_2,q)}{8b(1-d)} \phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
& - \int_{\tilde{\beta}_2}^{\tilde{\beta}_1} \int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty} \frac{g_4^2(z_1,z_2,q)}{8b(1-d)} \phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
& - \int_{\hat{\beta}_1}^{\infty} \int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{g_4^2(z_1,z_2,q)}{8b(1-d)} \phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \Bigg\}.
\end{aligned} \tag{14}$$

We have

$$\begin{aligned}
\frac{\partial\pi(q,\widetilde{S}^h)}{\partial q} \quad = \quad & -w + \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)} \times \Bigg\{ \int_{\hat{\beta}_2}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2} \frac{g_4(z_1,z_2,q)}{2(1-d)} \phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
& + \int_{\tilde{\beta}_2}^{\tilde{\beta}_1} \int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty} \frac{g_4(z_1,z_2,q)}{2(1-d)} \phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
& + \int_{\hat{\beta}_1}^{\infty} \int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{g_4(z_1,z_2,q)}{2(1-d)} \phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \Bigg\}.
\end{aligned} \tag{15}$$

Taking the derivative with respect to ρ yields

$$
\begin{aligned}
\frac{\partial^2 \pi(q,\widetilde{S}^h)}{\partial q \partial \rho} &= \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)} \Bigg\{ \int_{\hat{\beta}_2}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2} \frac{g_5(z_1,z_2)}{2(1-d)} \phi(z_1)\phi(z_2) \mathsf{d}z_1 \mathsf{d}z_2 \\
&\quad + \int_{\tilde{\beta}_2}^{\tilde{\beta}_1} \int_{z_1\gamma/\theta+\ell(q)/\tilde{\rho}}^{\infty} \frac{g_5(z_1,z_2)}{2(1-d)} \phi(z_1)\phi(z_2) \mathsf{d}z_2 \mathsf{d}z_1 \\
&\quad + \int_{\hat{\beta}_1}^{\infty} \int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{g_5(z_1,z_2)}{2(1-d)} \phi(z_1)\phi(z_2) \mathsf{d}z_1 \mathsf{d}z_2 \\
&\quad + \int_{\hat{\beta}_1}^{\infty} \frac{g_6(z_2,q)}{2(1-d)} h_2(z_2)\phi(z_2)\phi(z_2\theta+\gamma_2/\tilde{\rho}) \mathsf{d}z_2 \\
&\quad + \int_{\hat{\beta}_2}^{\infty} \frac{g_7(z_2,q)}{2(1-d)} h_1(z_2)\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde{\rho}) \mathsf{d}z_2 \Bigg\} \\
&\quad - \frac{\Delta_\rho(\sigma_1,\sigma_2,\rho)}{\Delta^2(\sigma_1,\sigma_2,\rho)} \Bigg\{ \int_{\hat{\beta}_2}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2} \frac{g_4(z_1,z_2,q)}{2(1-d)} \phi(z_1)\phi(z_2) \mathsf{d}z_1 \mathsf{d}z_2 \\
&\quad + \int_{\tilde{\beta}_2}^{\tilde{\beta}_1} \int_{z_1\gamma/\theta+\ell(q)/\tilde{\rho}}^{\infty} \frac{g_4(z_1,z_2,q)}{2(1-d)} \phi(z_1)\phi(z_2) \mathsf{d}z_2 \mathsf{d}z_1 \\
&\quad + \int_{\hat{\beta}_1}^{\infty} \int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{g_4(z_1,z_2,q)}{2(1-d)} \phi(z_1)\phi(z_2) \mathsf{d}z_1 \mathsf{d}z_2 \Bigg\},
\end{aligned} \tag{16}
$$

where

$$\Delta_\rho(\sigma_1,\sigma_2,\rho) = \frac{\mathsf{d}\Delta(\sigma_1,\sigma_2,\rho)}{\mathsf{d}\rho}.$$

By the integral transform given by (13), $\Delta(\sigma_1,\sigma_2,\rho)$ can be equally written as

$$\int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \phi(z_2)[\Phi(z_2\theta+\gamma_2/\tilde{\rho}) - \Phi(-z_2\theta-\gamma_1/\tilde{\rho})] \mathsf{d}z_2.$$

Then this gives

$$
\begin{aligned}
\Delta_\rho(\sigma_1,\sigma_2,\rho) &= \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} h_2(z_2)\phi(z_2)\phi(z_2\theta+\gamma_2/\tilde{\rho}) \mathsf{d}z_2 \\
&\quad + \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} h_1(z_2)\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde{\rho})] \mathsf{d}z_2.
\end{aligned}
$$

Counterexample: When $\mu_1 = 100, \mu_2 = 1000, d = 0.992, b = 0.2, \rho = 0.9999, \sigma_1 = 15, \sigma_2 = 100$ and $w = 7242.96$, we have the optimal $\tilde{q}^h = 830$, $\partial^2\pi(q,\widetilde{S}^h)/\partial q\partial\rho|_{q=\tilde{q}^h=830} = -0.00648758 < 0$ and $\partial^2\pi(q,\widetilde{S}^h)/\partial^2 q|_{q=\tilde{q}^h=830} = -47.4021$. Then $\mathsf{d}\tilde{q}^h/\mathsf{d}\rho|_{\tilde{q}^h=830} = -0.000136863 < 0$. Here

$$\iint_{\Omega_{1234}} f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 = 0.99999,\ \frac{\sigma_1}{\mu_1} = 0.15,\ \frac{\sigma_2}{\mu_2} = 0.1.$$

□

Proposition 5 (Chod and Rudi). *If $\boldsymbol{\xi}$ has a bivariate normal distribution and $d = 0$, then $\mathsf{d}\pi(\tilde{q}^h,\tilde{S}^h)/\mathsf{d}\rho < 0$.*

Claim 4 *The above Proposition 5 is false; there exists a counterexample. The expression of* $\mathsf{d}\pi(\tilde{q}^h, \tilde{S}^h)/\mathsf{d}\rho$ *given by C&R is erroneous.*

(Note that BLB are able to show $\mathsf{d}\pi(q^h, S^h)/\mathsf{d}\rho < 0$ when $\boldsymbol{\xi}$ is allowed to take negative values.)

Proof. Notice (18) in C&R has typos, we give the right expression of $\pi(q, \widetilde{S}^h)$ under $d = 0$ as following:

$$\pi(q, \widetilde{S}^h) \quad = \quad -wq + \mathsf{E}\left\{\frac{\xi_1^2 + \xi_2^2}{4b}\right\} - \mathsf{E}\left[\frac{(\xi_1 + \xi_2 - 2bq)^2}{8b}\mathsf{I}_{\Omega_{234}}\right].$$

Note that

$$\frac{\mathsf{d}\pi(\tilde{q}^h, \widetilde{S}^h)}{\mathsf{d}\rho} = \left.\frac{\partial\pi(q, \widetilde{S}^h)}{\partial\rho}\right|_{q=\tilde{q}^h} + \left.\frac{\partial\pi(q, \widetilde{S}^h)}{\partial q}\right|_{q=\tilde{q}^h} \frac{\partial\tilde{q}^h}{\partial\rho}.$$

Because $\partial\pi(q, \widetilde{S}^h)/\partial q|_{q=\tilde{q}^h} = 0$, we have

$$\frac{\mathsf{d}\pi(\tilde{q}^h, \widetilde{S}^h)}{\mathsf{d}\rho} = \left.\frac{\partial\pi(q, \widetilde{S}^h)}{\partial\rho}\right|_{q=\tilde{q}^h}. \tag{17}$$

Substitute $d = 0$ into (14), and take the derivative with respect to ρ, we obtain

$$\begin{aligned}
&\frac{\partial\pi(q, \widetilde{S}^h)}{\partial\rho} \\
&= \frac{-\Delta_\rho(\sigma_1, \sigma_2, \rho)}{\Delta^2(\sigma_1, \sigma_2, \rho)} \times \\
&\quad \times \left\{ \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{(\tilde{\rho}(\sigma_1 - \sigma_2)z_1 + \hat{\rho}(\sigma_1 + \sigma_2)z_2 + \mu_1 + \mu_2)^2}{8b}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \right.\\
&\quad + \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{(\tilde{\rho}(\sigma_1 + \sigma_2)z_1 + \hat{\rho}(\sigma_1 - \sigma_2)z_2 + \mu_1 - \mu_2)^2}{8b}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
&\quad - \int_{\hat{\beta}_2}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2} \frac{g_4^2(z_1, z_2, q)}{8b}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
&\quad - \int_{\tilde{\beta}_2}^{\tilde{\beta}_1} \int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty} \frac{g_4^2(z_1, z_2, q)}{8b}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
&\quad \left. - \int_{\hat{\beta}_1}^{\infty} \int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{g_4^2(z_1, z_2, q)}{8b}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \right\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\times \\
&\times\left\{\int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{(\tilde{\rho}(\sigma_1-\sigma_2)z_1+\hat{\rho}(\sigma_1+\sigma_2)z_2+\mu_1+\mu_2)}{4b}g_5(z_1,z_2)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right.\\
&\quad+\int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty}\frac{(g_6(z_2,q)+2bq)^2}{8b}h_2(z_2)\phi(z_2)\phi(z_2\theta+\gamma_2/\tilde{\rho})\mathsf{d}z_2\\
&\quad+\int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty}\frac{(g_7(z_2,q)+2bq)^2}{8b}h_1(z_2)\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde{\rho})\mathsf{d}z_2\\
&\quad+\int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{(\tilde{\rho}(\sigma_1+\sigma_2)z_1+\hat{\rho}(\sigma_1-\sigma_2)z_2+\mu_1-\mu_2)}{4b}\times\\
&\qquad\qquad\times\frac{(\sigma_1-\sigma_2)\tilde{\rho}z_2-(\sigma_1+\sigma_2)\hat{\rho}z_1}{4\tilde{\rho}\hat{\rho}}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\
&\quad+\int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty}\frac{(2\hat{\rho}\sigma_1z_2+(\sigma_1+\sigma_2)\gamma_2+\mu_1-\mu_2)^2}{8b}h_2(z_2)\phi(z_2)\phi(z_2\theta+\gamma_2/\tilde{\rho})\mathsf{d}z_2\\
&\quad+\int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty}\frac{(-2\hat{\rho}\sigma_2z_2-(\sigma_1+\sigma_2)\gamma_1+\mu_1-\mu_2)^2}{8b}h_1(z_2)\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde{\rho})\mathsf{d}z_2\\
&\quad-\int_{\hat{\beta}_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2}\frac{g_4(z_1,z_2,q)}{4b}g_5(z_1,z_2)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\
&\quad-\int_{\hat{\beta}_2}^{\tilde{\beta}_1}\int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty}\frac{g_4(z_1,z_2,q)}{4b}g_5(z_1,z_2)\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\
&\quad-\int_{\hat{\beta}_1}^{\infty}\int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{g_4(z_1,z_2,q)}{4b}g_5(z_1,z_2)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\
&\quad-\int_{\hat{\beta}_1}^{\infty}\frac{g_6^2(z_2,q)}{8b}h_2(z_2)\phi(z_2)\phi(z_2\theta+\gamma_2/\tilde{\rho})\mathsf{d}z_2\\
&\quad\left.-\int_{\hat{\beta}_2}^{\infty}\frac{g_7^2(z_2,q)}{8b}h_1(z_2)\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde{\rho})\mathsf{d}z_2\right\}.
\end{aligned}
\tag{18}
$$

Counterexample: When $\mu_1 = 1, \mu_2 = 0.05, b = 0.01, \rho = 0.5, \sigma_1 = 0.3, \sigma_2 = 0.023$ and $w = 0.498461$, we have the optimal $\tilde{q}^h = 3$ and $\partial\pi(q, \widetilde{S}^h)/\partial\rho|_{q=\tilde{q}^h=3} = 0.029046 > 0$. Then $\mathsf{d}\pi(\tilde{q}^h, \widetilde{S}^h)/\mathsf{d}\rho|_{\tilde{q}^h=3} = 0.029046 > 0$. Here

$$
\iint_{\Omega_{1234}} f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 = 0.984858,\ \frac{\sigma_1}{\mu_1} = 0.3,\ \frac{\sigma_2}{\mu_2} = 0.46.
$$

□

Proposition 9 (Chod and Rudi). *There exists a Stackelberg equilibrium capacity q^* maximizing the supplier's profit $\pi_s(q)$, characterized by the following necessary conditions:*

$$
\iint_{\Omega_{234}(q)} \frac{x_1+x_2-4bq}{2(1-d)} f(x_1,x_2)\mathsf{d}x_2\mathsf{d}x_1 = c.
$$

If, furthermore, the random variable $\mathbf{1}^T\boldsymbol{\xi}$ has an increasing generalized failure rate , the necessary condition is sufficient and q^* is unique. The corresponding equilibrium resource price

w^* *inducing the firm to order* q^* *is obtained from* $w = \iint_{\Omega_{234}(q)} \frac{x_1+x_2-2bq}{2(1-d)} f(x_1,x_2)\mathsf{d}x_2\mathsf{d}x_1$.

The proof of second part (the sufficient condition) of the above Proposition 9 given by C&R is incorrect. The following gives the explanation why their proof is not correct.

We make the following integral transform

$$\begin{cases} x_1 + x_2 = \tau, \\ x_1 - x_2 = \eta, \end{cases}$$

and let $g(\cdot)$ be the density function of $\zeta = \xi_1 + \xi_2$. Then, from (19) in C&R,

$$\begin{aligned} \frac{\mathsf{d}\pi_s(q)}{\mathsf{d}q} &= \iint_{\Omega_{234}(q)} \frac{x_1+x_2-4bq}{2(1-d)} f(x_1,x_2)\mathsf{d}x_2\mathsf{d}x_1 - c \\ &= \int_{2bq}^{\infty} \frac{\tau-4bq}{4(1-d)} \int_{-\tau}^{\tau} f(\frac{\tau+\eta}{2}, \frac{\tau-\eta}{2})\mathsf{d}\eta\mathsf{d}\tau - c \\ &\neq \int_{2bq}^{\infty} \frac{\tau-4bq}{2(1-d)} g(\tau)\mathsf{d}\tau - c \quad \text{(see P547, left column, line -12.)} \end{aligned}$$

This implies that

$$\begin{aligned} \frac{\mathsf{d}^2\pi_s(q)}{\mathsf{d}q^2} &= \frac{b^2 q}{1-d} \int_{-2bq}^{2bq} f(\frac{2bq+\eta}{2}, \frac{2bq-\eta}{2})\mathsf{d}\eta \\ &\quad - \int_{2bq}^{\infty} \frac{b}{1-d} \int_{-\tau}^{\tau} f(\frac{\tau+\eta}{2}, \frac{\tau-\eta}{2})\mathsf{d}\eta\mathsf{d}\tau \\ &\neq \frac{b}{1-d}\Big[2bq \cdot g(2bq) - 2\mathsf{Pr}(\zeta > 2bq)\Big] \text{(see P547, left column, line -11.)} \end{aligned}$$

Thus, the proof of C&R breaks down.

Proposition 10 (Chod and Rudi). *If* $\boldsymbol{\xi}$ *has a bivariate normal distribution, the equilibrium resource capacity and the supplier's profit satisfy the following relationships:*

(i) $\mathsf{d}q^*/\mathsf{d}\sigma_i \geq 0$ if and only if $\sigma_i \geq -\rho\sigma_j$,

(ii) $\mathsf{d}\pi_s(q^*, w^*)/\mathsf{d}\sigma_i \geq 0$ if and only if $\sigma_i \geq -\rho\sigma_j$,

(iii) $\mathsf{d}q^*/\mathsf{d}\rho \geq 0$, and

(iv) $\mathsf{d}\pi_s(q^*, w^*)/\mathsf{d}\rho \geq 0$.

Claim 5 *The general conclusions of Proposition 10 are false; there exist counterexamples.*

(Note that BLB are able to show part (ii) of the proposition under $\sigma_1 = \sigma_2$, when $\boldsymbol{\xi}$ is allowed to take negative values, but not the general case of $\sigma_1 \neq \sigma_2$ nor the other results.)

Proof Let $G(q) = \mathsf{d}\pi_s(q)/\mathsf{d}q$, the necessary condition for q^* is

$$G(q) = \iint_{\Omega_{234}(q)} \frac{\xi_1+\xi_2-4bq}{2(1-d)} f(\xi_1,\xi_2)\mathsf{d}\xi_2\mathsf{d}\xi_1 - c = 0.$$

Using the integral transform given by (13), we can rewrite $G(q)$ as

$$\begin{aligned} G(q) &= \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat{\beta}_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right. \\ &\quad+\int_{\tilde{\beta}_2}^{\tilde{\beta}_1}\int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\ &\quad\left.+\int_{\hat{\beta}_1}^{\infty}\int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right\}-c. \end{aligned} \tag{19}$$

The above equation is different from the counterpart in C&R:

$$G(q)=\int_{\frac{2bq-\mu_1-\mu_2}{\bar{\sigma}}}^{\infty}\frac{z\bar{\sigma}+\mu_1+\mu_2-4bq}{2(1-d)}\phi(z)\mathsf{d}z-c,\quad \text{(see P547, right column, line 10).}$$

where $\bar{\sigma}=\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_1\sigma_2\rho}$ (see P544, right column, line 9). By using this incorrect expression of $G(q)$, C&R can obtain the results in Proposition 10 (see P547, right column and P548, left column). In the remaining proof, we use (19) to arrive at different results.

For part (i), using implicit differentiation, we obtain $\mathsf{d}q^*/\mathsf{d}\sigma_i=-(\partial G/\partial\sigma_i)/(\partial G/\partial q)|_{q=q^*}$. Because $\pi_s(q)$ has an interior maximum at q^*, we have $\partial^2\pi_s(q)/\partial q^2|_{q=q^*}=\partial G(q)/\partial q|_{q=q^*}\le 0$. Thus, to give a counterexample to show (i) does not hold with $\sigma_1\ge-\rho\sigma_2$, it suffices to give a numerical example such that $\partial G(q)/\partial\sigma_1|_{q=q^*}<0$. To this end, we first derive $\partial G(q)/\partial\sigma_1$. We have

$$\begin{aligned} \frac{\partial G(q)}{\partial\sigma_1} &= \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat{\beta}_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2}\frac{z_1\tilde{\rho}+z_2\hat{\rho}}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right. \\ &\quad+\int_{\tilde{\beta}_2}^{\tilde{\beta}_1}\int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty}\frac{z_1\tilde{\rho}+z_2\hat{\rho}}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\ &\quad+\int_{\hat{\beta}_1}^{\infty}\int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{z_1\tilde{\rho}+z_2\hat{\rho}}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\ &\quad+\int_{\hat{\beta}_2}^{\infty}-\gamma_1\frac{2z_2\sigma_2\hat{\rho}+\gamma_1\sigma_2+\mu_2-4bq}{2(1-d)\sigma_1\tilde{\rho}}\phi(z_2)\phi(-z_2\theta-\frac{\gamma_1}{\tilde{\rho}})\mathsf{d}z_2 \\ &\quad\left.+\int_{\tilde{\beta}_2}^{\tilde{\beta}_1}-\frac{bq}{1-d}\left(\frac{2z_1\sigma_2}{\theta(\sigma_1+\sigma_2)^2}+\frac{\ell(q)}{\hat{\rho}(\sigma_1+\sigma_2)}\right)\phi(z_1)\phi\left(z_1\frac{\gamma}{\theta}+\frac{\ell(q)}{\hat{\rho}}\right)\mathsf{d}z_1\right\} \\ &\quad-\frac{\Delta_{\sigma_1}(\sigma_1,\sigma_2,\rho)}{\Delta^2(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat{\beta}_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right. \\ &\qquad+\int_{\tilde{\beta}_2}^{\tilde{\beta}_1}\int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\ &\qquad\left.+\int_{\hat{\beta}_1}^{\infty}\int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right\}, \end{aligned}$$

where

$$\Delta_{\sigma_1}(\sigma_1,\sigma_2,\rho)=\frac{\mathsf{d}\Delta(\sigma_1,\sigma_2,\rho)}{\mathsf{d}\sigma_1}=-\int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty}\frac{\gamma_1}{\sigma_1\tilde{\rho}}\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde{\rho})\mathsf{d}z_2.$$

Counterexample Let $\mu_1 = 20,\ \mu_2 = 15,\ \sigma_1 = 5,\ \sigma_2 = 3,\ \rho = 0.1,\ d = 0.5,\ b = 0.2,$ and $c = 19.0007$. We have $q^* = 20$, $\partial G(q)/\partial\sigma_1|_{q=q^*=20} = -0.121233$ and $\partial G(q)/\partial q|_{q=q^*=20} = -0.591631$. Then $\mathsf{d}q^*/\mathsf{d}\sigma_1|_{q^*=20} = -0.204913 < 0$. Here

$$\iint_{\Omega_{1234}} f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 = 0.999968,\ \frac{\sigma_1}{\mu_1} = 0.25,\ \frac{\sigma_2}{\mu_2} = 0.2.$$

For part (ii), it follows from $\pi_s(q) = (w(q) - c)q$ and

$$w(q) = \iint_{\Omega_{234}(q)} \frac{\xi_1 + \xi_2 - 2bq}{2(1-d)} f(\xi_1,\xi_2)\mathsf{d}\xi_2\mathsf{d}\xi_1$$

that

$$\pi_s(q) = q\iint_{\Omega_{234}(q)} \frac{\xi_1 + \xi_2 - 2bq}{2(1-d)} f(\xi_1,\xi_2)\mathsf{d}\xi_2\mathsf{d}\xi_1 - cq. \tag{20}$$

On the other hand,

$$\frac{\mathsf{d}\pi_s(q^*,w^*)}{d\sigma_i} = \left.\frac{\partial\pi_s(q,w(q))}{\partial\sigma_i}\right|_{q=q^*} + \left.\frac{\partial\pi_s(q,w(q))}{\partial q}\right|_{q=q^*}\cdot\frac{\partial q^*}{\partial\sigma_i}.$$

Because $\partial\pi_s(q,w(q))/\partial q|_{q^*} = 0$, we have

$$\frac{\mathsf{d}\pi_s(q^*,w^*)}{d\sigma_i} = \left.\frac{\partial\pi_s(q,w(q))}{\partial\sigma_i}\right|_{q=q^*}. \tag{21}$$

To give a counterexample to show part (ii) fails hold with $\sigma_1 \geq -\rho\sigma_2$, we first derive $\partial\pi_s(q,w(q))/\partial\sigma_1$. By the integral transform given by (13), we have

$$\begin{aligned}\pi_s(q,w(q)) &= \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat\beta_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde\rho}^{\tilde\beta_2} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right.\\ &\quad+\int_{\tilde\beta_2}^{\tilde\beta_1}\int_{z_1\gamma/\theta+\ell(q)/\hat\rho}^{\infty} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &\quad\left.+\int_{\hat\beta_1}^{\infty}\int_{\tilde\beta_1}^{z_2\theta+\gamma_2/\tilde\rho} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right\} - cq,\end{aligned} \tag{22}$$

so

$$\begin{aligned}\frac{\partial\pi_s(q,w(q))}{\partial\sigma_1} &= \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat\beta_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde\rho}^{\tilde\beta_2} q\frac{z_1\tilde\rho + z_2\hat\rho}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right.\\ &\quad+\int_{\tilde\beta_2}^{\tilde\beta_1}\int_{z_1\gamma/\theta+\ell(q)/\hat\rho}^{\infty} q\frac{z_1\tilde\rho + z_2\hat\rho}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &\quad+\int_{\hat\beta_1}^{\infty}\int_{\tilde\beta_1}^{z_2\theta+\gamma_2/\tilde\rho} q\frac{z_1\tilde\rho + z_2\hat\rho}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &\quad\left.+\int_{\hat\beta_2}^{\infty} -q\gamma_1\frac{2z_2\sigma_2\hat\rho + \gamma_1\sigma_2 + \mu_2 - 2bq}{2(1-d)\sigma_1\tilde\rho}\phi(z_1)\phi(-z_2\theta - \frac{\gamma_1}{\tilde\rho})\mathsf{d}z_2\right\}\end{aligned}$$

$$
-\frac{\Delta_{\sigma_1}(\sigma_1,\sigma_2,\rho)}{\Delta^2(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat\beta_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde\rho}^{\tilde\beta_2} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right.
$$

$$
+\int_{\tilde\beta_2}^{\tilde\beta_1}\int_{z_1\gamma/\theta+\ell(q)/\hat\rho}^{\infty} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1
$$

$$
\left.+\int_{\hat\beta_1}^{\infty}\int_{\tilde\beta_1}^{z_2\theta+\gamma_2/\tilde\rho} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right\}.
$$

Counterexample Letting $\mu_1=20,\ \mu_2=15,\ \sigma_1=5,\ \sigma_2=3,\ \rho=0.1, d=0.5,\ b=0.2$, and $c=19.0007$, we obtain $q^*=20$ and $\mathsf{d}\pi_s(q^*,w^*)/d\sigma_1|_{q=20}=-2.40872<0$. Here

$$
\iint_{\Omega_{1234}} f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2=0.999968,\ \frac{\sigma_1}{\mu_1}=0.25,\ \frac{\sigma_2}{\mu_2}=0.2.
$$

For part (iii), similar to part (i), we have that $\mathsf{d}q^*/\mathsf{d}\rho=-\partial G/\partial\rho/(\partial G/\partial q)|_{q=q^*}$. Because $\pi_s(q)$ has an interior maximum at q^*, then $\partial^2\pi_s(q)/\partial q^2|_{q=q^*}=\partial G/\partial q|_{q=q^*}\le 0$. Thus to give a counterexample to show that (iii) does not hold, it suffices to give a numerical example such that $\partial G(q)/(\partial\rho)|_{q=q^*}<0$. We have $\partial G/\partial\rho$ as follows:

$$
\begin{aligned}
\frac{\partial G(q)}{\partial\rho} \;=\; & \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat\beta_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde\rho}^{\tilde\beta_2}\frac{g_5(z_1,z_2)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right.\\
&+\int_{\tilde\beta_2}^{\tilde\beta_1}\int_{z_1\gamma/\theta+\ell(q)/\hat\rho}^{\infty}\frac{g_5(z_1,z_2)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\
&+\int_{\hat\beta_1}^{\infty}\int_{\tilde\beta_1}^{z_2\theta+\gamma_2/\tilde\rho}\frac{g_5(z_1,z_2)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\
&+\int_{\hat\beta_1}^{\infty}\frac{g_6(z_2,q)-2bq}{2(1-d)}h_2(z_2)\phi(z_2)\phi(z_2\theta+\gamma_2/\tilde\rho)\mathsf{d}z_2\\
&+\int_{\hat\beta_2}^{\infty}\frac{g_7(z_2,q)-2bq}{2(1-d)}h_1(z_2)\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde\rho)\mathsf{d}z_2\\
&\left.+\int_{\tilde\beta_2}^{\tilde\beta_1}\frac{-2bq}{2(1-d)}\frac{1}{(1+\rho)^2}(z_1\gamma\theta+\ell(q)\hat\rho)\phi(z_1)\phi(z_1\gamma/\theta+\ell(q)/\hat\rho)\mathsf{d}z_1\right\}\\
&-\frac{\Delta_{\rho}(\sigma_1,\sigma_2,\rho)}{\Delta^2(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat\beta_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde\rho}^{\tilde\beta_2}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right.\\
&+\int_{\tilde\beta_2}^{\tilde\beta_1}\int_{z_1\gamma/\theta+\ell(q)/\hat\rho}^{\infty}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\
&\left.+\int_{\hat\beta_1}^{\infty}\int_{\tilde\beta_1}^{z_2\theta+\gamma_2/\tilde\rho}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right\}.
\end{aligned}
$$

Counterexample Let $\mu_1=25,\ \mu_2=15,\ \sigma_1=5,\ \sigma_2=3,\ \rho=0.9992,\ d=0.5,\ b=0.0001$, and $c=39.9952$. We have $q^*=12$, $\partial G(q)/\partial\rho|_{q=q^*=12}=-0.000583854$ and $\partial G(q)/\partial q|_{q=q^*=12}=$

-0.0004. Then $\mathsf{d}q^*/\mathsf{d}\rho|_{q^*=12} = -1.45963 < 0$. Here

$$\iint_{\Omega_{1234}} f(\xi_1, \xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 = 0.999999,\ \frac{\sigma_1}{\mu_1} = 0.2,\ \frac{\sigma_2}{\mu_2} = 0.2.$$

Finally, we consider part (iv) . Note that

$$\frac{\mathsf{d}\pi_s(q^*, w^*)}{d\rho} = \left.\frac{\partial\pi_s(q, w(q))}{\partial\rho}\right|_{q=q^*} + \left.\frac{\partial\pi_s(q, w(q))}{\partial q}\right|_{q=q^*} \cdot \frac{\partial q^*}{\partial\rho}.$$

Because $\partial\pi_s(q, w(q))/\partial q|_{q=q^*} = 0$, it suffices to consider $\partial\pi_s(q, w(q))/\partial\rho|_{q=q^*}$. From (15) and (22), we have

$$\frac{\partial\pi_s(q, w(q))}{\partial\rho} = q\frac{\partial^2\pi(q, \widetilde{S}^h)}{\partial q\partial\rho}.$$

Counterexample Letting $\mu_1 = 100, \mu_2 = 1000, b = 0.2, \rho = 0.9999, d = 0.992, \sigma_1 = 15, \sigma_2 = 100$ and $w = 7242.96$, we obtain $\tilde{q}^h = 830$ and $\partial^2\pi(\tilde{q}^h, \widetilde{S}^h)/\partial q\partial\rho = -0.519006 < 0$. Then $\mathsf{d}\pi_s(q^*, w^*)/\mathsf{d}\rho|_{q^*=830} = -430.775 < 0$. Here

$$\iint_{\Omega_{1234}} f(\xi_1, \xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 = 0.99999,\ \frac{\sigma_1}{\mu_1} = 0.15,\ \frac{\sigma_2}{\mu_2} = 0.1.$$

$\square$

References

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