On-Line Appendix:
On "Resource Flexibility with Responsive Pricing"

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1 Notation and Preliminaries

We assume $\boldsymbol{\xi}$ follows a BVN distribution. We inherit all the notations from C&R, and we refer the reader to C&R for their definitions. To simplify exposition, we introduce the following additional notation:

$$\hat{\rho} = \sqrt{\frac{1+\rho}{2}}, \quad \tilde{\rho} = \sqrt{\frac{1-\rho}{2}}, \quad \hat{\sigma} = 2\sigma\hat{\rho}, \quad \tilde{\sigma} = 2\sigma\tilde{\rho}, \quad \theta = \frac{\hat{\rho}}{\tilde{\rho}},$$
$$\hat{b} = \frac{b}{\hat{\sigma}}, \quad \tilde{b} = \frac{b}{\tilde{\sigma}}, \quad \hat{\mu}_i = \frac{\mu_i}{\hat{\sigma}}, \ \tilde{\mu}_i = \frac{\mu_i}{\tilde{\sigma}}, \ i = 1, 2,$$
$$\hat{\alpha} = 2q\hat{b} - \hat{\mu}_1 - \hat{\mu}_2, \ \tilde{\alpha} = 2q\tilde{b} - \tilde{\mu}_1 + \tilde{\mu}_2, \ \tilde{\alpha}_1 = -2q\tilde{b} - \tilde{\mu}_1 + \tilde{\mu}_2,$$
$$g_1(z,q) = \frac{2bq - \mu_1 - \mu_2 - \hat{\sigma}z}{2(1-d)}, \ g_2(z,q) = \frac{-2bq + \mu_1 - \mu_2 + \tilde{\sigma}z}{2(1+d)},$$
$$g_3(z,q) = \frac{-2bq - \mu_1 + \mu_2 - \tilde{\sigma}z}{2(1+d)},$$
$$g_4(z_1, z_2, q) = \tilde{\rho}(\sigma_1 - \sigma_2)z_1 + \hat{\rho}(\sigma_1 + \sigma_2)z_2 + \mu_1 + \mu_2 - 2bq,$$
$$g_5(z_1, z_2) = \frac{(\sigma_1 + \sigma_2)\tilde{\rho}z_2 - (\sigma_1 - \sigma_2)\hat{\rho}z_1}{4\tilde{\rho}\hat{\rho}},$$
$$g_6(z,q) = 2\hat{\rho}\sigma_1 z + (\sigma_1 - \sigma_2)\gamma_2 + \mu_1 + \mu_2 - 2bq,$$
$$g_7(z,q) = 2\hat{\rho}\sigma_2 z - (\sigma_1 - \sigma_2)\gamma_1 + \mu_1 + \mu_2 - 2bq,$$
$$h_i(z) = \frac{z + \gamma_i\hat{\rho}}{\theta(1-\rho)^2}, \quad \gamma_i = \frac{\mu_i}{\sigma_i}, \quad b_i = \frac{b}{\sigma_i}, \quad i = 1, 2,$$

$$\hat{\beta}_1 = \frac{2qb_1 - \gamma_1 - \gamma_2}{2\hat{\rho}},\ \tilde{\beta}_1 = \frac{2qb_1 - \gamma_1 + \gamma_2}{2\tilde{\rho}},$$
$$\hat{\beta}_2 = \frac{2qb_2 - \gamma_1 - \gamma_2}{2\hat{\rho}},\ \tilde{\beta}_2 = \frac{-2qb_2 - \gamma_1 + \gamma_2}{2\tilde{\rho}},$$
$$\gamma = \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2}, \quad \ell(q) = \frac{2bq - \mu_1 - \mu_2}{\sigma_1 + \sigma_2},$$
$$\Delta(\sigma_1, \sigma_2, \rho) = \mathsf{Pr}(\xi_1 \geq 0, \xi_2 \geq 0) = \int_{-\hat{\mu}_1 - \hat{\mu}_2}^{\infty} \phi(z) \left[\Phi(z\theta + 2\tilde{\mu}_2) - \Phi(-z\theta - 2\tilde{\mu}_1)\right] \mathsf{d}z.$$

A common error in the derivation in C&R can be summarized as follows. When establishing the comparative statics, the following four steps were taken: 1) Assume that the demand intercepts $\boldsymbol{\xi}$ follow a BVN distribution. 2) Express $\boldsymbol{\xi}$ in terms of two independent standard normal random variables. 3) Reexpress the expected profit function π according to the variable transformation in the previous step. 4) Take the derivative of π with respect to the parameter of interest, observe the sign of the derivative, and obtain comparative statics. Because the BVN random variables can take negative values, a further assumption is made so that the region corresponding to the negative values of these variables can be ignored. Unfortunately, some calculation errors occurred in step 3 by allowing only *some* (but not all) negative values of $\boldsymbol{\xi}$. This miscalculation, in effect, yields an approximation of π, denoted by $\widehat{\pi}$. Consequently, in step 4, the conclusion on the comparative statics is based on the derivatives of $\widehat{\pi}$. But, the derivatives of $\widehat{\pi}$ can be very different from those of π. This can be seen from the following example.

Example 1. For any $\varepsilon > 0$, let $\varepsilon_n = \frac{n\varepsilon}{3}$, $n = 0, 1, \cdots$. Define $H_1(x) = x$, $x \in [0,\ \infty)$,

$$H_2(x) = \begin{cases} x + 2(x - \varepsilon_{2n}), & x \in [\varepsilon_{2n},\ \varepsilon_{2n+1}], \\ x + 2(\varepsilon_{2n+2} - x), & x \in [\varepsilon_{2n+1},\ \varepsilon_{2n+2}], \quad n = 0, 1, \cdots. \end{cases}$$

Then, $|H_1(x) - H_2(x)| < \varepsilon$, but $|\frac{\mathsf{d}H_1(x)}{\mathsf{d}x} - \frac{\mathsf{d}H_2(x)}{\mathsf{d}x}| > 2$.

2 Corrections and Counterexamples

Proposition 2 (Chod and Rudi) *If $\boldsymbol{\xi}$ has a bivariate normal distribution with $\sigma_1 = \sigma_2 = \sigma$, then $\mathsf{d}q^h/\mathsf{d}\sigma > 0$.*

Claim 1 *The proof of the above Proposition 2 in C&R is incorrect.*

(Note that BLB are able to show this result when $\boldsymbol{\xi}$ is allowed to take negative values.)

Proof. We first derive the expression for $\mathsf{d}q^h/\mathsf{d}\sigma$ under the truncated BVN distribution and then point out errors. By implicit differentiation, we have

$$\frac{\mathsf{d}q^h}{\mathsf{d}\sigma} = -\frac{\partial^2\pi/\partial q\partial\sigma}{\partial^2\pi/\partial^2 q}\bigg|_{q=q^h}.$$

Because $\partial^2\pi(q,S^h)/\partial^2 q < 0$, it suffices to prove $\partial^2\pi(q,S^h)/\partial q\partial\sigma > 0$.

Note that Chod and Rudi explicitly assume that the regions of the parameter space yielding negative demand curve intercepts can be ignored. This is to assume that

$$1 = \iint_{\Omega_{1234}} f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2, \tag{1}$$

where $\Omega_{1234} = \cup_{i=1}^4\Omega_i$ (see C&R for the definitions of the areas Ω_i, and see Figure 1 for an illustration). A precise account for this assumption is to assume $\boldsymbol{\xi}$ follows a truncated BVN distribution. That is,

$$\begin{aligned} f(\xi_1,\xi_2) &= \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ &\quad\times\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{\xi_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{\xi_1-\mu_1}{\sigma_1}\frac{\xi_2-\mu_2}{\sigma_2} + \left(\frac{\xi_2-\mu_2}{\sigma_2}\right)^2\right]\right\}. \end{aligned}$$

Let Ω_{234} be the union of Ω_2, Ω_3 and Ω_4. From C&R equation (16), we have

$$\begin{aligned} \frac{\partial\pi(q,S^h)}{\partial q} &= -w + \iint_{\Omega_2}\frac{\xi_1+\xi_2-2bq}{2(1-d)}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 + \iint_{\Omega_3}\frac{\xi_1+\xi_2 d-2bq}{1-d^2}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 \\ &\quad+\iint_{\Omega_4}\frac{\xi_1 d+\xi_2-2bq}{1-d^2}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 \\ &= -w + \iint_{\Omega_{234}}\frac{\xi_1+\xi_2-2bq}{2(1-d)}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 + \iint_{\Omega_3}\frac{\xi_1-\xi_2-2bq}{2(1+d)}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 \\ &\quad+\iint_{\Omega_4}\frac{\xi_2-\xi_1-2bq}{2(1+d)}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2. \end{aligned} \tag{2}$$

Now, make the following variable change so to transform the integration from the $\boldsymbol{\xi}$-space to the $\mathbf{Z}$-space, where $\mathbf{Z} = (Z_1, Z_2)$ are two independent standard normal random variables:

$$\begin{cases} z_1\hat{\sigma} + z_2\tilde{\sigma} = -2\mu_1 + 2\xi_1, \\ z_1\hat{\sigma} - z_2\tilde{\sigma} = -2\mu_2 + 2\xi_2. \end{cases} \tag{3}$$

Figure 2 illustrates this space transformation, indicating the changing positions of the five division lines of the $\boldsymbol{\xi}$-space in the $\mathbf{Z}$-space. Note that $\hat{\Omega}_i$ in the $\mathbf{Z}$-space, corresponds Ω_i in the $\boldsymbol{\xi}$-space ($i = 1,2,3,4$), while $\hat{\Omega}_i$ ($i = 5,6,7,8,9$) in the $\mathbf{Z}$-space correspond to the parameter regions in the $\boldsymbol{\xi}$-space that lead to negative demand curve intercepts.

We next reexpress $\partial\pi(q,S^h)/\partial q$ by the integrals on the **Z**-space. It is in this step that C&R made errors (see P545, left column, lines 9-12). Let ϕ and Φ denote the standard normal p.d.f and c.d.f., respectively. The correct expression is:

$$
\begin{aligned}
\frac{\partial\pi(q,S^h)}{\partial q} &= -\frac{1}{\Delta(\sigma,\sigma,\rho)}\bigg\{\int_{\hat{\alpha}}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2} g_1(z_1,q)\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
&\qquad -\int_{\tilde{\alpha}}^{\infty}\int_{z_2\theta^{-1}-2\hat{\mu}_2}^{\infty} g_2(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
&\qquad -\int_{-\infty}^{\tilde{\alpha}_1}\int_{-z_2\theta^{-1}-2\hat{\mu}_1}^{\infty} g_3(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\bigg\} - w \\
&= -\frac{1}{\Delta(\sigma,\sigma,\rho)}\bigg\{\int_{\hat{\alpha}}^{\infty} g_1(z_1,q)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\phi(z_1)\mathsf{d}z_1 \\
&\qquad -\int_{\tilde{\alpha}}^{\infty} g_2(z_2,q)\phi(z_2)[1-\Phi(z_2\theta^{-1}-2\hat{\mu}_2)]\mathsf{d}z_2 \\
&\qquad -\int_{-\infty}^{\tilde{\alpha}_1} g_3(z_2,q)\phi(z_2)[1-\Phi(-z_2\theta^{-1}-2\hat{\mu}_1)]\mathsf{d}z_2\bigg\} - w.
\end{aligned} \tag{4}
$$

From (4), we have

$$
\begin{aligned}
\frac{\partial^2\pi(q,S^h)}{\partial q\partial\sigma} &= \frac{1}{\Delta(\sigma,\sigma,\rho)}\bigg[\frac{2\tilde{\mu}_2}{\sigma}\int_{\hat{\alpha}}^{\infty} g_1(z_1,q)\phi(z_1)\phi(z_1\theta+2\tilde{\mu}_2)\mathsf{d}z_1 \\
&\qquad +\frac{2\tilde{\mu}_1}{\sigma}\int_{\hat{\alpha}}^{\infty} g_1(z_1,q)\phi(z_1)\phi(-z_1\theta-2\tilde{\mu}_1)\mathsf{d}z_1 \\
&\qquad +\frac{\hat{\rho}}{1-d}\int_{\hat{\alpha}}^{\infty} z_1\phi(z_1)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\mathsf{d}z_1 \\
&\qquad +\frac{\tilde{\rho}}{1+d}\int_{\tilde{\alpha}}^{\infty} z_2\phi(z_2)[1-\Phi(z_2\theta^{-1}-2\hat{\mu}_2)]\mathsf{d}z_2 \\
&\qquad -\frac{2\hat{\mu}_2}{\sigma}\int_{\tilde{\alpha}}^{\infty} g_2(z_2,q)\phi(z_2)\phi(z_2\theta^{-1}-2\hat{\mu}_2)\mathsf{d}z_2 \\
&\qquad -\frac{\tilde{\rho}}{1+d}\int_{-\infty}^{\tilde{\alpha}_1} z_2\phi(z_2)[1-\Phi(-z_2\theta^{-1}-2\hat{\mu}_1)]\mathsf{d}z_2 \\
&\qquad -\frac{2\hat{\mu}_1}{\sigma}\int_{-\infty}^{\tilde{\alpha}_1} g_3(z_2,q)\phi(z_2)\phi(-z_2\theta^{-1}-2\hat{\mu}_1)\mathsf{d}z_2\bigg] \\
&\quad +\frac{\Delta_\sigma(\sigma,\sigma,\rho)}{\Delta^2(\sigma,\sigma,\rho)}\bigg[\int_{\hat{\alpha}}^{\infty} g_1(z_1,q)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\phi(z_1)\mathsf{d}z_1 \\
&\qquad +\int_{\tilde{\alpha}}^{\infty} g_2(z_2,q)\phi(z_2)[1-\Phi(z_2\theta^{-1}-2\hat{\mu}_2)]\mathsf{d}z_2 \\
&\qquad +\int_{-\infty}^{\tilde{\alpha}_1} g_3(z_2,q)\phi(z_2)[1-\Phi(-z_2\theta^{-1}-2\hat{\mu}_1)]\mathsf{d}z_2\bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_\sigma(\sigma,\sigma,\rho) &= \frac{\mathsf{d}\Delta(\sigma,\sigma,\rho)}{\mathsf{d}\sigma} \\
&= -\frac{2\tilde{\mu}_2}{\sigma}\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\phi(z_1)\phi(z_1\theta+2\tilde{\mu}_2)\mathsf{d}z_1 - \frac{2\tilde{\mu}_1}{\sigma}\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\phi(z_1)\phi(-z_1\theta-2\tilde{\mu}_1)\mathsf{d}z_1.
\end{aligned}
$$

This expression contains several more terms than the one obtained by C&R. From this, we cannot show $\partial^2\pi(q,S^h)/\partial q\partial\sigma$ is positive.

We now provide an explanation of the errors in C&R. Note that in the derivation of (4), we excluded the regions in which ξ_i are negative, i.e., $\hat{\Omega}_i$ $(i=5,\cdots,9)$. In contrast, C&R's expression included these regions, a contradiction to their nonnegative demand assumption. To see this, observe that if the coefficients of variation are not extremely large so that $\Delta(\sigma,\sigma,\rho)\approx 1$, then

$$\iint_{\hat{\Omega}_6\cup\hat{\Omega}_8\cup\hat{\Omega}_9}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\approx 0, \tag{5}$$

$$\iint_{\hat{\Omega}_7\cup\hat{\Omega}_8}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\approx 0, \tag{6}$$

$$\iint_{\hat{\Omega}_5\cup\hat{\Omega}_6}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\approx 0. \tag{7}$$

This leads to the following approximation of (2):

$$\begin{aligned}
\frac{\partial\pi(q,S^h)}{\partial q} \approx\ & -w+\frac{\mu_1+\mu_2-2bq}{2(1-d)}+\iint_{\hat{\Omega}_1}g_1(z_1,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\
&+\iint_{\hat{\Omega}_6\cup\hat{\Omega}_8\cup\hat{\Omega}_9}g_1(z_1,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\
&+\iint_{\hat{\Omega}_3}g_2(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2+\iint_{\hat{\Omega}_5\cup\hat{\Omega}_6}g_2(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\
&+\iint_{\hat{\Omega}_4}g_3(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2+\iint_{\hat{\Omega}_7\cup\hat{\Omega}_8}g_3(z_2,q)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\
=\ & -w+\frac{\mu_1+\mu_2-2bq}{2(1-d)}\\
&+\int_{-\infty}^{\hat{\alpha}}g_1(z,q)\phi(z)\mathsf{d}z+\int_{\tilde{\alpha}}^{\infty}g_2(z,q)\phi(z)\mathsf{d}z\\
&+\int_{-\infty}^{\tilde{\alpha}_1}g_3(z,q)\phi(z)\mathsf{d}z=H_2.
\end{aligned} \tag{8}$$

(See C&R, p545, left column, lines 14-16). Taking derivative on both side of (8), C&R state that

$$\begin{aligned}
\frac{\partial H_2}{\partial\sigma} =\ & \frac{\mathsf{d}}{\mathsf{d}\sigma}\Bigg[-w+\frac{\mu_1+\mu_2-2bq}{2(1-d)}+\int_{-\infty}^{\hat{\alpha}}g_1(z,q)\phi(z)\mathsf{d}z\\
&+\int_{\tilde{\alpha}}^{\infty}g_2(z,q)\phi(z)\mathsf{d}z+\int_{-\infty}^{\tilde{\alpha}_1}g_3(z,q)\phi(z)\mathsf{d}z\Bigg]\\
=\ & -\int_{-\infty}^{\hat{\alpha}}\frac{z\sqrt{(1+\rho)/2}}{1-d}\phi(z)\mathsf{d}z+\int_{\tilde{\alpha}}^{\infty}\frac{z\sqrt{(1-\rho)/2}}{1+d}\phi(z)\mathsf{d}z\\
&-\int_{-\infty}^{\tilde{\alpha}_1}\frac{z\sqrt{(1-\rho)/2}}{1+d}\phi(z)\mathsf{d}z>0.
\end{aligned} \tag{9}$$

(see C&R, p545, left column). However, as argued in Example 1 in Section 2, $H_2 \approx \partial\pi(q,S^h)/\partial q$ does not imply $\partial H_2/\partial\sigma = \partial^2\pi(q,S^h)/\partial q\partial\sigma$.

Proposition 3 (Chod and Rudi) *If $\boldsymbol{\xi}$ has a bivariate normal distribution with $\sigma_1=\sigma_2=\sigma$, then $\mathsf{d}\pi(\tilde{q}^h,\tilde{S}^h)/\mathsf{d}\sigma > 0$.*

Claim 2 *The above Proposition 3 is false; there exists a counterexample. The expression of $\mathsf{d}\pi(\tilde{q}^h,\tilde{S}^h)/\mathsf{d}\sigma$ given by C&R is erroneous.*

(Note that BLB are able to show $\mathsf{d}\pi(q^h,S^h)/\mathsf{d}\sigma > 0$ when $\boldsymbol{\xi}$ is BVN which is allowed to take negative values.)

Proof. We first drive the expression for $\mathsf{d}\pi(\tilde{q}^h,\tilde{S}^h)/\mathsf{d}\sigma$. Note that

$$\frac{\mathsf{d}\pi(\tilde{q}^h,\widetilde{S}^h)}{\mathsf{d}\sigma} = \left.\frac{\partial\pi(q,\widetilde{S}^h)}{\partial\sigma}\right|_{q=\tilde{q}^h} + \left.\frac{\partial\pi(q,\widetilde{S}^h)}{\partial q}\right|_{q=\tilde{q}^h}\cdot\frac{\partial\tilde{q}^h}{\partial\sigma}$$

Because $\partial\pi(q,\widetilde{S}^h)/\partial q|_{q=\tilde{q}^h}=0$, so

$$\frac{\mathsf{d}\pi(\tilde{q}^h,\widetilde{S}^h)}{\mathsf{d}\sigma} = \left.\frac{\partial\pi(q,\widetilde{S}^h)}{\partial\sigma}\right|_{q=\tilde{q}^h}. \tag{10}$$

We notice $\pi(q,\widetilde{S}^h)$ given by C&R has typos (see P545, right column, lines 16-18); the correct expression can be written as

$$\begin{aligned}
\pi(q,\widetilde{S}^h) &= -wq + \mathsf{E}\left\{\frac{\xi_1^2+\xi_2^2+2d\xi_1\xi_2}{4b(1-d^2)}\mathsf{I}_{\Omega_1}\right\} + \mathsf{E}\left\{\left[\frac{(\xi_1-\xi_2)^2}{8b(1+d)} + \frac{q(\xi_1+\xi_2-bq)}{2(1-d)}\right]\mathsf{I}_{\Omega_{234}}\right\}\\
&= -wq + \mathsf{E}\left\{\frac{\xi_1^2+\xi_2^2+2d\xi_1\xi_2}{4b(1-d^2)}\mathsf{I}_{\Omega_{1234}}\right\}\\
&\quad + \mathsf{E}\left\{\left[\frac{(\xi_1-\xi_2)^2}{8b(1+d)} + \frac{q(\xi_1+\xi_2-bq)}{2(1-d)} - \frac{\xi_1^2+\xi_2^2+2d\xi_1\xi_2}{4b(1-d^2)}\right]\mathsf{I}_{\Omega_{234}}\right\}\\
&= -wq + \mathsf{E}\left\{\frac{\frac{1+d}{2}(\xi_1+\xi_2)^2 + \frac{1-d}{2}(\xi_1-\xi_2)^2}{4b(1-d^2)}\mathsf{I}_{\Omega_{1234}}\right\} - \mathsf{E}\left[\frac{(\xi_1+\xi_2-2bq)^2}{8b(1-d)}\mathsf{I}_{\Omega_{234}}\right].
\end{aligned} \tag{11}$$

By the integral transform given by (3), we have

$$\begin{aligned}
\pi(q,\widetilde{S}^h) &= -wq + \frac{1}{\Delta(\sigma,\sigma,\rho)}\left\{\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2}\frac{(z_1\hat{\sigma}+\mu_1+\mu_2)^2}{8b(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\right.\\
&\quad + \int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2}\frac{(z_2\tilde{\sigma}+\mu_1-\mu_2)^2}{8b(1+d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\
&\quad \left. - \int_{\hat{\alpha}}^{\infty}\frac{1-d}{2b}g_1^2(z_1,q)\phi(z_1)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\mathsf{d}z_1\right\}.
\end{aligned} \tag{12}$$

This leads to

$$
\begin{aligned}
\frac{\partial \pi(q,\widetilde{S}^h)}{\partial \sigma} \quad = \quad & \frac{1}{\Delta(\sigma,\sigma,\rho)}\left\{\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty} \frac{-\tilde{\mu}_2}{2b(1-d^2)\sigma}(z_1\hat{\sigma}+\mu_1+\mu_2)^2\phi(z_1)\phi(z_1\theta+2\tilde{\mu}_2)\mathsf{d}z_1\right. \\
& + \int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty} \frac{-\tilde{\mu}_1}{2b(1-d^2)\sigma}(z_1\hat{\sigma}+\mu_1+\mu_2)^2\phi(z_1)\phi(-z_1\theta-2\tilde{\mu}_1)\mathsf{d}z_1 \\
& + \int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2} \frac{z_1\hat{\rho}(z_1\hat{\sigma}+\mu_1+\mu_2)}{2b(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
& + \int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2} \frac{z_2\tilde{\rho}(z_2\tilde{\sigma}+\mu_1-\mu_2)}{2b(1+d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
& + \int_{\hat{\alpha}}^{\infty} \frac{\hat{\rho}}{b}g_1(z_1,q)z_1\phi(z_1)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\mathsf{d}z_1 \\
& + \int_{\hat{\alpha}}^{\infty} \frac{1-d}{b}\frac{\tilde{\mu}_2}{\sigma}g_1^2(z_1,q)\phi(z_1)\phi(z_1\theta+2\tilde{\mu}_2)\mathsf{d}z_1 \\
& \left. + \int_{\hat{\alpha}}^{\infty} \frac{1-d}{b}\frac{\tilde{\mu}_1}{\sigma}g_1^2(z_1,q)\phi(z_1)\phi(-z_1\theta-2\tilde{\mu}_1)\mathsf{d}z_1\right\} \\
& - \frac{\Delta_\sigma(\sigma,\sigma,\rho)}{\Delta^2(\sigma,\sigma,\rho)}\left\{\int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2} \frac{(z_1\hat{\sigma}+\mu_1+\mu_2)^2}{8b(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\right. \\
& + \int_{-\hat{\mu}_1-\hat{\mu}_2}^{\infty}\int_{-z_1\theta-2\tilde{\mu}_1}^{z_1\theta+2\tilde{\mu}_2} \frac{(z_2\tilde{\sigma}+\mu_1-\mu_2)^2}{8b(1+d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
& \left. - \int_{\hat{\alpha}}^{\infty} \frac{1-d}{2b}g_1^2(z_1,q)\phi(z_1)[\Phi(z_1\theta+2\tilde{\mu}_2)-\Phi(-z_1\theta-2\tilde{\mu}_1)]\mathsf{d}z_1\right\}.
\end{aligned}
$$

Counterexample: Letting $\mu_1 = 230, \mu_2 = 20, d = 0.5, b = 0.5, \rho = 0.8, \sigma_1 = \sigma_2 = \sigma = 4$ and $w = 213.333$, we have $\tilde{q}^h = 80$ and $\partial\pi(q,\widetilde{S}^h)/\partial\sigma|_{q=\tilde{q}^h=80} = -7.19496 < 0$. Also, $\mathsf{d}\pi(\tilde{q}^h,\widetilde{S}^h)/\mathsf{d}\sigma|_{\tilde{q}^h=80} = -7.19496 < 0$. Here

$$\iint_{\Omega_{1234}} f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 = 1,\ \frac{\sigma_1}{\mu_1} = 0.0173913,\ \frac{\sigma_2}{\mu_2} = 0.2.$$

□

Proposition 4 (Chod and Rudi). *If $\boldsymbol{\xi}$ has a bivariate normal distribution, then $\mathsf{d}\tilde{q}^h/\mathsf{d}\rho > 0$.*

Claim 3 *The above Proposition 4 is false; there exists a counterexample. The expression of $\mathsf{d}\tilde{q}^h/\mathsf{d}\rho$ given by C&R is erroneous.*

(Note that BLB do not discuss this result under BVN.)

Proof. By implicit differentiation, we have

$$\frac{\mathsf{d}\tilde{q}^h}{\mathsf{d}\rho} = -\left.\frac{\partial^2\pi/\partial q\partial\rho}{\partial^2\pi/\partial^2 q}\right|_{\tilde{q}^h}.$$

Because $\partial^2\pi(q,\widetilde{S}^h)/\partial^2 q < 0$, it remains to consider $\partial^2\pi(q,\widetilde{S}^h)/\partial q\partial\rho$. Using the following integral transform (see P544, right column, lines 13-14)

$$\left\{\begin{array}{l}\xi_1 = \tilde{\rho}\sigma_1 z_1 + \hat{\rho}\sigma_1 z_2 + \mu_1,\\ \xi_2 = -\tilde{\rho}\sigma_2 z_1 + \hat{\rho}\sigma_2 z_2 + \mu_2,\end{array}\right. \tag{13}$$

Figure 4 illustrates this space transformation, we can rewrite $\pi(q,\widetilde{S}^h)$ in (11) as

$$\begin{aligned}\pi(q,\widetilde{S}^h) &= -wq + \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\times\\ &\times\Bigg\{\int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{(\tilde{\rho}(\sigma_1-\sigma_2)z_1+\hat{\rho}(\sigma_1+\sigma_2)z_2+\mu_1+\mu_2)^2}{8b(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &+\int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{(\tilde{\rho}(\sigma_1+\sigma_2)z_1+\hat{\rho}(\sigma_1-\sigma_2)z_2+\mu_1-\mu_2)^2}{8b(1+d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &-\int_{\hat{\beta}_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2}\frac{g_4^2(z_1,z_2,q)}{8b(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &-\int_{\tilde{\beta}_2}^{\tilde{\beta}_1}\int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty}\frac{g_4^2(z_1,z_2,q)}{8b(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\ &-\int_{\hat{\beta}_1}^{\infty}\int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{g_4^2(z_1,z_2,q)}{8b(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\Bigg\}.\end{aligned}\tag{14}$$

We have

$$\begin{aligned}\frac{\partial\pi(q,\widetilde{S}^h)}{\partial q} &= -w + \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\times\Bigg\{\int_{\hat{\beta}_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2}\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &+\int_{\tilde{\beta}_2}^{\tilde{\beta}_1}\int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty}\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\ &+\int_{\hat{\beta}_1}^{\infty}\int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\Bigg\}.\end{aligned}\tag{15}$$

Taking the derivative with respect to ρ yields

$$\begin{aligned}\frac{\partial^2\pi(q,\widetilde{S}^h)}{\partial q\partial\rho} &= \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\Bigg\{\int_{\hat{\beta}_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2}\frac{g_5(z_1,z_2)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &+\int_{\tilde{\beta}_2}^{\tilde{\beta}_1}\int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty}\frac{g_5(z_1,z_2)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\ &+\int_{\hat{\beta}_1}^{\infty}\int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{g_5(z_1,z_2)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\ &+\int_{\hat{\beta}_1}^{\infty}\frac{g_6(z_2,q)}{2(1-d)}h_2(z_2)\phi(z_2)\phi(z_2\theta+\gamma_2/\tilde{\rho})\mathsf{d}z_2\\ &+\int_{\hat{\beta}_2}^{\infty}\frac{g_7(z_2,q)}{2(1-d)}h_1(z_2)\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde{\rho})\mathsf{d}z_2\Bigg\}\end{aligned}$$

$$-\frac{\Delta_\rho(\sigma_1,\sigma_2,\rho)}{\Delta^2(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat\beta_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde\rho}^{\tilde\beta_2}\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right.$$
$$+\int_{\tilde\beta_2}^{\tilde\beta_1}\int_{z_1\gamma/\theta+\ell(q)/\hat\rho}^{\infty}\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1$$
$$\left.+\int_{\hat\beta_1}^{\infty}\int_{\tilde\beta_1}^{z_2\theta+\gamma_2/\tilde\rho}\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right\},\tag{16}$$

where

$$\Delta_\rho(\sigma_1,\sigma_2,\rho)=\frac{\mathsf{d}\Delta(\sigma_1,\sigma_2,\rho)}{\mathsf{d}\rho}.$$

By the integral transform given by (13), $\Delta(\sigma_1,\sigma_2,\rho)$ can be equally written as

$$\int_{(-\gamma_1-\gamma_2)/2\hat\rho}^{\infty}\phi(z_2)[\Phi(z_2\theta+\gamma_2/\tilde\rho)-\Phi(-z_2\theta-\gamma_1/\tilde\rho)]\mathsf{d}z_2.$$

Then this gives

$$\begin{aligned}\Delta_\rho(\sigma_1,\sigma_2,\rho) &= \int_{(-\gamma_1-\gamma_2)/2\hat\rho}^{\infty}h_2(z_2)\phi(z_2)\phi(z_2\theta+\gamma_2/\tilde\rho)\mathsf{d}z_2\\ &\quad+\int_{(-\gamma_1-\gamma_2)/2\hat\rho}^{\infty}h_1(z_2)\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde\rho)]\mathsf{d}z_2.\end{aligned}$$

Counterexample: When $\mu_1=100,\mu_2=1000,d=0.992,b=0.2,\rho=0.9999,\sigma_1=15,\sigma_2=100$ and $w=7242.96$, we have the optimal $\tilde q^h=830$, $\partial^2\pi(q,\widetilde S^h)/\partial q\partial\rho|_{q=\tilde q^h=830}=-0.00648758$, and $\partial^2\pi(q,\widetilde S^h)/\partial^2 q|_{q=\tilde q^h=830}=-47.4021$. Then $\mathsf{d}\tilde q^h/\mathsf{d}\rho|_{\tilde q^h=830}=-0.000136863<0$. Here

$$\iint_{\Omega_{1234}}f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2=0.99999,\ \frac{\sigma_1}{\mu_1}=0.15,\ \frac{\sigma_2}{\mu_2}=0.1.$$

□

Proposition 5 (Chod and Rudi). *If* $\boldsymbol\xi$ *has a bivariate normal distribution and* $d=0$*, then* $\mathsf{d}\pi(\tilde q^h,\tilde S^h)/\mathsf{d}\rho<0$.

Claim 4 *The above Proposition 5 is false; there exists a counterexample. The expression of* $\mathsf{d}\pi(\tilde q^h,\tilde S^h)/\mathsf{d}\rho$ *given by C&R is erroneous.*

(Note that BLB are able to show $\mathsf{d}\pi(q^h,S^h)/\mathsf{d}\rho<0$ when $\boldsymbol\xi$ is allowed to take negative values.)

Proof. Notice (18) in C&R has typos, we give the right expression of $\pi(q,\widetilde S^h)$ under $d=0$ as following:

$$\pi(q,\widetilde S^h)\ =\ -wq+\mathsf{E}\left\{\frac{\xi_1^2+\xi_2^2}{4b}\right\}-\mathsf{E}\left[\frac{(\xi_1+\xi_2-2bq)^2}{8b}\mathsf{I}_{\Omega_{234}}\right].$$

Note that

$$\frac{\mathsf{d}\pi(\tilde{q}^h, \widetilde{S}^h)}{\mathsf{d}\rho} = \left.\frac{\partial\pi(q, \widetilde{S}^h)}{\partial\rho}\right|_{q=\tilde{q}^h} + \left.\frac{\partial\pi(q, \widetilde{S}^h)}{\partial q}\right|_{q=\tilde{q}^h} \frac{\partial\tilde{q}^h}{\partial\rho}.$$

Because $\partial\pi(q, \widetilde{S}^h)/\partial q|_{q=\tilde{q}^h} = 0$, we have

$$\frac{\mathsf{d}\pi(\tilde{q}^h, \widetilde{S}^h)}{\mathsf{d}\rho} = \left.\frac{\partial\pi(q, \widetilde{S}^h)}{\partial\rho}\right|_{q=\tilde{q}^h}. \tag{17}$$

Substitute $d = 0$ into (14), and take the derivative with respect to ρ, we obtain

$$\begin{aligned}
&\frac{\partial\pi(q, \widetilde{S}^h)}{\partial\rho} \\
&= \frac{-\Delta_\rho(\sigma_1, \sigma_2, \rho)}{\Delta^2(\sigma_1, \sigma_2, \rho)} \times \\
&\times \Bigg\{ \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{(\tilde{\rho}(\sigma_1-\sigma_2)z_1 + \hat{\rho}(\sigma_1+\sigma_2)z_2 + \mu_1+\mu_2)^2}{8b} \phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
&+ \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{(\tilde{\rho}(\sigma_1+\sigma_2)z_1 + \hat{\rho}(\sigma_1-\sigma_2)z_2 + \mu_1-\mu_2)^2}{8b} \phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
&- \int_{\hat{\beta}_2}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2} \frac{g_4^2(z_1, z_2, q)}{8b} \phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
&- \int_{\tilde{\beta}_2}^{\tilde{\beta}_1} \int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty} \frac{g_4^2(z_1, z_2, q)}{8b} \phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
&- \int_{\hat{\beta}_1}^{\infty} \int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{g_4^2(z_1, z_2, q)}{8b} \phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \Bigg\} \\
&+ \frac{1}{\Delta(\sigma_1, \sigma_2, \rho)} \times \\
&\times \Bigg\{ \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{(\tilde{\rho}(\sigma_1-\sigma_2)z_1 + \hat{\rho}(\sigma_1+\sigma_2)z_2 + \mu_1+\mu_2)}{4b} g_5(z_1, z_2)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
&+ \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \frac{(g_6(z_2, q) + 2bq)^2}{8b} h_2(z_2)\phi(z_2)\phi(z_2\theta + \gamma_2/\tilde{\rho})\mathsf{d}z_2 \\
&+ \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \frac{(g_7(z_2, q) + 2bq)^2}{8b} h_1(z_2)\phi(z_2)\phi(-z_2\theta - \gamma_1/\tilde{\rho})\mathsf{d}z_2 \\
&+ \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{(\tilde{\rho}(\sigma_1+\sigma_2)z_1 + \hat{\rho}(\sigma_1-\sigma_2)z_2 + \mu_1-\mu_2)}{4b} \times \\
&\qquad\qquad \times \frac{(\sigma_1-\sigma_2)\tilde{\rho}z_2 - (\sigma_1+\sigma_2)\hat{\rho}z_1}{4\tilde{\rho}\hat{\rho}} \phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
&+ \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \frac{(2\hat{\rho}\sigma_1 z_2 + (\sigma_1+\sigma_2)\gamma_2 + \mu_1-\mu_2)^2}{8b} h_2(z_2)\phi(z_2)\phi(z_2\theta + \gamma_2/\tilde{\rho})\mathsf{d}z_2 \\
&+ \int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty} \frac{(-2\hat{\rho}\sigma_2 z_2 - (\sigma_1+\sigma_2)\gamma_1 + \mu_1-\mu_2)^2}{8b} h_1(z_2)\phi(z_2)\phi(-z_2\theta - \gamma_1/\tilde{\rho})\mathsf{d}z_2
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\hat{\beta}_2}^{\infty} \int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2} \frac{g_4(z_1,z_2,q)}{4b} g_5(z_1,z_2)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
&- \int_{\tilde{\beta}_2}^{\tilde{\beta}_1} \int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty} \frac{g_4(z_1,z_2,q)}{4b} g_5(z_1,z_2)\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
&- \int_{\hat{\beta}_1}^{\infty} \int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}} \frac{g_4(z_1,z_2,q)}{4b} g_5(z_1,z_2)\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
&- \int_{\hat{\beta}_1}^{\infty} \frac{g_6^2(z_2,q)}{8b} h_2(z_2)\phi(z_2)\phi(z_2\theta+\gamma_2/\tilde{\rho})\mathsf{d}z_2 \\
&- \int_{\hat{\beta}_2}^{\infty} \frac{g_7^2(z_2,q)}{8b} h_1(z_2)\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde{\rho})\mathsf{d}z_2 \Bigg\}.
\end{aligned} \tag{18}
$$

Counterexample: When $\mu_1 = 1, \mu_2 = 0.05, b = 0.01, \rho = 0.5, \sigma_1 = 0.3, \sigma_2 = 0.023$ and $w = 0.498461$, we have the optimal $\tilde{q}^h = 3$ and $\partial\pi(q,\widetilde{S}^h)/\partial\rho|_{q=\tilde{q}^h=3} = 0.029046 > 0$. Then $\mathsf{d}\pi(\tilde{q}^h,\widetilde{S}^h)/\mathsf{d}\rho|_{\tilde{q}^h=3} = 0.029046 > 0$. Here

$$
\iint_{\Omega_{1234}} f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 = 0.984858, \ \frac{\sigma_1}{\mu_1} = 0.3, \ \frac{\sigma_2}{\mu_2} = 0.46.
$$

□

Proposition 9 (Chod and Rudi). *There exists a Stackelberg equilibrium capacity q^* maximizing the supplier's profit $\pi_s(q)$, characterized by the following necessary conditions:*

$$
\iint_{\Omega_{234}(q)} \frac{x_1+x_2-4bq}{2(1-d)} f(x_1,x_2)\mathsf{d}x_2\mathsf{d}x_1 = c.
$$

If, furthermore, the random variable $\mathbf{1}^T\boldsymbol{\xi}$ has an increasing generalized failure rate , the necessary condition is sufficient and q^* is unique. The corresponding equilibrium resource price w^* inducing the firm to order q^* is obtained from $w = \iint_{\Omega_{234}(q)} \frac{x_1+x_2-2bq}{2(1-d)} f(x_1,x_2)\mathsf{d}x_2\mathsf{d}x_1$.

The proof of second part (the sufficient condition) of the above Proposition 9 given by C&R is incorrect. The following gives the explanation why their proof is not correct.

We make the following integral transform

$$
\begin{cases} x_1 + x_2 = \tau, \\ x_1 - x_2 = \eta, \end{cases}
$$

and let $g(\cdot)$ be the density function of $\zeta = \xi_1 + \xi_2$. Then, from (19) in C&R,

$$
\begin{aligned}
\frac{\mathsf{d}\pi_s(q)}{\mathsf{d}q} &= \iint_{\Omega_{234}(q)} \frac{x_1+x_2-4bq}{2(1-d)} f(x_1,x_2)\mathsf{d}x_2\mathsf{d}x_1 - c \\
&= \int_{2bq}^{\infty} \frac{\tau-4bq}{4(1-d)} \int_{-\tau}^{\tau} f(\frac{\tau+\eta}{2},\frac{\tau-\eta}{2})\mathsf{d}\eta\mathsf{d}\tau - c \\
&\neq \int_{2bq}^{\infty} \frac{\tau-4bq}{2(1-d)} g(\tau)\mathsf{d}\tau - c \quad \text{(see P547, left column, line -12.)}
\end{aligned}
$$

This implies that

$$\begin{aligned}\frac{\mathsf{d}^2\pi_s(q)}{\mathsf{d}q^2} &= \frac{b^2 q}{1-d}\int_{-2bq}^{2bq} f(\frac{2bq+\eta}{2},\frac{2bq-\eta}{2})\mathsf{d}\eta \\ &\quad -\int_{2bq}^{\infty}\frac{b}{1-d}\int_{-\tau}^{\tau} f(\frac{\tau+\eta}{2},\frac{\tau-\eta}{2})\mathsf{d}\eta\mathsf{d}\tau \\ &\neq \frac{b}{1-d}\Big[2bq\cdot g(2bq)-2\mathsf{Pr}(\zeta>2bq)\Big]\text{(see P547, left column, line -11.)}\end{aligned}$$

Thus, the proof of C&R breaks down.

Proposition 10 (Chod and Rudi). *If* $\boldsymbol{\xi}$ *has a bivariate normal distribution, the equilibrium resource capacity and the supplier's profit satisfy the following relationships:*

(i) $\mathsf{d}q^*/\mathsf{d}\sigma_i \geq 0$ if and only if $\sigma_i \geq -\rho\sigma_j$,

(ii) $\mathsf{d}\pi_s(q^*,w^*)/\mathsf{d}\sigma_i \geq 0$ if and only if $\sigma_i \geq -\rho\sigma_j$,

(iii) $\mathsf{d}q^*/\mathsf{d}\rho \geq 0$, and

(iv) $\mathsf{d}\pi_s(q^*,w^*)/\mathsf{d}\rho \geq 0$.

Claim 5 *The general conclusions of Proposition 10 are false; there exist counterexamples.*

(Note that BLB are able to show part (ii) of the proposition under $\sigma_1=\sigma_2$, when $\boldsymbol{\xi}$ is allowed to take negative values, but not the general case of $\sigma_1\neq\sigma_2$ nor the other results.)

Proof Let $G(q)=\mathsf{d}\pi_s(q)/\mathsf{d}q$, the necessary condition for q^* is

$$G(q)=\iint_{\Omega_{234}(q)}\frac{\xi_1+\xi_2-4bq}{2(1-d)}f(\xi_1,\xi_2)\mathsf{d}\xi_2\mathsf{d}\xi_1-c=0.$$

Using the integral transform given by (13), we can rewrite $G(q)$ as

$$\begin{aligned}G(q) &= \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat{\beta}_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right. \\ &\quad+\int_{\hat{\beta}_2}^{\tilde{\beta}_1}\int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\ &\quad\left.+\int_{\hat{\beta}_1}^{\infty}\int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right\}-c. \qquad (19)\end{aligned}$$

The above equation is different from the counterpart in C&R:

$$G(q)=\int_{\frac{2bq-\mu_1-\mu_2}{\bar{\sigma}}}^{\infty}\frac{z\bar{\sigma}+\mu_1+\mu_2-4bq}{2(1-d)}\phi(z)\mathsf{d}z-c,\quad\text{(see P547, right column, line 10).}$$

where $\bar{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho}$ (see P544, right column, line 9). By using this incorrect expression of $G(q)$, C&R can obtain the results in Proposition 10 (see P547, right column and P548, left column). In the remaining proof, we use (19) to arrive at different results.

For part (i), using implicit differentiation, we obtain $\mathsf{d}q^*/\mathsf{d}\sigma_i = -(\partial G/\partial\sigma_i)/(\partial G/\partial q)|_{q=q^*}$. Because $\pi_s(q)$ has an interior maximum at q^*, we have $\partial^2\pi_s(q)/\partial q^2|_{q=q^*} = \partial G(q)/\partial q|_{q=q^*} \le 0$. Thus, to give a counterexample to show (i) does not hold with $\sigma_1 \ge -\rho\sigma_2$, it suffices to give a numerical example such that $\partial G(q)/\partial\sigma_1|_{q=q^*} < 0$. To this end, we first derive $\partial G(q)/\partial\sigma_1$. We have

$$\begin{aligned}
\frac{\partial G(q)}{\partial\sigma_1} &= \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat{\beta}_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2}\frac{z_1\tilde{\rho}+z_2\hat{\rho}}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right.\\
&\quad+\int_{\tilde{\beta}_2}^{\tilde{\beta}_1}\int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty}\frac{z_1\tilde{\rho}+z_2\hat{\rho}}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\
&\quad+\int_{\hat{\beta}_1}^{\infty}\int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{z_1\tilde{\rho}+z_2\hat{\rho}}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\\
&\quad+\int_{\hat{\beta}_2}^{\infty}-\gamma_1\frac{2z_2\sigma_2\hat{\rho}+\gamma_1\sigma_2+\mu_2-4bq}{2(1-d)\sigma_1\tilde{\rho}}\phi(z_2)\phi(-z_2\theta-\frac{\gamma_1}{\tilde{\rho}})\mathsf{d}z_2\\
&\quad\left.+\int_{\tilde{\beta}_2}^{\tilde{\beta}_1}-\frac{bq}{1-d}\left(\frac{2z_1\sigma_2}{\theta(\sigma_1+\sigma_2)^2}+\frac{\ell(q)}{\hat{\rho}(\sigma_1+\sigma_2)}\right)\phi(z_1)\phi\left(z_1\frac{\gamma}{\theta}+\frac{\ell(q)}{\hat{\rho}}\right)\mathsf{d}z_1\right\}\\
&\quad-\frac{\Delta_{\sigma_1}(\sigma_1,\sigma_2,\rho)}{\Delta^2(\sigma_1,\sigma_2,\rho)}\left\{\int_{\hat{\beta}_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde{\rho}}^{\tilde{\beta}_2}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right.\\
&\qquad+\int_{\tilde{\beta}_2}^{\tilde{\beta}_1}\int_{z_1\gamma/\theta+\ell(q)/\hat{\rho}}^{\infty}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1\\
&\qquad\left.+\int_{\hat{\beta}_1}^{\infty}\int_{\tilde{\beta}_1}^{z_2\theta+\gamma_2/\tilde{\rho}}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\right\},
\end{aligned}$$

where

$$\Delta_{\sigma_1}(\sigma_1,\sigma_2,\rho) = \frac{\mathsf{d}\Delta(\sigma_1,\sigma_2,\rho)}{\mathsf{d}\sigma_1} = -\int_{(-\gamma_1-\gamma_2)/2\hat{\rho}}^{\infty}\frac{\gamma_1}{\sigma_1\tilde{\rho}}\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde{\rho})\mathsf{d}z_2.$$

Counterexample Let $\mu_1 = 20$, $\mu_2 = 15$, $\sigma_1 = 5$, $\sigma_2 = 3$, $\rho = 0.1$, $d = 0.5$, $b = 0.2$, and $c = 19.0007$. We have $q^* = 20$, $\partial G(q^*)/\partial\sigma_1 = -0.121233$ and $\partial G(q)/\partial q|_{q=q^*} = -0.591631$. Then $\mathsf{d}q^*/\mathsf{d}\sigma_1|_{q^*=20} = -0.204913 < 0$. Here

$$\iint_{\Omega_{1234}} f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 = 0.999968,\ \frac{\sigma_1}{\mu_1} = 0.25,\ \frac{\sigma_2}{\mu_2} = 0.2.$$

For part (ii), it follows from $\pi_s(q) = (w(q)-c)q$ and

$$w(q) = \iint_{\Omega_{234}(q)}\frac{\xi_1+\xi_2-2bq}{2(1-d)}f(\xi_1,\xi_2)\mathsf{d}\xi_2\mathsf{d}\xi_1$$

that

$$\pi_s(q) = q \iint_{\Omega_{234}(q)} \frac{\xi_1+\xi_2-2bq}{2(1-d)} f(\xi_1,\xi_2)\mathsf{d}\xi_2\mathsf{d}\xi_1 - cq. \tag{20}$$

On the other hand,

$$\frac{\mathsf{d}\pi_s(q^*,w^*)}{d\sigma_i} = \left.\frac{\partial\pi_s(q,w(q))}{\partial\sigma_i}\right|_{q=q^*} + \left.\frac{\partial\pi_s(q,w(q))}{\partial q}\right|_{q=q^*} \cdot \frac{\partial q^*}{\partial\sigma_i}.$$

Because $\partial\pi_s(q,w(q))/\partial q|_{q^*} = 0$, we have

$$\frac{\mathsf{d}\pi_s(q^*,w^*)}{d\sigma_i} = \left.\frac{\partial\pi_s(q,w(q))}{\partial\sigma_i}\right|_{q=q^*}. \tag{21}$$

To give a counterexample to show part (ii) fails hold with $\sigma_1 \geq -\rho\sigma_2$, we first derive $\partial\pi_s(q,w(q))/\partial\sigma_1$. By the integral transform given by (13), we have

$$\begin{aligned}\pi_s(q,w(q)) &= \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\Bigg\{\int_{\hat\beta_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde\rho}^{\tilde\beta_2} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\ &\quad + \int_{\tilde\beta_2}^{\tilde\beta_1}\int_{z_1\gamma/\theta+\ell(q)/\hat\rho}^{\infty} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\ &\quad + \int_{\hat\beta_1}^{\infty}\int_{\tilde\beta_1}^{z_2\theta+\gamma_2/\tilde\rho} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\Bigg\} - cq, \end{aligned} \tag{22}$$

so

$$\begin{aligned}\frac{\partial\pi_s(q,w(q))}{\partial\sigma_1} &= \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\Bigg\{\int_{\hat\beta_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde\rho}^{\tilde\beta_2} q\frac{z_1\tilde\rho+z_2\hat\rho}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\ &\quad + \int_{\tilde\beta_2}^{\tilde\beta_1}\int_{z_1\gamma/\theta+\ell(q)/\hat\rho}^{\infty} q\frac{z_1\tilde\rho+z_2\hat\rho}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\ &\quad + \int_{\hat\beta_1}^{\infty}\int_{\tilde\beta_1}^{z_2\theta+\gamma_2/\tilde\rho} q\frac{z_1\tilde\rho+z_2\hat\rho}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\ &\quad + \int_{\hat\beta_2}^{\infty} -q\gamma_1\frac{2z_2\sigma_2\hat\rho+\gamma_1\sigma_2+\mu_2-2bq}{2(1-d)\sigma_1\tilde\rho}\phi(z_1)\phi(-z_2\theta-\frac{\gamma_1}{\tilde\rho})\mathsf{d}z_2\Bigg\}\end{aligned}$$

$$\begin{aligned}&-\frac{\Delta_{\sigma_1}(\sigma_1,\sigma_2,\rho)}{\Delta^2(\sigma_1,\sigma_2,\rho)}\Bigg\{\int_{\hat\beta_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde\rho}^{\tilde\beta_2} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\ &\quad + \int_{\tilde\beta_2}^{\tilde\beta_1}\int_{z_1\gamma/\theta+\ell(q)/\hat\rho}^{\infty} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\ &\quad + \int_{\hat\beta_1}^{\infty}\int_{\tilde\beta_1}^{z_2\theta+\gamma_2/\tilde\rho} q\frac{g_4(z_1,z_2,q)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\Bigg\}.\end{aligned}$$

Counterexample Letting $\mu_1 = 20,\ \mu_2 = 15,\ \sigma_1 = 5,\ \sigma_2 = 3,\ \rho = 0.1, d = 0.5,\ b = 0.2$, and $c = 19.0007$, we obtain $q^* = 20$ and $\mathsf{d}\pi_s(q^*, w^*)/d\sigma_1|_{q^*=20} = -2.40872 < 0$. Here

$$\iint_{\Omega_{1234}} f(\xi_1, \xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 = 0.999968,\ \frac{\sigma_1}{\mu_1} = 0.25,\ \frac{\sigma_2}{\mu_2} = 0.2.$$

For part (iii), similar to part (i), we have that $\mathsf{d}q^*/\mathsf{d}\rho = -\partial G/\partial\rho/(\partial G/\partial q)|_{q=q^*}$. Because $\pi_s(q)$ has an interior maximum at q^*, then $\partial^2\pi_s(q)/\partial q^2|_{q=q^*} = \partial G/\partial q|_{q=q^*} \le 0$. Thus to give a counterexample to show that (iii) does not hold, it suffices to give a numerical example such that $\partial G(q)/(\partial\rho)|_{q=q^*} < 0$. We have $\partial G/\partial\rho$ as follows:

$$\begin{aligned}
\frac{\partial G(q)}{\partial\rho} \;=\; & \frac{1}{\Delta(\sigma_1,\sigma_2,\rho)}\Bigg\{\int_{\hat\beta_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde\rho}^{\tilde\beta_2}\frac{g_5(z_1,z_2)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
& + \int_{\tilde\beta_2}^{\tilde\beta_1}\int_{z_1\gamma/\theta+\ell(q)/\hat\rho}^{\infty}\frac{g_5(z_1,z_2)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
& + \int_{\hat\beta_1}^{\infty}\int_{\tilde\beta_1}^{z_2\theta+\gamma_2/\tilde\rho}\frac{g_5(z_1,z_2)}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
& + \int_{\hat\beta_1}^{\infty}\frac{g_6(z_2,q)-2bq}{2(1-d)}h_2(z_2)\phi(z_2)\phi(z_2\theta+\gamma_2/\tilde\rho)\mathsf{d}z_2 \\
& + \int_{\hat\beta_2}^{\infty}\frac{g_7(z_2,q)-2bq}{2(1-d)}h_1(z_2)\phi(z_2)\phi(-z_2\theta-\gamma_1/\tilde\rho)\mathsf{d}z_2 \\
& + \int_{\tilde\beta_2}^{\tilde\beta_1}\frac{-2bq}{2(1-d)}\frac{1}{(1+\rho)^2}(z_1\gamma\theta+\ell(q)\hat\rho)\phi(z_1)\phi(z_1\gamma/\theta+\ell(q)/\hat\rho)\mathsf{d}z_1\Bigg\} \\
& -\frac{\Delta_\rho(\sigma_1,\sigma_2,\rho)}{\Delta^2(\sigma_1,\sigma_2,\rho)}\Bigg\{\int_{\hat\beta_2}^{\infty}\int_{-z_2\theta-\gamma_1/\tilde\rho}^{\tilde\beta_2}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2 \\
& + \int_{\tilde\beta_2}^{\tilde\beta_1}\int_{z_1\gamma/\theta+\ell(q)/\hat\rho}^{\infty}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_2\mathsf{d}z_1 \\
& + \int_{\hat\beta_1}^{\infty}\int_{\tilde\beta_1}^{z_2\theta+\gamma_2/\tilde\rho}\frac{g_4(z_1,z_2,q)-2bq}{2(1-d)}\phi(z_1)\phi(z_2)\mathsf{d}z_1\mathsf{d}z_2\Bigg\}.
\end{aligned}$$

Counterexample Let $\mu_1 = 25,\ \mu_2 = 15,\ \sigma_1 = 5,\ \sigma_2 = 3,\ \rho = 0.9992,\ d = 0.5,\ b = 0.0001$, and $c = 39.9952$. We have $q^* = 12$, $\partial G(q^*)/\partial\rho = -0.000583854$ and $\partial G(q)/\partial q|_{q=q^*} = -0.0004$. Then $\mathsf{d}q^*/\mathsf{d}\rho|_{q^*=12} = -1.45963 < 0$. Here

$$\iint_{\Omega_{1234}} f(\xi_1, \xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2 = 0.999999,\ \frac{\sigma_1}{\mu_1} = 0.2,\ \frac{\sigma_2}{\mu_2} = 0.2.$$

Finally, we consider part (iv) . Note that

$$\frac{\mathsf{d}\pi_s(q^*, w^*)}{d\rho} = \frac{\partial\pi_s(q, w(q))}{\partial\rho}\bigg|_{q=q^*} + \frac{\partial\pi_s(q, w(q))}{\partial q}\bigg|_{q=q^*}\cdot\frac{\partial q^*}{\partial\rho}.$$

Because $\partial\pi_s(q,w(q))/\partial q|_{q=q^*}=0$, it suffices to consider $\partial\pi_s(q,w(q))/\partial\rho|_{q=q^*}$. From (15) and (22), we have

$$\frac{\partial\pi_s(q,w(q))}{\partial\rho} \quad = \quad q\frac{\partial^2\pi(q,\widetilde{S}^h)}{\partial q\partial\rho}.$$

Counterexample: Letting $\mu_1=100,\mu_2=1000,b=0.2,\rho=0.9999,d=0.992,\sigma_1=15,\sigma_2=100$ and $w=7242.96$, we obtain $\tilde{q}^h=830$ and $\partial^2\pi(\tilde{q}^h,\widetilde{S}^h)/\partial q\partial\rho=-0.519006<0$. Then $\mathsf{d}\pi_s(q^*,w^*)/\mathsf{d}\rho|_{q^*=830}=-430.775<0$. Here we have

$$\iint_{\Omega_{1234}} f(\xi_1,\xi_2)\mathsf{d}\xi_1\mathsf{d}\xi_2=0.99999,\ \frac{\sigma_1}{\mu_1}=0.15,\ \frac{\sigma_2}{\mu_2}=0.1.$$

□